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ITERATIVE SOLUTION OF A NONLINEAR OPERATOR EQUATION *

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ABSTRACT

Suppose $X = L_p$, $p \geq 2$, and K is a non-empty closed convex subset of X. Suppose $T:K \to X$ is a monotonic Lipschitzian mapping with Lipschitz constant $L \geq 1$ such that, for x in K and fixed f in X, the equation $x + Tx = f$ has a solution in K. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by $x_0 \in K$,

$$x_{n+1} = x_n + \lambda r_n ,$$

for $n \geq 1$, where $\lambda = [(p-1)L^2]^{-1}$ and $r_n = f - x_n - Tx_n$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of $x + Tx = f$ in K. Convergence is at least as fast as a geometric progression with ratio $(1-\lambda)^{1/2}$. A related result deals with covergence of the sequence $\{x_n\}_{n=0}^{\infty}$ when T is monotone and locally Lipschitzian.

1. INTRODUCTION

Let X be an arbitrary Banach space. An operator T with domain $D(T)$ and range $R(T)$ in X is said to be monotone [9] if

(1) $$\|x - y\| \leq \|x - y + t(Tx-Ty)\|$$

holds for each x, y in $D(T)$ and some $t > 0$. If (1) holds for all $t > 0$ then T is called accretive [2]. The accretive operators were introduced independently by F.E. Browder [2] and T. Kato [9]. An early fundamental result in the theory of accretive operators, due to Browder [2], states that the intial value problem

(2) $$\frac{du}{dt} + Tu = 0, \quad u(0) = u_0$$

is solvable when T is locally Lipschitzian and accretive on X. Utilizing the existence result for (2), Browder [1] has shown that if T is locally Lipschitzian and accretive then T is m-accretive, i.e., $(I+T)X = X$. This result was subsequently generalized by R.H. Martin [12] to the continuous accretive operators. For $X = H$, a Hilbert space, one of the earliest problems in the theory of monotone operators was to solve the equation $x + Tx = f$ for x, given an element f of H and a monotone operator T (see e.g. [5], [6], [8], [13], [14], [18], [19]). Recently W.G. Dotson [8] has shown that if $T:H \to H$ is monotone and has Lipschitz constant 1 (in this case T is called nonexpansive, [10]), an iterative process of the type introduced by W.R. Mann [11], under suitable conditions, converges strongly to the unique solution of the equation $x + Tx = f$. In [5] the author constructed an approximation method which converges strongly to the solution of this equation where $T:K \to H$ is a monotone Lipschitzian operator with Lipschitz constant $L \geq 1$, and where K is a non-empty closed convex subset of H. The result of [5] thus generalizes Dotson's theorem [8] both in the domain of definition of the operator and in the range of its Lipschitz constant. Also, a convergence theorem for the equation $x + Tx = f$ was proved in [5] when T was locally Lipschitzian and monotone.

Our objective in this paper is to construct an iteration process which converges strongly to the solution of the equation $x + Tx = f$ in spaces more general than Hilbert spaces. Our method will, in addition, give a rate of convergence faster than the rate of convergence established in [5]. Furthermore, using our iteration process, we shall prove a convergence theorem when T is locally Lipschitzian and monotone.

2. PRELIMINARIES

In the sequel we shall make use of the following results: For a Banach space X we shall denote by J the normalized duality mapping from X to 2^{X^*} given by

$$Jx = \{x^* \in X^* : \|x^*\|^2 = \|x\|^2 = \langle x, x^* \rangle\},$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If X^* is strictly convex, then J is single-valued and if X^* is uniformly convex then J is uniformly continuous on bounded sets (see e.g.,[18], [19]). It is easy to see that the duality map is positive homogeneous i.e., for any $\lambda > 0$, $J(\lambda x) = \lambda J(x)$, $x \in X$.

A Banach space X is called an upper weak parallelogram space with constant $b \geq 0$ (or briefly X is UWP(b)), in the terminology of [4], if

$$\|x+y\|^2 + b\|x-y\|^2 \geq 2\|x\|^2 + 2\|y\|^2 , \qquad (3)$$

holds for all x, y in X. It is proved in [4] that L_p and ℓ_p spaces, $p \geq 2$, are upper weak parallelogram spaces with $(p-1)$ as the smallest number b such that inequality (3) holds for all x, y in L_p (or ℓ_p). In the sequel we shall make use of a characterization of weak parallelogram spaces in terms of normalized duality mapping given in the following theorem:

THEOREM B (Bynum, [4]). Let X be a Banach space with normalized duality mapping, J. Then X is UWP(b) if and only if for each x, y in X and $j \in Jy$,

$$\|x+y\|^2 \leq b\|x\|^2 + \|y\|^2 + 2\langle x, j \rangle . \qquad (4)$$

For $X = L_p$ or ℓ_p $(p \geq 2)$, J is single-valued and inequality (4) can be re-stated as:

(5) $$\|x+y\|^2 \leq (p-1)\|x\|^2 + \|y\|^2 + 2 < x, J(y) >$$

for all x, y in X. The accretiveness (or monotonicity) condition for T defined in (1) can also be expressed in terms of the duality map J as follows (see [9]): For each x, y in $D(T)$ there is some $j \in J(x-y)$ such that

(6) $$\mathrm{Re} < Tx - Ty, j > \geq 0 ,$$

and obviously for single-valued J this reduces to

(7) $$\mathrm{Re} < Tx - Ty, J(x-y) > \geq 0 .$$

3. MAIN RESULTS

We prove the following theorems:

THEOREM 1 Suppose $X = L_p$, $p \geq 2$, and K is a non-empty closed convex subset of X. Suppose $T:K \to X$ is a monotonic Lipschitzian mapping with Lipschitz constant $L \geq 1$ such that, for x in K and fixed f in X, the equation $x + Tx = f$ has a solution in K. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by,

$$x_0 \in K,$$

$$x_{n+1} = x_n + \lambda r_n,$$

for $n \geq 1$, where $\lambda = \frac{1}{(p-1)L^2}$ and $r_n = f - x_n - Tx_n$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution of $x + Tx = f$ in K. Convergence is at least as fast as a geometric progression with ratio $(1-\lambda)^{1/2}$.

PROOF Define $G = (I+T)$ and observe that the monotonicity of T implies:

$$\begin{aligned} < Gx - Gy, J(x-y) > &= < (I+T)x - (I+T)y, J(x-y) > \\ &= \|x - y\|^2 + < Tx - Ty, J(x-y) > \\ &\geq \|x - y\|^2 , \end{aligned}$$

for all x, y in K. Furthermore, $r_n = f - x_n - Tx_n = f - Gx_n$. Observe also that,

(8) $$r_{n+1} = f - Gx_{n+1} = r_n + Gx_n - Gx_{n+1} = r_n - (Gx_{n+1} - Gx_n)$$

Since the duality map is positive homogeneous, using inequalities (5) and (8) we obtain:

$$\begin{aligned} \|r_{n+1}\|^2 &\leq \|r_n\|^2 - 2 < Gx_{n+1} - Gx_n, J(r_n) > \\ &\quad + (p-1)\|Gx_{n+1} - Gx_n\|^2 \\ &\leq \|r_n\|^2 - 2 < Gx_{n+1} - Gx_n, J(r_n) > \\ &\quad + (p-1)L^2\|x_{n+1} - x_n\|^2 \\ &= \|r_n\|^2 - 2 < Gx_{n+1} - Gx_n, J(r_n) > \\ &\quad + (p-1)\lambda^2L^2\|r_n\|^2 \\ &= \|r_n\|^2 - 2\lambda^{-1} < Gx_{n+1} - Gx_n, J(\lambda r_n) > \\ &\quad + (p-1)\lambda^2L^2\|r_n\|^2 \\ &\leq \|r_n\|^2 - 2\lambda\|r_n\|^2 + (p-1)\lambda^2L^2\|r_n\|^2 \\ &= (1 - 2\lambda + (p-1)\lambda^2L^2)\|r_n\|^2 = \left(1 - \frac{1}{(p-1)L^2}\right)\|r_n\|^2, \end{aligned}$$

and so $r_n \to 0$ as $n \to \infty$. Since G has continuous inverse we have $x_n \to x^*$, the unique solution of $Gx = f$.

The fact that convergence is at least as fast as a geometric progression with ratio $(1-\lambda)^{1/2}$ follows from the last inequality in the proof of Theorem 1.

REMARKS Theorem 1 is a significant improvement of the main result of [5] (which is itself a generalization of the results of [8]) in several ways.

1. The method used in [5] is an approximation method and not an iterative process because the sequence $\{x_n\}_{n=0}^{\infty}$ defined there is not actually iterated. Theorem 1 above provides an iteration process for approximating the solution of the equation $x + Tx = f$.

2. While Theorem 1 of [5] was proved for Hilbert spaces, our Theorem 1 above is proved in L_p spaces, $p \geq 2$.

3. The rate of convergence established for the approximation method in [5] is of the order $O(n^{-1/2})$ whereas convergence in Theorem 1 above is at least as fast as a geometric progression.

Before we prove our next theorem, we observe that if in Theorem 1, $K = X$ so that T maps X into X, the existence of a solution follows from [2].

DEFINITION Let $D(T)$ denote the domain of a map T. Then $T:D(T) \to X$ is called locally Lipschitzian with constant L if, for each s in $D(T)$, there is an $\epsilon > o$ such that

(9) $$\|Tx - Ty\| \leq L\|x - y\|$$

whenever $\|x - s\| \leq \epsilon$ and $\|y - s\| < \epsilon$. Let X be as in Theorem 1.

THEOREM 2 Suppose $T:D(T) \to X$ is a locally Lipschitzian monotone operator with open domain $D(T)$ in X and let $f \in X$. Suppose the equation $x + Tx = f$ has a solution q in $D(T)$. Then there is a neighbourhood B in $D(T)$ of q and a real number $r_1 > 0$ such that for any $r \geq r_1$ and any initial guess $x_1 \in B$, the sequence $\{x_n\}$ generated from x_1 by

(10) $$x_{n+1} = x_n + t_n r_n$$

for $n \geq 1$, where $t_n = (n+r)^{-1}$ and $r_n = f - x_n - Tx_n$, remains in B and converges strongly to q with $\|x_n - q\| = O(n^{-1/2})$.

PROOF By hypothesis $q \in D(T)$. Since T is locally Lipschitzian, given $\epsilon > 0$, choose $\hat{\epsilon}$ in $(0,\epsilon)$ such that (9) is satisfied. Let $B_1 = \{x \in X: \|x-q\| \leq \hat{\epsilon}\}$. $(I + T)$ is monotone and so $(I + T)$ is locally bounded at each interior point of its effective domain, [15]. Therefore, there exists $d > 0$ such that the ball with cente q and radius d, $B_d(q)$, is contained in $D(I + T) = D(T)$ and $T(B_d(q))$ is bounded. Set $B = B_1 \cap B_d(q)$. Then $T(B)$ is bounded and T is Lipschitzian on B. Set

$$r_1 = \frac{L^2(p-1)D_*^2}{d^2} \quad \text{where } D_* = \text{diameter of } T(B)$$

Then $r_1 > 0$ and

(11) $$D_* \leq L^{-1}(p-1)^{-1/2}\, dr^{1/2} \quad \text{if } r \geq r_1$$

Set $t_n = (n+r)^{-1}$ and $d_n = (n+r-1)^{-1/2}$ and observe that $t_n^2 + (1-t_n)^2 d_n^2 = d_{n+1}^2$. Starting with an initial guess $x_1 \in B$, define the sequence $\{x_n\}_{n=1}^{\infty}$

inductively by (10).

We assert: For al $n \geq 1$, x_n is well defined and

(12) $$\|x_n - q\| \leq d_n dr^{1/2}$$

For $n = 1$, x_1 is clearly well defined since it was chosen in B. Suppose now the assertion has been proved for $n = k$. Then

$$\|x_k - q\| \leq d_k dr^{1/2} \leq d_1 dr^{1/2} = d\ ,$$

so that $x_k \in B$. Now q is a solution implies $q + Tq = f$ so that, using the monotonicity of T, (5) and (7) we have:

$$\begin{aligned}\|x_{k+1} - q\|^2 &= \|x_k - q + t_k r_k\|^2 \\ &= \|(1-t_k)(x_k-q) + t_k(Tq - Tx_k)\|^2 \\ &\leq (1-t_k)^2\|x_k - q\|^2 + (p-1)t_k^2\|Tq - Tx_k\|^2 \\ &\qquad - 2t_k(1-t_k) < Tx_k - Tq,\ J(x_k-q)> \\ &\leq (p-1)L^2D_*^2 t_k^2 + (1-t_k)^2\|x_k - q\|^2 \\ &\leq [t_k^2 + (1-t_k)^2 d_k^2]d^2 r = d_{k+1}^2 d^2 r\end{aligned}$$

so that

$$\|x_{k+1} - q\| \leq d_{k+1} dr^{1/2}\ ,$$

completing the induction process. Since $d_n = O(n^{-1/2})$ the error estimate of the theorem is also established.

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REFERENCES

[1] BROWDER, F.E., Nonlinear elliptic boundary value problems, Bull. Amer. Math. Soc. 69, 862-874 (1963).

[2] BROWDER, F.E. Nonlinear mappings of nomexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. 73, 875-882 (1967).

[3] BRUCK, JR, R.E., The iterative solution of the equation $y \in x + Tx$ for a monotone operator T in Hilbert space, Bull. Amer. Math. Soc. 79, 1258-1262 (1973).

[4] BYNUM, W.L., Weak parallelogram laws for Banach spaces, Canad. Math. Bull. 19, 269-275 (1976).

[5] CHIDUME, C.E., An approximation method for monotone Lipschitzian operators in Hilbert space, J. Austral. Math. Soc. (Series A) 41, 59-63 (1986).

[6] CHIDUME, C.E., The iterative solution of the equation $f \in x + Tx$ for a monotone operator T in Lp spaces, J. Math. Anal. Appl. 531-537 (1986).

[7] CHIDUME, C.E., Iterative approximation of fixed points of Lipschitz strictly pseudo-contractive mappings, Proc. Amer. Math. Soc. 99, 283-288 (1987).

[8] DOTSON, W.G., An iterative process for nonlinear monotonic nonexpansive operators in Hilbert space, Math Comp. 32, 223-225 (1978).

[9] KATO, T., Nonlinear semigroups and evolution equations, J. Math. Soc. Japan, 19, 508-520 (1967).

[10] KIRK, W.A., A fixed point theorem for mappings which do not increase distance, Amer, Math. Monthly 72, 1004-1006 (1965).

[11] MANN,W.R., Mean value methods in iteration, Proc. Amer. Math. Soc. 4, 506-510 (1953).

[12] MARTIN, JR. R.H., A global existence theorem for autonomous differential equations in Banach spaces, Proc. Amer. Math. Soc. 26, 307-314 (1970).

[13] MINTY, G.J., Monotone (nonlinear) operators in Hilbert space, Duke Math. J. 29, 541-546 (1962).

[14] PETRYSHYN, W.V., On the extension and solution of nonlinear operator equations, Illinois J. Math. 10, 255-274 (1966).

[15] ROCKAFELLAR, R.T., Local boundedness of nonlinear monotone operators, Michigan Math. J. 16, 397-407 (1969).

[16] SMUL'YAN, V.L., Sur la dérivabilité de la norm dans l'espace de Banach, C.R. (Dokl.) Acad. Sci. URSS 27, 255-258 (1940).

[17] SMUL'YAN, V.L., Sur les topologies différentes dans l'espace de Banach, C.R. (Dokl.) Acad. Sci. URSS 23, 331-334 (1939).

[18] VAINBERG, M.M., On the convergence of the method of steepest descent for nonlinear equation, Sibirsk. Math. Z. 2, 201-220 (1961).

[19] ZARANTONELLO, E.H., Solving functional equations by contractive averaging, Technical Report No.160, U.S. Army Math. Res. Centre, Madison, Wisconsin, 1960.