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ALGEBRA OF PSEUDO-DIFFERENTIAL C*-OPERATORS

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ALGEBRA OF PSEUDO-DIFFERENTIAL C*-OPERATORS *

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1. INTRODUCTION

Various functional methods have been intensively applied to topology during the last decade. The technique of C*-algebras is quite useful in a number of problems connected with topological properties of manifolds. In particular, C*-algebras and their representations play an important role in K-theory and problems associated with it. Several authors have obtained interesting results concerning pseudo-differential operators in the framework of C*-algebras. Mishenko [5] and [6] interpreted several versions of the theory of pseudo-differential operators in terms of C*-algebras, that is, elliptic psuedo-differential operators on a compact manifold; pseudo-differential operators on a Euclidian space, operators with almost periodic functions etc. Mishenko and Fomenko [7] derived analogues of the well-known Atiyah-Singer index formulas in this situation.

In this paper we study the algebra of pseudo-differential operators in the framework of C*-algebras. We essentially prove that every pseudo-differential operator of order m admits an adjoint operator, in this case, which is again a pseudo-differential operator. Consequently, we get that the space of all pseudo-differential operators on a compact manifold is an involutive algebra. This kind of problem has been discussed, in the classical case, by Hörmander [1], Kohn and Nirenberg [2] and Kumano-go [3] and [4] among many others. The formulation of such a problem naturally arises when one seeks analogues of the Atiyah-Bott formulas for this case.

Regarding the general theory of pseudo-differential operators we refer to [10].

2. PRELIMINARIES

Let A be any C*-algebras with an identity and K a right A-module. Recall the definition of a Hilbert C*-module as in ([9] and [5]).

Definition 2.1 A pre-Hilbert C*-module is a right A-module K equipped with an inner product $< \cdot,\cdot >: K \times K \to A$ satisfying $\forall\, x, y, z \in K$, $a \in A$, $\lambda \in \mathbb{C}$ the following conditions:

i) $< x,x > \geq 0$; $< x,x > = 0$ only if $x = 0$:

ii) $< x, y > = < y, x >^*$;

iii) $< x, y + z > = < x, y > + < x, z >$; $< x, \lambda y > = \lambda < x, y >$;

iv) $< xa, y > = a^* < x, y >$; $< x, ya > = < x, y > a$.

For all $x \in K$, $\|x\|_K^2 = \|< x,x >\|$ defines a norm on a pre-Hilbert C^*-module K. Moreover, it satisfies $\|xa\|_K \le \|x\|_K \cdot \|a\|$ and $\| < x,y > \| \le \|x\|_K \cdot \|y\|_K$, $\forall x,y \in K$, $a \in A$. The inner product gives the homomorphism

$$\varphi: K \to K^* = \mathrm{Hom}_A(K, A)$$

which is not, however, an isomorphism for an arbitrary C^*-algebra A, in contrast to the case where A is the field of complex numbers.

For this reason, a supplementary condition is imposed.

v) the homomorphism $\varphi: K \to K^* = \mathrm{Hom}_A(K,A)$ is an isomorphism. This guarantees the existence of an adjoint to any bounded homomorphism of Hilbert C^*-modeles.

We shall use the following terminology. A pre-Hilbert C^*-module K which is complete with respect to the norm $\|\cdot\|_K$, together with an inner product satisfying (i) - (iv), is called a Hilbert C^*-module. If (v) is also satisfied, then K is called self-dual. The inner product $< \cdot,\cdot >$ will be called a Hermitian product. Denote by $\mathrm{Hom}_A^*(K_1,K_2)$ the space of A-homomorphisms $T : K_1 \to K_2$ having adjoints, that is, homomorphisms T^* such that

$$< Tx, y > = < x, T^*y > , \ x \in K_1 \ , \ y \in K_2 .$$

It is clear that $\mathrm{Hom}_A^*(K_1,K_2) \subset \mathrm{Hom}_A(K_1,K_2)$ is a closed subspace in the operator norm. Also, the space $\mathrm{End}_A^*(K) = \mathrm{Hom}_A^*(K,K)$ is a C^*-algebra. If we denote by $K^\# = \mathrm{Hom}_A^*(K,A)$, then the homomorphism $\varphi: K \to K^*$ induced by the inner product $< x,y >$ realizes the isomorphism $\varphi: K \to K^\# \subset K^*$.

Thus we have a natural catefory $\mathcal{M}$ whose objects are Hilbert C^*-modules and whose morphisms are operators in $\mathrm{Hom}_A^*(K_1, K_2)$.

Examples of such objects are numerous. For instance, the C^*-algebra A itself as well as the direct sum A^k of k copies of, with the Hermitian product given by $< x,y > = \sum_{i=1}^{k} x_i^* y_i$, is a Hilbert C^*-module. Also, these modules are self-dual. Furthermore, A^k is projective relative to epimorphisms.

Denote by $\ell_2(A)$ the space of sequences $x = (x_1, \dots x_n, \dots)$ $\forall x_n \in A$, which satisfy the condition that $\sum_{n=1}^{\infty} x_n^* x_n$ converges in A. We can define a Hermitian product in $\ell_2(A)$ by putting

$$< x, y > = \sum_{n=1}^{\infty} x_n^* y_n \ . \qquad (2.1)$$

and hence the norm by

$$\|x\|^2_{\ell_2(A)} = \| \sum_{n=1}^{\infty} x_n^* x_n \| \ . \qquad (2.2)$$

The convergence of the series (2.1) follows from an analogue of Cauchy's inequality for C^*-algebras (see [7]):

$$\| \sum_{n=1}^{\infty} x_n^* y_n \|^2 \le \| \sum_{n=1}^{\infty} x_n^* x_n \| \cdot \| \sum_{n=1}^{\infty} y_n^* y_n \| \ . \qquad (2.3)$$

Then $\ell_2(A)$ is a Hilbert C^*-module ([7]).

For the sake of simplicity we also assume that each Hilbert C^*-module K has at most a countable set of generators, that is, a countable subset whose A-linear span is dense in K.

Theorem 2.1 ([6]) Every countably generated Hilbert C^*-module is a projective object in the category $\mathcal{M}$ relative to epimorphisms.

If the Hilbert C^*-module K is finitely generated, then K is self-dual, for there is an epimorphism $f : A^k \to K$ having an adjoint. Then by Theorem 2.1, K is a direct summand in the module A^k and is, therefore, self-dual.

3. VECTOR A-BUNDLES AND PSEUDO-DIFFERENTIAL OPERATORS OVER C*-ALGEBRAS

Definition 3.1 Let A be a C*-algebra. By a vector A-bundle we mean a locally trivial fibre bundle $\xi = (E, p, M, F, G)$ where E is a total space, M is the base of the fibre bundle, $p : E \to M$ is a projection, F is a fibre of the fibre bundle which is finitely generated projective Hilbert C*-module and G is the structural group equal to $\mathrm{Aut}_A(F)$, the group of A-automorphisms of the Hilbert C*-module F. If F is a free, k-dimensional Hilbert C*-module, the $\mathrm{Aut}_A(F)$ is the same as the group of invertible k^{th}-order matrices with coefficients in A.

Suppose that the base M is a compact smooth manifold (of class C^∞). Denote by $\Gamma(\xi)$ the space of continuous sections of the bundle ξ. The space $\Gamma(\xi)$ is endowed with the natural structure of Hilbert C*-module, coinciding with the structure of the Hilbert C*-module in each fibre F when $\Gamma(\xi)$ is restricted to F. Each A-bundle ξ admits a fibre Hermitian product with values in A. Thus if $u_1, u_2 \in \Gamma(\xi)$ are two sections, then a continuous function $< u_1,u_2 > \in C(M,A)$ is defined, $C(M,A)$ being the algebra of continuous functions on M with values in A. Without loss of generality we can assume that the gluing functions of the vector bundle are smooth sections (of class C^∞), and denote by $\Gamma^\infty(\xi)$ the space of smooth sections (of class C^∞) of the bundle ξ. We can then choose the fibre Hermitian product to be smooth, i.e. for any two sections $u_1, u_2 \in \Gamma^\infty(\xi)$, the function $< u_1,u_2 > \in C^\infty(M,A)$, where $C^\infty(M,A)$ is the space of smooth functions (of class C^∞) with values in A.

We now want to introduce Sobolev norms in $\Gamma^\infty(\xi)$. First we consider the local situation.

Let $X \subset \mathbb{R}^n$ be a bounded open set, and $C_0^\infty(X,A)$ the ring of smooth functions (of class C^∞) with compact supports and with values in A. Let $S(\mathbb{R}^n,A)$ denote the space of C^∞-functions whose derivatives decrease faster than any power of $\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$ as $\|x\| \to \infty$.

For $u \in S(\mathbb{R}^n,A)$ we define the Fourier transform $\hat{u}(\xi)$ by

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)dx \quad , \tag{3.1}$$

where $\xi \in \mathbb{R}^n$, $x \cdot \xi = x_1 \xi_1 + \dots + x_n \xi_n$ and dx is the Lebesgue measure on $\mathbb{R}^n$.

The inverse Fourier transform can be defined as

$$u(x) = \int e^{ix\cdot\xi}\hat{u}(\xi)đ\xi \quad , \tag{3.2}$$

where $đ\xi = (2\pi)^{-n} d\xi$.

We denote by Δ the operator

$$\Delta = - \sum_{i=1}^{n} \frac{\partial^2}{(\partial x_i)^2} \quad .$$

For $u \in C_0^\infty(X,A)$ we put

$$\|u\|_s^2 = \| \int_X ((1 + \Delta)^s u^*(x)) u(x) dx \| \quad . \tag{3.3}$$

where s is any real number, and denote by $H_0^s(X,A)$ the completion of $C_0^\infty(X,A)$ relative to the Sobolev norm (3.3). Then

Lemma 3.1 ([7]) The space $H_0^s(X,A)$ is isomorphic to $\ell_2(A)$ as a Hilbert C*-module.

Note that a Hermitian product in $H_0^s(X,A)$ is given by

$$< u, v >_s = \int_X ((1 + \Delta)^s u^*(x)) v(x) dx \quad . \tag{3.4}$$

For $s = 0$, $H_0^s(X,A)$ equals $L_2(X,A)$ where $L_2(X,A)$ denotes the space of such measurable functions (i.e. classes) f, for which the integral $\int_X f^*(x) f(x) dx$ converges. Likewise, in $L_2(X,A)$ we have a Hermitian product defined by

$$< f, g > = \int_X f^*(x) g(x) dx, \tag{3.5}$$

and denote the induced norm by $\|\cdot\|_{L_2}$.

Let E be trivial A-bundle on the domain X with fibre P, where P is a finitely generated projective Hilbert C^*-module with a nondegenerate, positive definite inner product with values in A. For the sake of simplicity, we assume that P is A^k, where A^k is a direct sum of k-copies of A. Analogous to (3.3) we define Sobolev norms in the space $\Gamma_0^\infty(X,A^k)$ of sections with compact support by

$$\|u\|_s^2 = \| \int_X < (1+\Delta)^s u(x), u(x) > dx \| \quad . \tag{3.6}$$

The completion of the space relative to the Sobolev norm is denoted by $H_0^s(X,A^k)$. Then it follows trivially from Lemma 3.1 that the Hilbert C^*-module $H_0^s(X,A^k)$ is isomorphic to the module $\ell_2(A^k)$, the direct sum in the module $\ell_2(A) \times \ldots \times \ell_2(A)$.

Similarly one can verify that, $L_2(X,A^k)$ is the same as $H_0^s(x,A^k)$ for $s = 0$, in which the Hermitian product is given by

$$< f, g > = \int < f(x), g(x) > dx; \; < f(x), g(x) > =: \sum_{i=1}^{k} f_i^*(x)\, g_i(x) \quad . \tag{3.7}$$

We now define pseudo-differential A-operators in spaces of section of A-bundles E_1 and E_2. Let $\pi : T^* X \to X$ be the natural projection of a cotangent bundle. Consider the pre-images of the bundles $\pi^*(E_i)$, $i = 1, 2$, taking account of the fact that the inner product in each fibre is induced by the inner product in the fibres of E_i. Without loss of generality, we assume that $E_i = E$, $i = 1, 2$, with fibre A^k. We consider the A-homomorphisms of the bundles $a : \pi^*(E) \to \pi^*(E)$ as a family of A-homomorphisms

$$a(x, \xi) : A^k \to A^k$$

parametrized by points of the cotangent bundle $(x,\xi) \in T^*X$. Suppose that $a(x,\xi)$ satisfy the following conditions:

(a) $$\| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \| \leq C_{\alpha,\beta} < \xi >^{m-|\alpha|} \quad . \tag{3.8}$$

where $< \xi >$ stand for $(1+\sum_{i=1}^{n} \xi_i^2)^{1/2}$, α, β are multi-indices of non-negative integers and m being any real number.

(b) $a(x,\xi)$ have compact support in the variable x, that is, $\pi(\text{supp } a) \subset X$ is a compact set.

We shall call the homomorphism satisfying conditions (a) and (b) the <u>symbol of the pseudo-differential</u> A-operator T, which is defined by:

$$T\, u(x) = \int e^{ix\cdot\xi}\, a(x, \xi)\, \hat{u}(y)\, d\xi \quad , \tag{3.9}$$

where $u \in \Gamma_0^\infty(X,A^k)$.

The number m is called the <u>order of the operator</u> T. We shall denote by S^m the class of symbols satisfying the properties (a) and (b) as given above.

<u>Theorem 3.1</u> The operator T defined by (3.9) is a continuous map from $\Gamma_0^\infty(X,A^k)$ into itself.

The proof of Theorem 3.1 is an easy computation (see [8]).

By means of Theorem 3.1, we can extend the operator T as a continuous map of $S(\mathbb{R}^n,A^k)$ into $S(\mathbb{R}^n,A^k)$.

<u>Theorem 3.2</u> ([7]) Every pseudo-differential A-operator T of order m:

$$T : H_0^s(X, A^k) \to H_0^{s-m}(X, A^k)$$

is a bounded operator in the Sobolev norms.

It now follows trivially from Theorem 3.2 the following:

<u>Theorem 3.3</u> (L_2-continuity) A pseudo-differential A-operator T of order zero can be extended to a bounded map of $L_2(\mathbb{R}^n,A^k)$ into $L_2(\mathbb{R}^n,A^k)$, i.e., there exists a constant $C > 0$ such that

$$\| Tu \|_{L_2} \leq C \| u \|_{L_2} , \; u \in \Gamma_0^\infty (X, A^k) \quad .$$

Next we consider the adjoint of a pseudo-differential A-operator.

Note that

$$< u, v > = \int < u(x), v(x) > dx \quad ,$$

where $u, v \in \Gamma_0^\infty(X, A^k)$ and $< u(x), v(x) >$ equals $\sum_{i=1}^{k} u_i^*(x)\, v_i(x)$.

Theorem 3.4 Every pseudo-differential A-operator T of order m admits an adjoint operator, T^*, given by

$$< Tu, v > = < u, T^* v > , \; u, v \in \Gamma_0^\infty(X, A^k) \quad , \tag{3.10}$$

which is again a pseudo-differential A-operator.

Proof (see also [10]): We can write for $u, v \in \Gamma_0^\infty(X, A^k)$,

$$< Tu, v > = \iiint e^{-i(x-y)\cdot\xi}\, u^*(y) a^{\#}(x, \xi)\, v(x)\, dy đ\xi dx \quad .$$

$$= \int u^*(y) \left\{ \iint e^{i(y-x)\cdot\xi}\, a^{\#}(x, \xi)\, v(x)\, dx đ\xi \right\} dy \quad ,$$

where $a^{\#}(x,\xi) = (a_{ji}^*(x,\xi))$, the matrix of the symbol $a(x,\xi)$ being $a_{ij}(x,\xi)$. Hence it follows that

$$T^* v(x) = \iint e^{i(x-y)\cdot\xi}\, a^{\#}(y, \xi)\, v(y)\, dy đ\xi \quad , \tag{3.11}$$

which is evidently a pseudo-differential A-operator. One can see easily that (3.10) determines T^* uniquely.

Thus a pseudo-differential A-operator is a morphism in the category $\mathcal{M}$, described in the preceding section.

Theorem 3.5 Let T be a pseudo-differential A-operator with symbol $a(x,\xi)$ and let T^* be its adjoint operator. Then the symbol of T^* has the following expansion:

$$a^*(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!}\, \partial_\xi^\alpha D_x^\alpha a^{\#}(x, \xi) \tag{3.12}$$

where the asymptotic sum runs over all multi-indices α.

Theorem 3.6 (Composition) Let S and T be pseudo-differential A-operators with symbols $a(x,y) \in S^{m_1}$ and $b(x,\xi) \in S^{m_2}$, respectively. Then $R = ST$ is a pseudo-differential A-operator with symbol $r(x,\xi) \in S^{m_1+m_2}$, and one has the analogue of Leibniz's formula:

$$r(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!}\, \partial_\xi^\alpha b(x, \xi) \cdot D_\beta^\alpha a(x, \xi). \tag{3.13}$$

Proof of Theorem 3.6 One can write

$$R\, u(x) = \iint e^{i(x-y)\xi}\, b(x, \xi) \int e^{iy\cdot\eta}\, a(y,\eta)\, \hat{u}(\eta)\, đ\eta dy đ\xi$$

$$= \int r(x, \eta) e^{ix\cdot\eta}\, \hat{u}(\eta)\, đ\eta \quad ,$$

where

$$r(x, \eta) = \iint e^{i(x-y)\cdot(\xi-\eta)}\, b(x, \xi)\, a(y, \eta)\, dy đ\xi.$$

It can be shown easily, as in the classical case, that $r(x,\eta) \in S^{m_1+m_2}$, and by Taylor's formula

$$r(x, \eta) = \iint \left(\sum_{|\alpha| \le N} \frac{1}{\alpha!}\, \partial_\eta^\alpha b(x, \eta)(\xi-\eta)^\alpha \right) \cdot a(y, \eta) e^{i(x-y)(\xi-\eta)} dy đ\xi$$

$$+ r_N.$$

Integrating first with respect to y and then ξ, one verifies that

$$\iint e^{i(x-y)(\xi-\eta)}\, a(y, \eta)(\xi-\eta)^\alpha\, dy đ\xi = D_x^\alpha a(x, \eta).$$

Hence

$$r(x, \eta) = \sum_{|\alpha| \le N} \frac{1}{\alpha!}\, \partial_\eta^\alpha b(x, \eta)\, D_x^\alpha a(x, \eta) + r_N.$$

It remains to show that $r_N \in S^{m_1+m_2-N}$. This can be done in a similar way as the corresponding assertion in the case of classical pseudo-differential operators, provided we take into consideration the standard estimates while applying to a C^*-algebra.

The proof of Theorem 3.5 uses Taylor's formula as above, and can be proved as in the classical case. Therefore, we omit the proof.

As a consequence of Theorems 3.2, 3.4, 3.5 and 3.6, the class $S^\infty = \bigcup_{m\in\mathbb{R}} S^m$ makes an involutive algebra in the sense: if $T_i \in S^{m_i}$, $i = 1, 2$, then $T_1 + T_2 \in S^m$ for $m = \max(m_1, m_2)$ and $T_1 \cdot T_2 \in S^{m_1+m_2}$.

In order to define a pseudo-differential A-operator in sections of the A-bundles E_1 and E_2 on a compact manifold M, we consider the atlas $\{X_\alpha\}$ of charts of the manifold M, in each of which the bundles E_1 and E_2 are trivial. Let $a : \pi^*(E_1) \to \pi^*(E_2)$ be a A-homomorphism satisfying (3.8). Let $\{\varphi_\alpha\}$ be a partition of unity subordinated to the covering $\{X_\alpha\}$, and let ψ_α be functions such that $\psi_\alpha|\text{supp}\,\varphi_\alpha \equiv 1$ and $\text{supp}\,\psi_\alpha \subset X_\alpha$. We then put

$$Tu(x) = \sum_\alpha [T_\alpha(\varphi_\alpha u)](x) \quad ; \tag{3.14}$$

where $u \in \Gamma^\infty(E_1)$ and T_α is the pseudo-differential A-operator defined by (3.9) in the chart X_α by means of the sumbol $a_\alpha(x,\xi) = a(x,\xi)\,\psi_\alpha(x)$.

Using the partition of unity φ_α and (3.4), we define Sobolev's norms in the space of sections $\Gamma^\infty(M,E_1)$. Let $u_1\ u \in \Gamma^\infty(M,E_1)$ be arbitrary sections. We put

$$\langle u_1,u_2\rangle_s = \sum_\alpha \int ((1+\Delta_\alpha)^s\ \ \varphi_\alpha(x)\,u_1(x),\ \ \varphi_\alpha(x)\,u_2(x))\,dx,$$

$$||u||_s^2 = ||\langle u,u\rangle_s||. \tag{3.15}$$

The completion, relative to the Sobolev norm, of the space of sections $\Gamma^\infty(M,E_i)$ will be denoted by $H^s(M,E_i)$. In general, if an operator $T : H^s(M,E_1) \to H^{s-m}(M,E_2)$ is bounded for all s, then we shall say that the operator T is of order m. We now formulate some necessary propositions.

Theorem 3.7 The pseudo-differential A-operator $T : H^s(M,E_1) \to H^{s-m}(M,E_2)$ of order m defined by (3.14) is bounded in the Sobolev norms.

Theorem 3.8 The definition of pseudo-differential A-operator (3.14) does not depend on the partition of unity φ_α, the functions ψ_α, or the local coordinates system to within operators of smaller order.

Theorem 3.9 Let a_1 and a_2 be symbols of the pseudo-differential A-operator T_1 and T_2 respectively. Then the operator T_3 and $T_2\,T_1$, where $a_3 = a_2\,a_1$, (i.e., composition of symbols), differ by an operator of lower order.

The proof of Theorems 3.7 - 3.9 is entirely analogous to that of the same propositions for the case of classical pseudo-differential operators, provided account is taken of the special features in applying the standard estimates in case of a C^*-algebra, for instance, which were introduced in the proof of Lemma 3.2 [7].

Likewise, with obvious modifications one can extend the Theorem 3.4 to compact manifold:

Theorem 3.10 Every pseudo-differential A-operator T,

$$T : H^s(M,E_1) \to H^{s-m}(M,E_2)$$

of order m has an adjoint, T^*, which is also a pseudo-differential A-operator.

Summarizing all this, we have

Theorem 3.11 The class $S^\infty = \bigcup_m S^m$ of pseudo-differential A-operators on a compact manifold M, is an involutive algebra. Here S^m denotes the class of pseudo-differential A-operators $T : H^s(M,E_1) \to H^{s-m}(M,E_2)$ of order m.

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