TRITA-PFU-87-03

MHD EQUILIBRIUM WITH TOROIDAL ROTATION

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Stockholm, March 1987

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Abstract The present work attempts to formulate the equilibrium of axisymmetric plasma with purely toroidal flow within ideal MHD theory. In general, the inertial term $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$ caused by plasma flow is so complicated that the equilibrium equation is completely different from the Grad-Shafranov equation. However, in the case of purely toroidal flow the equilibrium equation can be simplified so that it resembles the Grad-Shafranov equation. Generally one arbitrary two-variable function and two arbitrary single variable functions, instead of only four single-variable functions [1, 2], are allowed in the new equilibrium equations. Also, the boundary conditions of the rotating* equilibrium are the same as those of the static equilibrium. So numerically one can calculate the rotating equilibrium as a static equilibrium.

* In this paper, for convenience, we always assume that
rotating - with purely toroidal fluid flow
static - without any fluid flow.

1. Introduction

Equilibrium with plasma flow is not a new topic [3, 4], but an increasing interest towards it seems to have risen since fluid velocities approaching the thermal velocity have been observed in PLT [5, 6] and PDX [7] in the presence of unbalanced neutral beam injection.

Much work has been done in the area with an emphasis on flow effects on equilibrium. Green and Zehrfeld [8] estimated the shift of the magnetic axis due to plasma flow in the large aspect ratio limit. Maschke and Perrin [1] gave some exact polynomial solutions to the equilibrium equation with purely toroidal flow in order to discuss effects on the static field configuration. The conclusions are quite natural: fluid flow effects are important only when the Mach number M approaches unity. However, when the poloidal rotation is taken into consideration [9, 10], the conclusion is quite different. It seems that small velocity (M = 0.06) can cause drastical asymmetry in poloidal density distribution when the velocity is nearly perpendicular to the magnetic field, whereas the effects upon the magnetic field still remain small. It is, however, physically questionable to assume that the velocity is perpendicular to the magnetic field. According to the PDX report [7], poloidal rotation damping times are much shorter than toroidal damping times, so practically one can expect a toroidal rotation dominated equilibrium. Also one needs to explain the difference between the toroidal velocity profile calculated under that supposition ([9] Fig.3c, 4c; [4] Fig.4) and the experimental results ([7] p.1650). In spite of these problems, physically it is still significant to reveal that a poloidal rotation might have an important effect on the density profile of the equilibrium.

Instead of studying the physical behaviour of rotating plasma, we attempt to formulate the toroidally rotating plasma equilibrium within the ideal MHD theory without limiting to fusion research. Although, the physical background of our work is plasma rotation under a neutral beam injection. The formulation of the rotating equilibrium has already been done by several authors [1, 2, 4, 14] with different points of view, but we try to do this in such a way that the similarity of the rotating equilibrium to the static equilibrium is emphasized. So the equilibrium equation has been re-derived with a similar form to the Grad-Shafranov equation, and in order to make a thorough comparison with the static problem, the condition for a plasma boundary to be a flux surface in the presence of the plasma flow is discussed in detail. Also, we point out that the rotating equilibrium equation generally has the degrees of freedom of one arbitrary two-variable function and two arbitrary single-variable functions. This result is more general than previous works where only four arbitrary single-variable functions are allowed [1, 2, 9, 11].

2. The equilibrium equation and its degrees of freedom

The starting equations are the stationary ideal MHD equations:

$$\nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$
$$\rho\, (\mathbf{u}\cdot\nabla)\, \mathbf{u} = \mathbf{j} \times \mathbf{B} - \nabla p \tag{2}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$
$$\nabla \times \mathbf{E} = \mathbf{0} \tag{4}$$
$$\mu_0\, \mathbf{j} = \nabla \times \mathbf{B} \tag{5}$$
$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{0} \tag{6}$$
$$\mathbf{u} \cdot \nabla S = 0 \tag{7}$$

here $\mathbf{u}$ is the fluid velocity, $\mathbf{B}$ is the magnetic field, $\mathbf{j}$ is the current density, $\mathbf{E}$ is the electric field, p and ρ are the pressure and mass density of the fluid respectively, μ_0 is the permeability of vacuum and S is the entropy. The rationalized MKS units are used.

Cylindrical coordinates (R, ϕ, z), $(\mathbf{e}_R, \mathbf{e}_\phi, \mathbf{e}_z)$ are chosen.

Two fundamental hypotheses

(H1) Axisymmetry $$\frac{\partial}{\partial\phi} = 0 \tag{8}$$

(H2) Purely toroidal flow $$\mathbf{u} = u_\phi\, \mathbf{e}_\phi \tag{9}$$

Introducing the magnetic flux function ψ of the magnetic field $\mathbf{B}$

$$\mathbf{B} = B_\phi\, \mathbf{e}_\phi + R^{-1}\, \nabla\psi \times \mathbf{e}_\phi \tag{10}$$

and the current density

$$\mu_0 \mathbf{j} = -\, R^{-1}\, \pounds\psi\, \mathbf{e}_\phi + R^{-1}\, \nabla I \times \mathbf{e}_\phi \tag{11}$$

From the equations (1) - (7) and the hypotheses (8) and (9) we arrive at the vectorial equilibrium equation [1]

$$(\pounds\psi + I\, I')\nabla\psi + \mu_0\, R^2\, \nabla p - \rho\, \mu_0\, R^3\, \omega^2\, \nabla R = 0, \tag{12}$$

in which angular velocity ω and poloidal current I have to be functions of ψ:

$$I = I(\psi) = B_\phi R \qquad (13)$$

$$\omega = \omega(\psi) = u_\phi / R, \qquad (14)$$

where

$$\pounds = R^2 \nabla \cdot R^{-2} \nabla \qquad (15)$$

$$I' = dI / d\psi. \qquad (16)$$

All the information about ψ in equations (1) - (7) under the hypotheses (8) and (9) are summarized in the equations (12) - (14).

From the following point of view and (12) we can derive the equilibrium equation.

Since in the vector equation (12) ∇p has components only in directions of $\nabla\psi$ and ∇R and z does not appear explicitly in the coefficients of the equation, it is reasonable to assume that pressure p is a function of ψ and R:

$$p = f(\psi, R) \qquad (17)$$

it means that instead of space arguments R and z we can choose ψ and R as independent arguments to describe pressure distribution. The same conclusion can be made to other physical quantities such as the density due to the following equation (19). So the symbols such as $(\partial p/\partial\psi)_R$ and $(\partial p/\partial R)_\psi$ have meaning.

Equilibrium equation

The $\nabla\psi$ and ∇R components of (12) give

$$\pounds\psi + I\,I' + \mu_0 R^2 (\partial p / \partial\psi)_R = 0 \qquad (18)$$

$$(\partial p / \partial R)_\psi = \rho R \omega^2(\psi) \qquad (19)$$

respectively.

Since the equations (18) and (19) are the components of momentum equation (12), both of them are 'equilibrium equations' in the sense of force balance. However, in the MHD equilibrium theory the magnetic field configuration ψ is of main interest, so eq. (18) can be considered as a new equilibrium equation for ψ, and (19) serves as an assistant relation to solve for the density ρ.

Degrees of freedom of the equilibrium equation

In this section we try to answer or explain the questions
(i) Where do the degrees of freedom come from and how many are they?
(ii) What are the forms of these degrees of freedom?

Origin of degrees of freedom. The viewpoint taken here is that *if there are more unknowns than equations, then there are degrees of freedom, and the number of degrees of freedom = number of unknowns - number of equations.*

In the Grad-Shafranov equation, we have three unknowns: ψ (poloidal field $\mathbf{B}_p$), $I(\psi)$ (toroidal field B_ϕ) and pressure p, but only one relation — the Grad-Shafranov equation itself. We have, therefore, two extra degrees of freedom, two arbitrary single-variable functions, $I(\psi)$ and $p(\psi)$ in the Grad-Shafranov equation.

Quite differently from the static case, the rotating equilibrium involves mass density ρ and velocity $u_\phi = \omega R$. So in this case we have five unknowns, ψ, p, ρ, ω and I, and only two relations, Eqs. (18) and (19). The obvious deduction is that we have three degrees of freedom in Eqs. (18) and (19), they are one-variable arbitrary functions $I(\psi)$ and $\omega(\psi)$ and another degree of freedom for p waiting to be determined.

The forms of degrees of freedom. Here we start from the viewpoint that *every degree of freedom should be an arbitrary two-variable function (double degree of freedom) unless there is a condition to limit it.* The toroidal component of the momentum equation (2)

$$\mu_0 \, \mathbf{j} \times \mathbf{B} \cdot \mathbf{e}_\phi = - \nabla I \times \nabla\psi \cdot \mathbf{e}_\phi \, / \, R^2 = 0 \qquad (20)$$

implies that $I = I(\psi)$, and Ohm's law (6) and Faraday's law (4) give

$$\nabla\Phi = - \mathbf{E} = \mathbf{u} \times \mathbf{B} = u_\phi \, \nabla\psi \, / \, R = \omega(\psi) \, \nabla\psi, \qquad (21)$$

and $\omega = \omega(\psi)$. Here Φ is the electric potential. So I and ω have to be arbitrary single-variable functions (single degree of freedom). There is, however, no limitation on p. Thus for p *there must exist a double degree of freedom.* The next problem is the appearance of this degree of freedom. One might think that $p(\psi,R)$ itself could be an arbitrary two-variable function. $p(\psi,R)$ could be treated as an arbitrary two-variable function in Eq. (18) if ψ were the only quantity to solve, because in an ideal MHD model there is no limit to hamper his. But the problem is that in this case one can not see the relation between ψ and the rotation, which gives the most important information of rotating plasmas.

One way to formulate this problem is to take this double degree of freedom in the form of a relation between p, ρ and ψ

$$F(p, \rho, \psi) = 0 \tag{22}$$

here F is an arbitrary function. In this form the physics can be seen immediately.

From (22) one can derive the form which the pressure has to take. This can be discussed in two cases:

(1) $\partial F/\partial p = 0$, Eq. (22) reduces to a single degree of freedom $\rho = \rho(\psi)$. Integrating directly Eq. (19) gives

$$p = p_0(\psi) + \rho(\psi)\,\omega^2 R^2/2 \tag{23}$$

(2) generally $\rho \neq \rho(\psi)$, we then have a double degree of freedom

$$p = f(\rho,\psi) \tag{24}$$

here f is an arbitrary function. Inserting in Eq.(19) and integrating gives

$$\int \frac{1}{\rho}\frac{\partial f}{\partial \rho}\,d\rho = \theta_f(\psi) + \frac{1}{2} R^2 \omega^2(\psi) \tag{25}$$

where $\theta_f(\psi)$ is the integration constant with respect to energy. From (25) we can solve for ρ

$$\rho = \rho(\,R^2\omega^2/2 + \theta_f(\psi), \psi) \tag{26}$$

substituted into Eq. (24) yields

$$p = p(\,R^2\omega^2/2 + \theta_f(\psi), \psi). \tag{27}$$

Taking into account the arbitrariness of the function f, we can write

$$p = g(\, R^2\omega^2 / 2,\, \psi). \tag{28}$$

Here g (ξ, η) is an arbitrary function. And from Eq. (19) we solve for ρ

$$\rho(\psi,R) = (\, \partial g/\partial\xi \,)_{\,\xi\, =\, R^2\, \omega^2 / 2}. \tag{29}$$

Actually (28) takes (23) as a special case, so we conclude that (28) is the general form of the pressure. From it we can see how the pressure depends on R and $\omega(\psi)$.

Some important reduced cases of $p(\psi,R)$

Corresponding to different choices of $f(\rho,\psi)$ in (24), we can obtain degenerate forms of $p(\psi,R)$. Here are some of the most important cases which usually serve as basic assumptions in many works [1, 2, 9, 11].

(1) Entropy S is an arbitrary surface quantity

$$S = S(\psi) \tag{30}$$

$$p = A(\psi)\, \rho^{\gamma} = f(\rho,\psi) \tag{31}$$

here γ is the adiabatic index and A is a definite function of entropy S and consequently an arbitrary function of ψ. We find

$$\frac{\partial f}{\partial \rho} = \gamma\, \rho^{\gamma - 1}\, A\,(\psi) \tag{32}$$

Here the integral (25) has a definite physical meaning -- enthalpy i

$$i = \int \frac{1}{\rho}\, \frac{\partial f}{\partial \rho}\, d\rho = \frac{\gamma A}{\gamma - 1}\, \rho^{\gamma-1} = \theta_s(\psi) + \frac{1}{2}\, R^2\, \omega^2 \tag{33}$$

From (33) one can solve for ρ, then p

$$\rho(\psi,R) = \left(\frac{\gamma-1}{\gamma A(\psi)} \left(\theta_S(\psi) + \frac{1}{2} R^2 \omega^2 \right)\right)^{\frac{1}{\gamma-1}} \tag{34}$$

$$p(\psi,R) = A(\psi) \left(\frac{\gamma-1}{\gamma A(\psi)} \left(\theta_S(\psi) + \frac{1}{2} R^2 \omega^2 \right)\right)^{\frac{\gamma}{\gamma-1}} \tag{35}$$

here $A(\psi)$, $\theta_s(\psi)$, $\omega(\psi)$ and $I(\psi)$ are independent arbitrary surface quantities.

(2) Temperature is a surface quantity

$$T = T(\psi) \tag{36}$$

$$p = \rho R_g T(\psi) = f(\rho,\psi) \tag{37}$$

here T is temperature and R_g is the gas constant. The integral (25) becomes

$$\int \frac{1}{\rho} \frac{\partial f}{\partial \rho} d\rho = R_g T(\psi) \ln \frac{\rho}{\rho_0} = \theta_T(\psi) + \frac{1}{2} R^2 \omega^2 \tag{38}$$

in which ρ_0 can be an arbitrary function of ψ, but it is not independent and can be combined into the arbitrary function $\theta_T(\psi)$. The result is

$$\rho(\psi,R) = \rho_0(\psi) \exp \frac{R^2 \omega^2(\psi)}{2 R_g T(\psi)} \tag{39}$$

$$p(\psi,R) = p_0(\psi) \exp \frac{R^2 \omega^2(\psi)}{2 R_g T(\psi)} \tag{40}$$

Here the independent surface quantities should be $p_0(\psi)$, $\omega(\psi)$, $T(\psi)$ and $I(\psi)$.

3. Boundary conditions of the rotating equilibrium

A complete comparison to static equilibrium demands the flux surface condition for the plasma boundary.

It is obvious that the rotating equilibrium should have the same conductor-plasma or conductor-vacuum conditions as a static equilibrium, because a perfect conductor boundary must be a flux surface whether there is plasma flow or not. However, the plasma-vacuum interface conditions need to be discussed.

Conditions for a plasma boundary to be a flux surface.

By definition, a plasma boundary is a constant pressure surface. So we need only to check that if $\mathbf{B}\cdot\nabla p$ is zero at the plasma boundary. Taking the scalar product of $\mathbf{B}$ with the momentum equation (2), under the pure toroidal plasma flow assumption, we can obtain

$$\mathbf{B}\cdot\nabla p = \frac{\rho\, u_\phi^2\, B_R}{R} \tag{41}$$

From this relation we can conclude that the necessary and sufficient condition for a plasma boundary to be a flux surface is that either the plasma density or the velocity vanishes at the plasma boundary. Physically speaking, however, it is always reasonable to assume zero density or velocity condition at the plasma boundary. Therefore we can still use the flux surface condition in the case of the rotating equilibrium.

Conductor - vacuum boundary value problem.

The conductor -vacuum boundary value problem for a rotating equilibrium can be presented in the same way as a static equilibrium [12], since they have similar equilibrium equations and the same boundary conditions.

Let Γ be the plasma boundary in R-z plane and C the conducting wall, then inside Γ

$$\pounds\psi + I\,I' + \mu_0 R^2 \left(\frac{\partial p}{\partial \psi}\right)_R = 0 \tag{42}$$

$$p(\psi,R) = g\left(\frac{\omega^2(\psi)\,R^2}{2}, \psi\right) \tag{43}$$

where $I(\psi)$, $\omega(\psi)$ and $g(\xi,\eta)$ are arbitrary functions.

Outside Γ

$$\pounds\psi = 0 \tag{44}$$

$$I'(\psi) = 0 \tag{45}$$

$$p(\psi, R) = 0\,. \tag{46}$$

Conditions at the plasma boundary Γ and conductor boundary C

At Γ:

$$\psi_\Gamma = \psi_i = \psi_e = \text{const.} \tag{47}$$

$$\frac{\partial \psi_e}{\partial n}\Big|_\Gamma = \frac{\partial \psi_i}{\partial n}\Big|_\Gamma \tag{48}$$

$$B_{\phi i\Gamma} = B_{\phi e\Gamma} \tag{49}$$

$$p(\psi_\Gamma) = 0 \tag{50}$$

At C:

$$\psi_{eC} = \text{const.} \tag{51}$$

Here ψ_i and ψ_e denote ψ inside and outside Γ respectively. Equation (47) is the flux surface condition which actually includes the condition of continuity of the normal magnetic field, and (48) (49) reflect continuity of the tangent magnetic field. The continuity of the pressure is included in the selection of the form of pressure at the plasma boundary. For instance, if the temperature $T(\psi)$ is a surface function, in Eq.(40) one can choose $p_0(\psi_\Gamma) = 0$ to ensure both flux surface and continuous pressure conditions at the plasma boundary.

The conductor-vacuum boundary problem can be posed in the following way: given the conductor boundary C, position of limiter and arbitrary functions $\omega(\psi)$, $I(\psi)$ and $g(\xi,\eta)$, we can solve ψ, u_ϕ, ρ and the shape of the cross section of the plasma column Γ.

4. Discussions and the numerical example

We can summarize our main results into the following conclusions:
(i) The boundary-value problems for the surface function ψ are the same for the rotating and the static equilibria, because of the similarity of the equilibrium equation and the identity of the surface condition. This similarity leads to many resemblances of rotating problems with those of static. Therefore, lots of results and methods in the static equilibrium can be applied to rotating equilibria. Particularly numerical methods in the static equilibrium can be used for the rotating case almost without correction, for instance the method of Sheffield [13]. It implies that a rotating equilibrium problem can be transformed to a static one to some extent;
(ii) Since rotating equilibrium involves both density and velocity distributions, it has more degrees of freedom than static equilibrium. Generally one arbitrary two-variable function and two arbitrary single-variable functions are allowed in a pure toroidally rotating equilibrium.

Directly from the momentum Eq. (12) one can see that the third term, which represents the contribution of velocity on the equilibrium, appears in the form of ω^2. This means that the direction of velocity, and consequently the direction of neutral beam injection (whether co- or counter-injection) should have no effect on the magnetic field configuration within an ideal MHD theory. This has been confirmed by the experiments on PLT [6]. However, it should be noted that the polarity of the electric field is still related to the direction of velocity due to $\mathbf{E} = - \mathbf{u} \times \mathbf{B}$.

Numerical example

There should be nothing new in numerical methods except for a different form for $p(\psi,R)$, which is chosen as Eq.(40) in our example. That means we choose $T = T(\psi)$.

Numerical results show that the toroidal flow has two main effects on the magnetic field configuration:

(1) An outward movement of both pressure and magnetic flux contours in the major radius direction due to the centrifugal effect of the plasma rotation. This can be described in shifts of both magnetic and pressure maximum axes in Figure1, keeping the outermost contour fixed (conducting wall boundary). We note that the shift of the pressure axis is larger than that of the magnetic axis. This leads to
(2) A radially outwards deviation of the pressure contour from the flux contour, this is shown in Figure 2.

The experimental results with the neutral beam injection from PLT (T_e = 1.65 KeV, u_ϕ = 1.3×10^7cm, M = 0.28; [6]) and PDX ($T_e \geq 1.4$ KeV, $u_\phi = 1.0 \times 10^7$cm, $M \leq 0.23$; [7]) both give Mach numbers M < 0.3, but from the figures one can see that the flow effects on the magnetic configuration are quite small under these Mach numbers. One can therefore, expect that purely toroidal rotation has no serious effects on the pressure profile and magnetic field configuration. However, as shown in [9], a small poloidal rotation with a much lower Mach number than 1 might cause measurable effects on the density distribution.

5. Acknowledgements

The author would like to express his thanks to Prof. Xu Fu for supervising him to complete the thesis on which this paper based, to Dr. M. Tendler for providing valuable literature and information, to Drs. P. Brunsell, M. Faghihi, P. Karlsson and J. Scheffel for inspiring him to complete the report.

He is indebted to Drs. Jan Scheffel and John Tonks for correcting the English.

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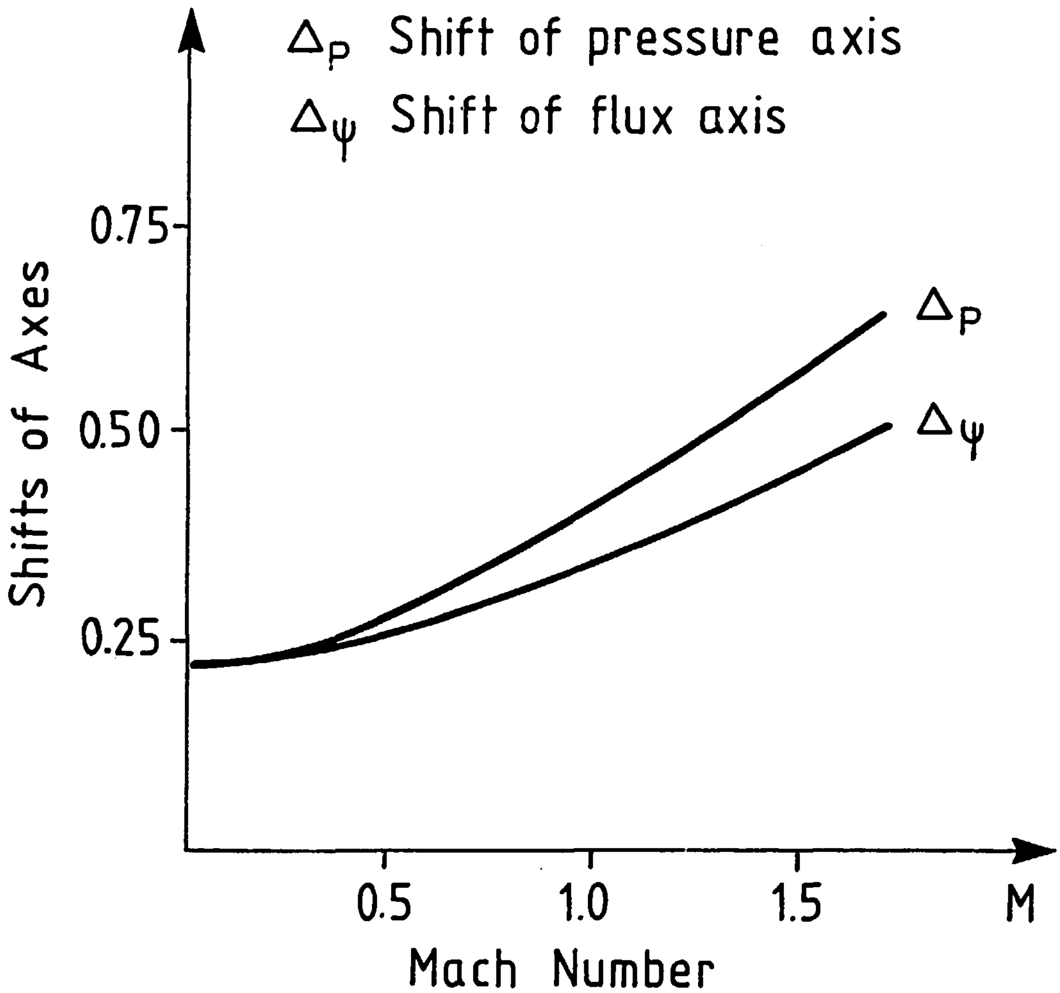


Fig.1 Shifts of Axis via Mach Number

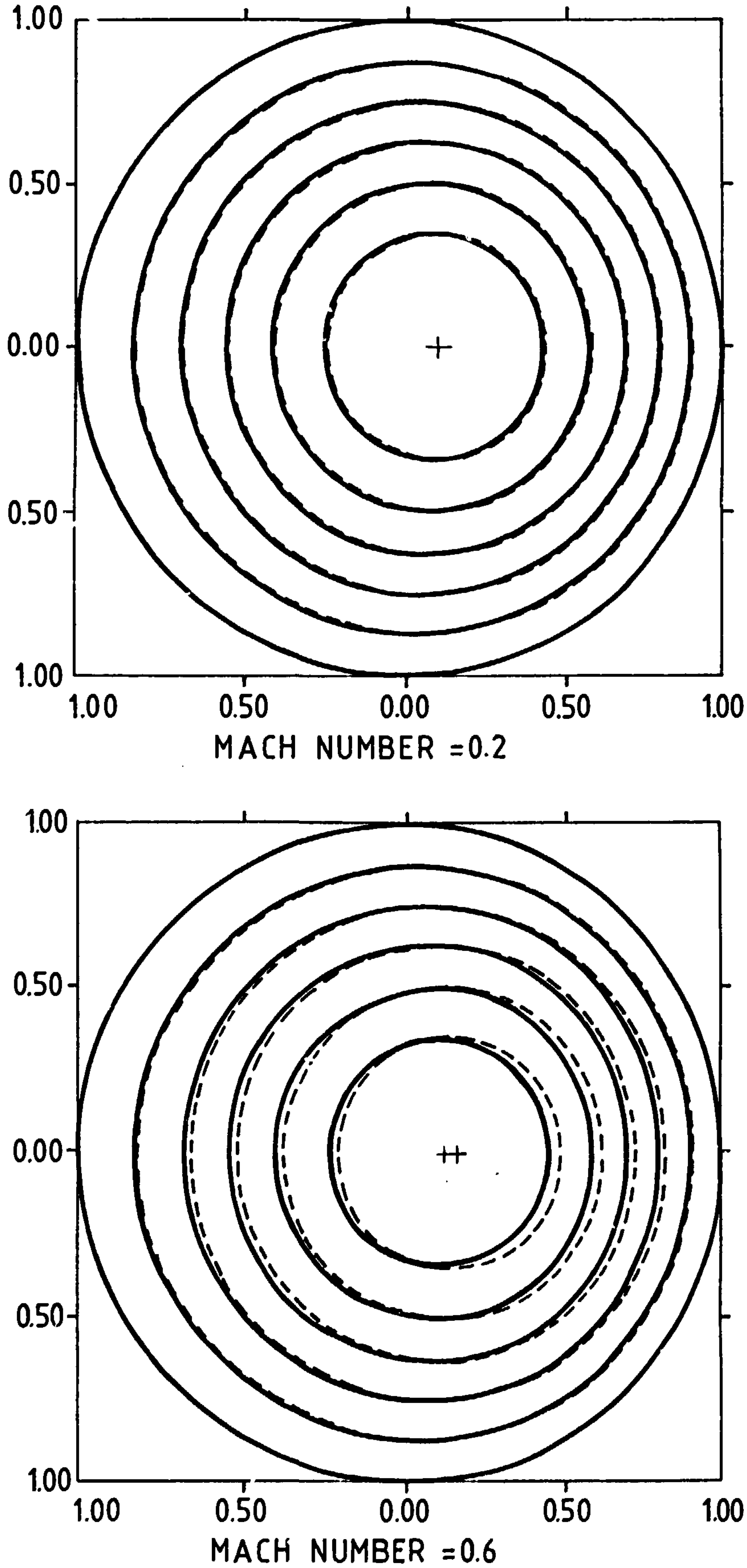
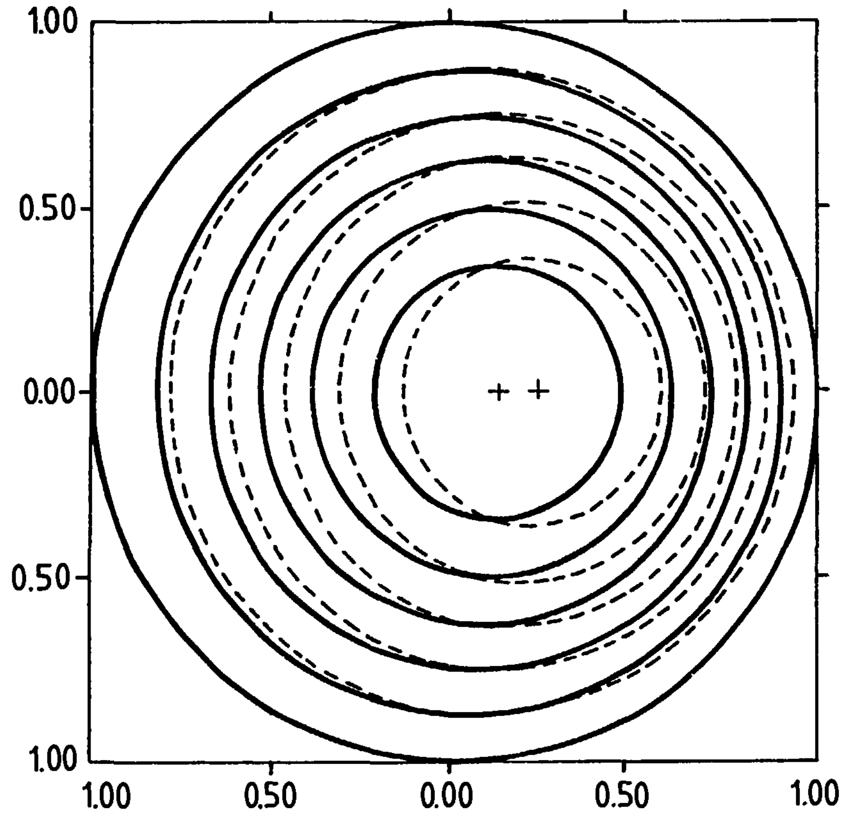


Fig.2 Separation of Flux Surfaces from Pressure Contours

------ pressure contours

——— flux contours



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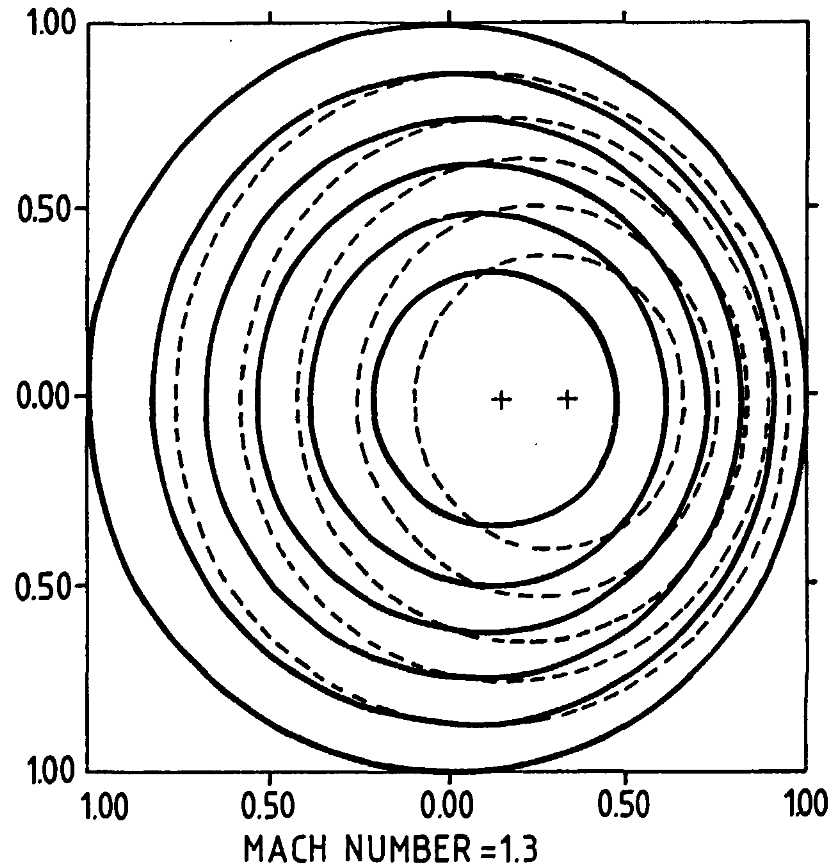

MACH NUMBER = 1.3

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Key Words: Magnetic confinement of plasma, MHD equilibrium, rotating plasma, Grad-Shafranov equation.