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INHOMOGENEITIES IN A FRIEDMANN UNIVERSE

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INHOMOGENEITIES IN A FRIEDMANN UNIVERSE *

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ABSTRACT

One of the outstanding problems in cosmology is the growth of inhomogeneities, which are characterized by an anisotropic pressure and density distribution. Following a method developed by McVittie (1967, 1968) it has been possible to find time-dependent spherically symmetric solutions of Einstein's field equations containing an arbitrary pressure and density distribution which connect smoothly to a Friedmann universe for any desired equation of state.

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1. INTRODUCTION

Recent redshift surveys (DeLapparent et al., 1986) and discovery of voids (Kirshner et al., 1981) pose serious constraints on models of large scale structure formation (Davis et al., 1985). When carried to its logical conclusion this might even influence the homogeneity of the universe as a whole.

The description of such inhomogeneities immediately raises the question of boundary conditions at the surface of these inhomogeneities. While the universe as a whole can be described by a Robertson-Walker line element which is isotropic and will contain only a time-dependent pressure and density distribution, this is not the case for the inhomogeneities, which at least are radial dependent as well. Unless one limits oneself to pressure-less dust, which seems physically unrealistic, this raises a problem of matching the pressure distribution at the boundary. (The corresponding question of matching the density distribution can be overcome by assuming a discontinuity at the boundary, although one would prefer a smooth cross-over there as well.) In addition, one would like to have a line element for the interior solution, whose gravitational potentials connect smoothly to the corresponding values outside the distribution.

Some time ago, McVittie (1967, 1968) had investigated a large class of time-dependent spherically symmetric solutions of Einstein's field equations containing a non-vanishing pressure and density distribution which moreover, have time as well as radial dependence. It is the purpose of this note to show that his solution can be adapted to our problem of finding interior solutions which connect smoothly to a Friedmann universe for any desired equation of state (at the boundary). It is shown that this can be achieved by a transformation of the time coordinate. The local time of the interior solution is not the universal time defined by the exterior one. From the point of view of the exterior observer, the time is uniquely defined (since $g_{44} = 1$ for a Friedmann universe). In addition to requiring that the portentials match at the boundary we demand that under this transformation the pressure of the interior distribution connects smoothly to the one of the exterior solution at the boundary. This results in a number of conditions and limits the permitted values of the constants occurring in McVittie's solutions (loc.cit.). Consequently, the number of possible cases is reduced from the four, listed by McVittie, to two particular ones in our case. This, incidentally, also guarantees that the density distribution carries smoothly from the interior one to that pertaining to the exterior at the boundary, but in no way puts any limitation on a possible equation of state connecting the pressure and density distribution.

2. INTERIOR SOLUTION

The line element inside the material in comoving coordinates is given by

$$ds^2 = y^2dt^2 - (c^2/R_0^2)\ e^x\ S^2\ [dr^2 + f^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad , \qquad (1)$$

where $x = x(z)$, $y = y(z)$ with $y_z = 2(1-y)$ dimensionless functions of the variable z defined by $e^z = Q(r)/S(t)$. $Q = Q(r)$ and $f = f(r)$ are dimensionless functions of r, $S = S(t)$. The condition for an isotropic distribution

$$T_1^1 = T_2^2 = T_3^3$$

and considering r and z as independent variables results in three equations for the functions f, Q and y. (McVittie, 1967) Solving these one finds that f has one of the three forms

$$f = C^{-1}\sin(Cr+e), \qquad f = r + e, \qquad f = C^{-1}\sinh(Cr+e) \qquad (2)$$

where C and e are two constants in the nature of a scale factor and phase respectively. At the boundary of the inhomogeneity $r = r_b$ we can set, without loss of generality, $Q(r_b) = Q_b = 1$ so that

$$Q^b = 1 + (b/C^2A)[\cos(Cr_b+e) - \cos(Cr+e)] \qquad (3)$$

where A is a constant of integration, but only certain values of the constant b are allowed. The remaining functions are found to be

$$y = (qu + p)/(1 - u), \qquad e^x = \tilde{c}^2u^m(u - 1)^n \qquad (4)$$

where

$$u = K\ e^{-sz} = K(S/Q)^s \qquad (4a)$$

$\tilde{c}$ and K are constants of integration, while again the constants p, q, m, n and s can only take certain values, which form four distinctive groups corresponding to the four possible classes of solution obtained by McVittie (loc. cit.).

The pressure and density are given by

$$\kappa p/c^2 = -y^{-1}\{2S_{tt}/S + (3y-2)(S_t/S)^2 + (c^2/R_0^2)\, e^{-x}\, S^{-2}\, [y\, B_1 + 2(y^2-y-y_z)\, B_2 + (1-y)(y^2-y-2y_z)\, B_3]\} \tag{5}$$

and

$$\kappa\rho = 3(S_t/S)^2 + (c^2/R_0)^2\, e^{-x}\, S^{-2}\, \{3\, B_1 - 6(1-y)\, B_2 + [2y_z - (1-y)(2a-1-y)]\, B_3\} \tag{6}$$

where we have set

$$B_1 = (1 - f_r^2)\, f^{-2},\ B_2 = (f_r/f)(Q_r/Q),\ B_3 = (Q_r/Q)^2$$

and a is again one of the set of constants with determined values.

3. BOUNDARY CONDITIONS

We now require that the solutions given by (2), (3) and (4) for the line element (1) at the boundary $r = r_b$ go over into that corresponding to a Friedmann universe

$$ds^2 = dT^2 - (R/R_0)^2[dX^2 + F^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{7}$$

where

$$F = \sin X, \qquad F = X, \qquad F = \sinh X$$

corresponding to $k = 1, 0, -1$ where k determines the curvature of the space and $R = R(X)$; and more important, that at the boundary the pressure $p(r_b) = p_b$ be the one obtained by solving the field equations for (7), viz.

$$\kappa p/c^2 = -2\ddot{R}/R - \dot{R}^2/R^2 - kc^2/R^2 \tag{8}$$

where a dot indicates differentiation with respect to T.

Since $g_{44} \neq 1$ for the interior solution, its local time is not the universal time defined by the exterior solution. From (4) it follows that

$$y_b = (qU - p)/(U - 1), \quad U = K\,S^s. \tag{9}$$

Hence, we shall set

$$y_b = dT/dt \qquad \text{or} \qquad T = \int y_b dt. \tag{10}$$

Comparing (1) with (7) shows that the remaining conditions to be satisfied are

$$[S^2(r)\; e^{x(t)}]_b = R^2(T), \qquad (dr/f)_b = dX/F. \tag{11}$$

From (4) and (11) it then follows that

$$R = \tilde{c}(KS^s - 1)^{\frac{1}{2}n}\, S^p \quad . \tag{12}$$

The second condition is automatically satisfied by a change of scale of the radial distance, i.e. $X = Cr + e$. However, if we prefer to keep the same r-coordinate then

$$X = r, \qquad C = 1, \qquad e = 0$$

so that (3) becomes

$$Q^b = 1 + (b/A)(\cos r_b - \cos r) \tag{13}$$

The question which remains to be answered is whether, under the transformation (10), the pressure (5) transforms into (8) and if any additional conditions are to be met in order to effect that transformation. We first note that the derivatives in (5) are given with respect to t, while those in (8) are with respect to T. With the help of (10) we find

$$\dot{R} = R'/y_b, \quad R'/R = (S'/S)\, y_b \quad \text{so that} \quad \dot{R}/R = S'/S$$

from which it follows immediately the terms involving S''/S and $(S'/S)^2$ identically transform into the corresponding terms of (5). What remains are the terms multiplying $1/R^2$

$$kc^2/R^2 = (c^2e^{-x}/S^2y)_b\{y_b\, B_{1b} - 2[y_b^2 - y_b - (y_z)_b]\, B_{2b}$$
$$- (1-y_b)[y_b^2 - y_b - 2(y_z)_b]\, B_{3b}\} \tag{14}$$

where B_{ib} $(i = 1,2,3)$ denote the values of B_i at r_b. Noting that $B_i = 1$ if $k = 1$ (and $B_1 = -1$ if $k = -1$) and using again (11) it is seen that the transformation is complete provided the last two terms on the right-hand side of (14) vanish. This will result in a cubic equation in y_b (or U) and if we demand that all its coefficients vanish, we obtain the following possible values for the various constants:

i) $a = b = \frac{1}{2},\ p = 1,\ q = 0,\ m = 0,\ n = 4,\ s = -\frac{1}{2}$

$B_{2b} = 0$ so that $\cos r_b = 0$ or $r_b = \frac{1}{2}\pi$ and $B_3 = A^{-2}$

ii) $a = 3,\ b = -2,\ p = q = -1,\ m = 4,\ n = 4,\ s = 1$

or $p = q = +1,\ m = 0,\ n = 4,\ s = -1$

$B_{2b} + B_{3b} = 0$, so that $\cos r_b = \frac{1}{2}[A \pm (A^2+4)^{1/2}],\ A^{-2} = B_{2b}(B_{2b}-1)$.

Comparison with the four cases listed by McVittie shows that our two possible solutions are included in his first two.

Turning now to the density (6) it is readily seen that it also reduces at the boundary to the Friedmann value. It follows that for either of the above two possibilities

$$\kappa\rho = 3(S'/S)^2 + 3c^2/R^2$$

which is exactly the Friedmann value. Thus, our interior solution goes smoothly into the exterior Friedmann solution at the boundary with the pressure and density taking their corresponding values. The behaviour of the scale function S for various possible equations of state will be discussed in a future work.

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