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QUASI-ADEQUATE SEMIGROUPS

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ABSTRACT

The least fundamental adequate good congruence on an arbitrary type W semigroup S is described as well as the largest superabundant full subsemigroup of S and the largest full subsemigroup of S which is a band of cancellative monoids. Weak type W semigroups are defined by replacing the idempotent-connected property in type W by one of its consequences and a structure theorem is obtained for such semigroups.

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INTRODUCTION

There is a structure theorem for type W semigroups [3] - a certain class of quasi-adequate semigroups - which generalizes that of Hall [6] for orthodox semigroups. The Hall construction of the orthodox semigroup W_B (see [7]) has been used in the structure theorem of [3]. Recently, Venkatesan [11] gave a structure theorem for orthodox semigroup without the use of the Hall construction W_B. One of the two objectives of this paper is to extend the structure theorem of Venkatesan [11] to a certain class of quasi-adequate semigroups. The other objective is to describe for a type W semigroup S, the least fundamental adequate good congruence on S, the largest superabundant full subsemigroup of S and the largest full subsemigroup of S which is a band of cancellative monoids.

After the preliminaries in which we collect some basic information from the literature, comments are made in Section 2 on the idempotent-separating good homomorphic images of particular quasi-adequate semigroups. This is a preparation for the last two sections where in the third one we investigate a class of quasi-adequate semigroups using fundamental quasi-adequate semigroups and adequate semigroups. The main result of this section is similar to that of Venkatesan [11] and close to El-Qallali and Fountain's work [3]. However, we give a direct proof of the result independent of the Hall construction W_B. The class of quasi-adequate semigroups in the present approach may properly contain the class of type W semigroups. In the last section we extend Venkatesan's result [12] to type W semigroups.

We use the notation and terminology of [7] and assume some familiarity with the contents of [3] and [5].

1. PRELIMINARIES

We begin by recalling some of the basic facts about the relations L^* and R^*. Let a, b be elements of a semigroup S. We say that aL^*b if and only if a and b are related by Green's relation L in some oversemigroup of S. The relation R^* is defined dually and the relation H^* is the intersection of L^* and R^*. Evidently L^* is a right congruence and R^* is a left congruence on S. For any result about L^* there is a dual result for R^*. The following Lemma from [9] and [10] gives an alternative description of L^*.

Lemma 1.1 Let a, b be elements of a semigroup S. Then the following conditions are equivalent:

(1) $a \, L^* \, b$

(2) for all $x, y \in S^1$: $ax = ay$ if and only if $bx = by$.

As an easy but useful consequence of Lemma 1.1 we have

Corollary 1.2 Let a be an element of a semigroup S and e be an idempotent of S. Then the following conditions are equivalent:

(1) $a \, L^* \, e$

(2) $ae = a$ and for all $x, y \in S^1$; $ax = ay$ implies $ex = ey$.

It follows from Corollary 1.2 and its dual that:

Corollary 1.3 If e is an idempotent in a semigroup S, then H^*_e is a cancellative monoid.

In particular, we emphasize that on any semigroup S, we have $L \subseteq L^*$. It is well known and easy to show that for regular elements a, b in S; $a \, L^* \, b$ if and only if $a \, L \, b$, thus if S is a regular semigroup, then $L^* = L$.

Recall that, an abundant semigroup is one in which each L^*-class and each R^*-class contains an idempotent, and it is superabundant if each H^*-class contains an idempotent. An abundant semigroup is said to be quasi-adequate if its idempotents form a subsemigroup and it is adequate if the idempotents commute (see [3], [4] and [5]).

A semigroup homomorphism $\rho: S \to T$ is said to be a good homomorphism if for all elements a, b of S; $a \, L^*(S) b$ implies $a\rho \, L^*(T) \, b\rho$ and that $a \, R^*(S) \, b$ implies $a\rho \, R^*(T) \, b\rho$. A congruence τ on a semigroup S is said to be a good congruence if the natural homomorphism from S onto S/τ is a good homomorphism. From [2] we quote:

Lemma 1.4 Let S be an abundant semigroup. If ρ is a semigroup homomorphism from S into T (a congruence on S), then the following conditions are equivalent:

(1) ρ is good

(2) for any element a of S, there are idempotents e, f with $e \, L^* \, a$, $f \, R^* \, a$ in S such that $e\rho \, L^* \, a\rho$ and $f\rho \, R^* \, a\rho$ in $T(S/\rho)$.

On quasi-adequate semigroups we have from [3] a generalization of the main part of Lallements's Lemma [7, Lemma II.4.6] as follows:

Proposition 1.5 If S is a quasi-adequate semigroup with set B of idempotents and ρ is a good congruence on S (a semigroup good homomorphism from S onto T), then S/ρ (T) is a quasi-adequate semigroup whose set of idempotents is $\{e\rho;\ e \in B\}$.

Let S be an abundant semigroup and E be its set of idempotents. The relations; μ_L, μ_R and μ have been defined in [2] by the following rules:

$(a, b) \in \mu_L$ if and only if $(ea, eb) \in L^*$ for all $e \in E$,

$(a, b) \in \mu_R$ if and only if $(ae, be) \in R^*$ for all $e \in E$

and $\mu = \mu_L \cap \mu_R$.

S is called fundamental if μ is the identity relation on S. The main property of μ is stated in the following proposition.

Proposition [2] 1.6 μ is the largest congruence on S contained in H^*.

If a is an element of S, then a^* and a^+ denote typical idempotents in L^*_a and R^*_a respectively. S is said to be idempotent-connected if for each element a in S and for some (hence for all) a^+, a^*, there exists a bijection $\alpha: \langle a^+ \rangle \to \langle a^* \rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^+ \rangle$. A type A semigroup is an idempotent-connected adequate semigroup. In this case α is defined by $x\alpha = (xa)^*$ [4].

Proposition [3] 1.7 Let S be an idempotent-connected quasi-adequate semigroup and $\theta: S \to T$ be a good homomorphism onto an adequate semigroup T. Then T is a type A semigroup.

Moreover, we conclude from [2] the following result.

Lemma 1.8 If S is an idempotent-connected abundant semigroup, then μ is good on S.

For the rest of the section, S is a quasi-adequate semigroup with band B of idempotents. Following [7], we denote the ε-class in B of an idempotent e by $E(e)$. The relation δ is defined in [3] on S by the rule:

$a \, \delta \, b$ if and only iff $E(a^+) \, a \, E(a^*) = E(b^+) \, b \, E(b^*)$

for some a^+, a^*, b^+, b^*.

It is clear that δ is an equivalence relation. Some of its properties have been examined in [3], in particular we have

Proposition 1.9 (i) For any elements a and b in S; $a \delta b$ if and only if $a = ebf$ for some $e \in E(b^+)$, $f \in E(b^*)$. (ii) $H^* \cap \delta = 1$, (iii) δ is a congruence on S if and only if for any elements a and b in S;

$$a\, E(a^*)\, E(b^+) \subseteq E((ab)^+)\, ab\, E((ab)^*).$$

A congruence relation ρ on S is called an adequate congruence if S/ρ is an adequate semigroup. We have from [3]:

Proposition 1.10 If δ is a congruence, then it is the minimum adequate good congruence on S.

If S is idempotent-connected on which δ is a congruence, then as a consequence of Proposition 1.10 and Proposition 1.7, we have

Corollary 1.11 If S is idempotent-connected on which δ is a congruence, then S/δ is type A.

2. *-HOMOMORPHISMS

In this section certain properties of idempotent-separating good homomorphisms on quasi-adequate semigroups are investigated. It has been noted recently [8], the idempotent-connected property is preserved under good epimorphisms. We show (Theorem 2.3) that also the equivalence relation δ is a congruence on any idempotent-separating good homomorphic image of a quasi-adequate semigroup S whenever δ is a congruence on S.

Throughout this section, S will denote a quasi-adequate semigroup with a band B of idempotents. First we say, a semigroup homomorphism $\varphi: S \to T$ is a *-homomorphism if for any elements a and b in S, $a\, L^*(S)\, b$ if and only if $a\varphi L^*(T) b\varphi$ where as $a\, R^*(S) b$ if and only if $a\varphi\, R^*(T)\, b\varphi$. In fact *-homomorphisms on S are just the idempotent-separating good homomorphisms as shown in the following Lemma:

Lemma 2.1 Let $\varphi : S \to T$ be a semigroup homomorphism. Then the following conditions are equivalent:

(1) φ is a *-homomorphism

(2) φ is an idempotent-separating good homomorphism.

Proof If (1) holds and e, f are in B such that $e\varphi = f\varphi$, then $e\, L\, f$ and $e\, R\, f$ which implies $e\, H\, f$. Therefore $e = f$ and (2) holds. On the other hand, if (2) holds and a, b are in S such that $a\varphi L^*(T)\, b\varphi$, $a^*\, L^*(S)\, a$, $b^*\, L^*(S)\, b$ for some a^*, b^*; then $a^*\varphi L^*(T)\, a\varphi$, $b^*\varphi L^*(T)\, b\varphi$ and $a^*\varphi\, L b^*\varphi$, that is, $(a^*b^*)\varphi = a^*\varphi$ and $(b^*a^*)\varphi = b^*\varphi$. But a^*b^*, b^*a^* are also in B, then $a^*b^* = a^*$, $b^*a^* = b^*$ and $a\, L^*(S)\, b$.

By a similar argument; $a\varphi R^*(T)\, b\varphi$ implies $a\, R^*(S)\, b$. The reverse implications are guaranteed by the hypothesis. Hence (1) holds.

As an immediate consequence of Lemma 2.1 and Proposition 1.5 we have:

Corollary 2.2 If θ is a *-homomorphism from S onto a semigroup T, then T is a quasi-adequate semigroup whose band of idempotents is $\{e\theta;\ e \in B\}$ and isomorphic to B.

Part (1) of the following Theorem is a special case of [8, Theorem 1.10]. For the convenience of the reader we give a proof of the result.

Theorem 2.3 Let θ be a *-homomorphism from S onto a semigroup T.
(1) If S is idempotent-connected, then also is T.
(2) If δ is a congruence on S, then $\delta(T) = \rho$ is also a congruence on T.

Proof (1) Let $t \in T$ and choose $s \in S$ so that $s\theta = t$. For some s^*, s^+; $s\, L^*(S)\, s^*$, $s\, R^*(S)\, s^+$ which implies $s\theta L^*(T)\, s^*\theta$, $s\theta R^*(T)\, s^+\theta$. Denote $s^*\theta$ by t^* and $s^+\theta$ by t^+. S is idempotent-connected. Then there is a connecting-isomorphism $\alpha: \langle s^+ \rangle \to \langle s^* \rangle$, that is, α is a bijection with $xs = s(x\alpha)$ for any $x \in \langle s^+ \rangle$. Notice that for any $y \in \langle t^+ \rangle$, there exists a unique x in B such that $x\theta = y$. Since

$$(s^+ x s^+)\theta = s^+\theta\, x\theta\, s^+\theta = t^+ y t^+ = y = x\theta$$

which implies $s^+ x s^+ = x$, then $x \in \langle s^+ \rangle$. Define $\varphi: \langle t^+ \rangle \to \langle t^* \rangle$ by $y\varphi = (x\alpha)\theta$ for any $y \in \langle t^+ \rangle$, where $x \in \langle s^+ \rangle$ and $y = x\theta$. Since for any $x \in \langle s^+ \rangle$:

$$t^*(x\alpha)\theta t^* = (s^*(x\alpha)s^*)\theta = (x\alpha)\theta \quad (x\alpha \in \langle s^* \rangle)$$

then $(x\alpha)\theta \in \langle t^* \rangle$ and φ is a well-defined map. In fact $\varphi = \theta^{-1}|_{\langle t^+ \rangle}\, \alpha\, \theta$ which is readily a bijection. Finally, for any $y \in \langle t^+ \rangle$, $x \in \langle s^+ \rangle$ with $y = x\theta$ we get;

$$yt = x\theta\, s\theta = (xs)\theta = (s(x\alpha))\theta = s\theta(x\alpha)\theta = t(y\varphi) .$$

Therefore, φ is a connecting isomorphism and T is an idempotent-connected quasi-adequate semigroup.

(2) Let u, v be in T such that $(u, v) \in \rho$, that is, $u = gvh$ for some $g \in E(u^+)$, $h \in E(v^*)$ (Proposition 1.9 (i)). Let a, b be in S and e, f be in B such that $a\theta = u$, $b\theta = v$, $e\theta = g$ and $f\theta = h$, that is, $a\theta = (ebf)\theta$. One can notice immediately that $e\theta \in E(b^+\theta)$, $f\theta \in E(b^*\theta)$. Thus, $e \in E(b^+)$, $f \in E(b^*)$. Let ω be in T and choose c in S be such that $c\theta = \omega$ and get $(ac)\theta = (ebfc)\theta$. Since

$$bfc = bfc^+c \in bE(b^*)\, E(c^+)c$$

then by Proposition 1.9 (iii), there exist $z \in E((bc)^+)$, $h \in E((bc^*)$ such that $bfc = zbch$. Thus $ebfc = ezbch$. It is evident $(e, b^+) \in \varepsilon$, $(z, (bc)^+) \in \varepsilon$ so that $(ez, b^+(bc)^+) \in \varepsilon$. Since $(bc)^+ \, \mathcal{R}^* \, bc$, then $b^+(bc)^+ \, \mathcal{R}^* \, b^+bc = bc$ which implies $(b^+(bc)^+, (bc)^+) \in \varepsilon$. Therefore, $(ez, (bc)^+) \in \varepsilon$. Put $ez = g$ to get

$$ebfc = gbch \quad \text{where} \quad g \in E((bc)^+),\ h \in E((bc)^*).$$

Hence

$$u\omega = a\theta\, c\theta = (ac)\theta = (ebfc)\theta = (gbch)\theta$$

$$= g\theta\, b\theta\, c\theta\, h\theta = (g\theta)\, v\omega(h\theta)$$

where

$$g\theta \in E((bc)^+\theta) = E(((bc)\theta)^+) = E((v\omega)^+)$$

and

$$h\theta \in E((bc)^*\theta) = E(((bc)\theta)^*) = E((v\omega)^*).$$

Therefore, $(u\omega, v\omega) \in \rho$. By a similar argument $(\omega u, \omega v) \in \rho$. Hence ρ is a congruence on T.

A congruence τ on S is a <u>*-congruence</u> if the natural homomorphism from S onto S/τ is a *-homomorphism. A similar statement as Lemma 2.1 and Theorem 2.3 holds for *-congruences.

The congruence μ is an idempotent-separating congruence on S (Proposition 1.6). If S is idempotent-connected, the μ is good (Lemma 1.8), that is, μ is a *-congruence and we have as a direct application of Theorem 2.3, the following corollary:

<u>Corollary 2.4</u> (i) If S is idempotent-connected, then S/μ is an idempotent-connected quasi-adequate semigroup. (ii) If δ is a congruence and μ is good on S, then $\delta(S/\mu)$ is a congruence on S/μ.

It has been noted in [4] that μ is not always good on S. Therefore, it seems worthwhile to impose the goodness condition on μ which presumably weakens the requirement of idempotent-connected property. This condition is taken up in the next section where we need the following result whose straighforward proof is omitted.

<u>Proposition 2.5</u> If μ is good on S, then S/μ is fundamental.

3. WEAK TYPE W SEMIGROUPS

A <u>type W semigroup</u> is defined in [3] as an idempotent-connected quasi-adequate semigroup on which the equivalence relation δ is a congruence. El-Qallali and Fountain [3] gave a structure theorem for type W semigroups which generalizes that of Hall for orthodox semigroups [6]. Their approach is based on [7, Chapter IV]. The components of the construction are; a band B, a type A semigroup T whose semilattice of idempotents isomorphic to B/ε and a *-homomorphism from T to the inverse semigroup W_B/γ where W_B is the Hall semigroup. We refer the reader to [3] for further details. Venkatesan [11] describes the structure of orthodox semigroups in terms of fundamental orthodox semigroups, inverse semigroups and idempotent-separating homomorphisms without using Hall's semigroup W_B.

We define a <u>weak type W semigroup</u> to be a quasi-adequate semigroup on which μ is good and δ is a congruence. If S is a type W semigroup, then clearly it is a weak type W semigroup. We have not been able to determine whether or not every weak type W is type W. The objective of this section is to establish a structure theorem for weak type W semigroups which generalizes that of Venkatesan for orthodox semigroups [11]. The components used in this construction are: a fundamental quasi-adequate semigroup F on which δ is a congruence, an adequate semigroup Q and a $*$-homomorphism from Q onto F/δ.

Recall from the previous sections that if S is a weak type W semigroup, the S/δ is an adequate semigroup and S/μ is a fundamental quasi-adequate semigroup on which $\delta(S/\mu)$ is a congruence. To achieve our goal we start with the following proposition.

Proposition 3.1 If S is a weak type W semigroup, then S can be embedded in the direct product $S/\mu \times S/\delta$.

Proof Define the map $\varphi: S \to S/\mu \times S/\delta$ by $s\varphi = (s\mu, s\delta)$. φ is a homomorphism and if $s_1\varphi = s_2\varphi$ for some s_1, s_2 in S, then $(s_1, s_2) \in \mu \cap \delta$ which implies by Proposition 1.6 and Proposition 1.9 (ii) that $s_1 = s_2$ and φ is one-to-one.

Proposition 3.1 shows that any weak type W semigroup is isomorphic to a subdirect product of a fundamental quasi-adequate semigroup on which the equivalence relation δ is a congruence and an adequate semigroup. Our aim now is to locate the copy of S in a direct product of a more general form.

Let S be a weak type W semigroup. Let Q be an adequate semigroup isomorphic to S/δ and F be a fundamental quasi-adequate semigroup isomorphic to S/μ. Write $\rho = \delta(F)$. ρ is a congruence on F. Let $\alpha: Q \to S/\delta$ and $\beta: F \to S/\mu$ be isomorphisms. If $g \in Q$, $g\alpha = x\delta$ for some $x \in S$ and $x\mu \in S/\mu$, then there exists an element a in F such that $a\beta = x\mu$ and $a\rho$ in F/ρ is well-determined by g. To justify this claim, let $g\alpha = y\delta$ for some $y \in S$ and $y\mu = b\beta$ for some b in F. Then $(x, y) \in \delta$, that is, $x = eyf$ for some $e \in E(y^+)$, $f \in E(y^*)$. Therefore $x\mu = e\mu\, y\mu\, f\mu$ where $e\mu \in E(y^+\mu)$, $f\mu \in E(y^*\mu)$. Let g and h be idempotents in F such that $g\beta = e\mu$, $h\beta = f\mu$. Then $a\beta = (gbh)\beta$ and $a = gbh$. Since $e\mu \in E(y^+\mu)$, $y\mu = b\beta$, then $y^+\mu \,\mathcal{R}\, b^+\mu$ which implies $E(y^+\mu) = E(b^+\beta)$ and $g\beta \in E(b^+\beta)$, that is, $g \in E(b^+)$. Similarly $h \in E(b^*)$. It follows that $(a, b) \in \rho$ and the claim is justified. Therefore we have a map $\psi: Q \to F/\rho$ defined by $g\psi = a\rho$ for any g in Q, where $g\alpha = x\delta$ and $x\mu = a\beta$ for some x in S.

Lemma 3.2 ψ is a $*$-homomorphism from Q onto F/ρ.

Proof It is clear from the definition that ψ is an epimorphism. Let g be in Q and $g\psi = a\rho$ for some $a \in F$ where there exists x in S such that $g\alpha = x\delta$, $a\beta = x\mu$. It follows from the fact that: $\rho, \alpha, \delta, \beta, \mu$ are good and S/δ, F/ρ are adequate:

$$(g\psi)^* = a^*\rho, \quad g^*\alpha = x^*\delta, \quad a^*\beta \;L\; x^*\mu \quad .$$

Let e be an idempotent in F such that $e\beta = x^*\mu$. Then $g^*\psi = e\rho$ and $a^* \,\mathcal{L}\, e$. Thus $a^*\rho = e\rho$ and $g^*\psi = a^*\rho = (g\psi)^*$ which implies $g^*\psi \;\mathcal{L}^*(F/\rho)\; g\psi$. Similarly $g^+\psi \;\mathcal{R}^*(F/\rho)\; g\psi$.

By Lemma 1.4; ψ is good.

The same argument as in [11] can be carried out to show that ψ is idempotent-separating and thus ψ is a $*$-homomorphism.

We retain the above notations and put:

$$T = \{(a, g) \in F \times Q \,;\, a\rho = g\psi\}$$

T is readily a subsemigroup of $F \times Q$. In fact we have:

Lemma 3.3 T is isomorphic to S.

Proof Let (a, g) be an element in T. Let x be an element in S such that $g\alpha = x\delta$, $x\mu = a\beta$. If also y is in S with $g\alpha = y\delta$, $y\mu = a\beta$, then $(x, y) \in \delta \cap \mu$ and $x = y$. Therefore we have a map $\lambda: T \to S$ defined by $(a, g)\lambda = x$ for any (a, g) in T where x is in S and $g\alpha = x\delta$, $x\mu = a\beta$. It is easy to check that λ is an isomorphism.

In particular, if $Q = S/\delta$ and $F = S/\mu$, then the $*$-homomorphism $\psi: S/\delta \to F/\delta$ is defined by $(x\delta)\psi = (x\mu)\rho$ for any element x in S, and $T = \{(x\mu, x\delta);\, x \in S\}$ is the copy of S in the direct product $S/\mu \times S/\delta$.

It can be noted that the following result has been proved.

Proposition 3.4 Let S be a weak Type W semigroup, Q be an adequate semigroup isomorphic to S/δ and F be a fundamental quasi-adequate semigroup isomorphic to S/μ. Then the equivalence relation $\delta(F) = \rho$ is a congruence on F and there is a $*$-homomorphism ψ from Q onto F/ρ such that S is isomorphic to $T = \{(a, g) \in F \times Q;\, a\rho = g\psi\}$.

Our objective now is to prove a converse of Proposition 3.4, that is, to describe a specific construction and thus establish a structure theorem for weak type W semigroups.

Let Q be an adequate semigroup and F be a fundamental quasi-adequate semigroup on which $\delta(F) = \rho$ is a congruence. Assume there is a $*$-homomorphism ψ from Q onto F/ρ. Define a subset S of the direct product $F \times Q$ by;

$$S = \{(a, g) \in F \times Q;\ a\rho = g\psi\}$$

S is readily a subsemigroup of $F \times Q$.

The following sequence of results provides considerably more information about S.

<u>Lemma 3.5</u> S is a quasi-adequate semigroup.

<u>Proof</u> For any element (a, g) in S; we get $a^*\rho = (a\rho)^* = (g\psi)^* = g^*\psi$ and (a^*, g^*) is an idempotent in S. Since $(a^*, g^*)\ L^*(F \times Q)(a, g)$. Then $(a^*, g^*)\ L^*(S)(a, g)$. Similarly $(a^+, g^+)\ R^*(S)(a, g)$. It is noticable that the set of idempotents in S forms a subsemigroup of S and thus the result holds.

The following corollary is an immediate consequence from the definition of S and the proof of Lemma 3.5.

<u>Corollary 3.6</u> (i) For any idempotent f in $Q(F)$ there exists an idempotent e in $F(Q)$ such that $(e, f)((f, e))$ is an idempotent in S.
(ii) $L^*(S) = L^*(F \times Q) \cap (S \times S)$ and $R^*(S) = R^*(F \times Q) \cap (S \times S)$.

<u>Lemma 3.7</u> For any elements $(a, g), (b, p)$ in S. $((a, g), (b, p)) \in \mu$ if and only if $a = b$.

<u>Proof</u> Let e be an idempotent in F and choose an idempotent f in Q such that (e, f) in S. Then for any elements $(a, g), (b, p)$ in S;

$$\begin{aligned}((a, g), (b,p)) \in \mu_L &\Rightarrow (e, f)(a, g)\ L^*(S)(e, f)(b, p) \quad \text{(definition of } \mu_L)\\ &\Rightarrow (e, f)(a, g)\ L^*(F \times Q)(e, f)(b, g) \quad \text{(Corollary 3.6 (ii))}\\ &\Rightarrow ea\ L^*(F)\ eb\\ &\Rightarrow (a, b) \in \mu_L(F)\end{aligned}$$

Similarly,

$$((a, g), (b, p)) \in \mu_R \Rightarrow (a, b) \in \mu_R(F)$$

Therefore

$$((a, g), (b, p)) \in \mu \Rightarrow (a, b) \in \mu_L(F) \cap \mu_R(F) = \mu(F) \quad .$$

Hence $a = b$ because F is fundamental.

On the other hand, if $(a, g), (b, p)$ are elements in S such that $a = b$, then $a\rho = b\rho$ and $g\psi = p\psi$. Let (e, f) be an idempotent in S, that is, $e\rho = f\psi$. Then $f\psi\, g\psi = f\psi\, p\psi$ which implies $fg\ L^*(Q) fp$ (ψ is a $*$-homomorphism) and $ea = eb$. Therefore $(ea, fg) L^*(F \times Q)(eb, fp)$ and $(e, f)(a, g) L^*(S)(e, f)(b, p)$.

By a similar argument we get $(a, g)(e, f) R^*(S)(b, p)(e, f)$. Hence $((a, g), (b, p)) \in \mu$.

<u>Corollary 3.8</u> μ is good on S.

<u>Proof</u> Let (a, g) be an element in S; $(a, g)L^*(S)(a^*, g^*)$. Then for any elements $(b, p), (c, v)$ in S.

$$\begin{aligned}((a, g)(b, p), (a, g)(c, v)) \in \mu &\Rightarrow ((ab, gp), (ac, pv)) \in \mu\\ &\Rightarrow ab = ac \quad \text{(Lemma 3.7)}\\ &\Rightarrow a^*b = a^*c \quad \text{(Corollary 1.2)}\\ &\Rightarrow ((a^*b, g^*p), (a^*c, g^*v)) \in \mu \quad \text{(Lemma 3.7)}\\ &\Rightarrow ((a^*, g^*)(b, p), (a^*, g^*)(c, v) \in \mu\end{aligned}$$

Similarly

$$((b, p)(a, g), (c, v)(a, g)) \in \mu \Rightarrow ((b, p)(a^+, g^+), (c, v)(a^+, g^+) \in \mu$$

Thus the result follows by Corollary 1.2 and Lemma 1.4.

<u>Lemma 3.9</u> For any elements $(a, g), (b, p)$ in S $((a, g), (b, g)) \in \delta$ if and only if $g = p$.

<u>Proof</u> Let $(a, g), (b, p)$ be elements in S such that $((a, g), (b, p)) \in \delta$. Then $(a, g) = (e, f)(b, p)(q, h)$ for some $(e, f) \in E((b, p)^+) = E(b^+, p^+)$ and $(q, h) \in E((b, p)^*) = E(b^*, p^*)$.

It follows that $f \in E(P^+)$, $h \in E(p^*)$. Since Q is adequate, then $f = P^+$, $h = P^*$ and $g = fph = p^+ p p^* = p$.

On the other hand if $g = p$ in the elements (a, g), (b, p) of S, then $g\psi = p\psi$ and $a\rho = b\rho$, that is, $a = ebf$ for some $e \in E(b^+)$, $f \in E(b^*)$. Notice that: $p^*\psi = b^*\rho = f\rho$ and (f, p^*) is an idempotent in S. Moreover, $(f, p^*) \in E(b^*, p^*) = E((b, p)^*)$.

Similarly; $(e, p^+) \in E((b, p)^+)$. Now from $a = ebf$ we get $(a, g) = (e, p^+)(b, p)(f, p^*)$ which implies $((a, g), (b, p)) \in \delta$.

As an immediate consequence of Lemma 3.9 we have;

Corollary 3.10 δ is a congruence on S.

Moreover,

Lemma 3.11 S/μ is isomorphic to F and S/δ is isomorphic to Q.

Proof Define $\gamma: S \to F$ by $(a, g)\gamma = a$ for any element (a, g) in S. Clearly, γ is a homomorphism from S onto F. If (a, g), (b, p) are elements in S, then

$$(a, g)\gamma = (b, p)\gamma \iff a = b \iff ((a, g), (b, p)) \in \mu \quad \text{(Lemma 3.7)}$$

Hence the kernel of γ is μ and S/μ is isomorphic to F.

By a similar argument it follows that S/δ is isomorphic to Q.

Now a converse of Proposition 3.4 is evident and in conclusion we have

Theorem 3.12 Let Q be an adequate semigroup and F be a fundamental quasi-adequate semigroup on which $\delta(F) = \rho$ is a congruence and there is a *-homomorphism ψ from Q onto F/ρ. Then

$$S = \{(a, g) \in F \times Q;\ a\rho = g\psi\}$$

is a weak type W semigroup.

Conversely, every such semigroup can be constructed in this way.

4. SUBSEMIGROUPS OF A TYPE W SEMIGROUP

As it has been mentioned in the previous section, a type W semigroup S is an idempotent-connected quasi-adequate semigroup on which the

equivalence relation δ is a congruence. In this section we describe the least fundamental adequate good congruence on S, the largest superabundant full subsemigroup of S and the largest full subsemigroup of S which is a band of cancellative monoids. These results generalize those of Venkatesan [12] for orthodox semigroups.

Let S be a type W semigroup with band B of idempotents. It follows from the results in the first and second sections that: μ is a good congruence on S, S/μ is a fundamental quasi-adequate semigroup whose band of idempotents is $\{e\mu: e \in B\}$, δ is the minimum adequate good congruence on S, S/δ is a type A semigroup whose semilattice of idempotents is $\{e\delta; e \in B\}$, and $\mu(S/\delta)$ is a good congruence on S/δ. These facts are used in the discussion frequently without comment.

Consider the natural homomorphisms $\delta^{\#}: S \to S/\delta$, $\mu(S/\delta)^{\#}: S/\delta \to (S/\delta)/\mu$ and define λ to be the kernel of the composition $\delta^{\#}\cdot\mu(S/\delta)^{\#}$.

Since the composition of good homomorphisms is good and a surjective homomorphism is good if and only if its kernel is a good congruence. We conclude that λ is a good congruence on S and thus S/λ is a quasi-adequate semigroup whose band of idempotents is $\{e\lambda;\ e \in B\}$. Some more properties of λ will be investigated in order to achieve part of our objective. We start with the following Lemma which can be deduced directly from the definition.

Lemma 4.1 For any elements x, y in S, the following statements are equivalent:

(1) $(x, y) \in \lambda$

(2) $(x\delta, y\delta) \in \mu(S/\delta)$

(3) $(ex)^*\delta = (ey)^*\delta$ and $(ex)^+\delta = (ye)^+\delta$ for any $e \in B$.

Corollary 4.2 $\lambda|B = \delta|B$

Proof For any g, h in B with $(g, h) \in \lambda$, that is, in particular $eg\delta = eh\delta$ for any $e \in B$. Take in turn $e = g$ and $e = h$ to get $g\delta = gh\delta = hg\delta = h\delta$ and $(g, h) \in \delta$. Conversely, if g, h in B such that $(g, h) \in \delta$, then for any $e \in B$; $(eg)^*\delta = (eg)\delta = (eh)\delta = (eh)^*\delta$ and $(ge)^+\delta = (he)^+\delta$. Thus $(g, h) \in \lambda$.

The following proposition establishes one of the main properties of λ.

Proposition 4.3 λ is the least fundamental good congruence on S.

Proof Let $g\lambda$, $h\lambda$ be idempotents in S/λ for which g, h in B. Then $g\delta$, $h\delta$ are idempotents in S/δ so that $gh\delta = hg\delta$ and by Corollary 4.2 $(gh, hg) \in \lambda$. Thus S/λ is adequate and λ is an adequate good congruence on S. To show that S/λ is fundamental, let e be in B and x, y be in S. Then

$$e\lambda\, x\lambda\ \mathcal{L}^*(S/\lambda)\ e\lambda\, y\lambda \implies (ex)^*\lambda = (ey)^*\lambda$$

$$\implies (ex)^*\delta = (ey)^*\delta \qquad \text{(Corollary 4.2)}$$

Likewise

$$x\lambda\, e\lambda\ \mathcal{R}^*(S/\lambda)\ y\lambda\, e\lambda \implies (xe)^+\delta = (ye)^+\delta$$

Hence

$$(x\lambda, y\lambda) \in \mu(S/\lambda) \implies (ex)^*\delta = (ey)^*\delta \text{ and } (xe)^+\delta = (ye)^+\delta \quad \forall e \in B$$

$$\implies (x, y) \in \lambda \qquad \text{(Lemma 4.1)}$$

that is, λ is a fundamental adequate good congruence on S. Now let π be a fundamental adequate good congruence on S. Then for any x, y in S;

$$(x, y) \in \lambda \implies (ex)^*\delta = (ey)^*\delta,\ (xe)^+\delta = (ye)^+\delta \qquad e \quad B$$

$$\implies (ex)^*\pi = (ey)^*\pi,\ (xe)^+\pi = (ye)^+\pi \qquad (\rho \subseteq \pi)$$

$$\implies (ex)\pi\ \mathcal{L}^*(S/\pi)(ey)\pi,\ (xe)\pi\ \mathcal{R}^*(S/\pi)(ye)\pi \qquad (\pi \text{ is good})$$

$$\implies (x\pi, y\pi) \in \mu(S/\pi) \qquad (\text{definition of } \mu(S/\pi))$$

$$\implies x\pi = y\pi \qquad (S/\pi \text{ is fundamental})$$

Hence the result.

As an immediate consequence we have

Corollary 4.4 S/λ is type A and $\delta \subseteq \lambda$.

By a full subsemigroup of S we mean simply one which contains B. Recall from [2] that any full subsemigroup U of S satisfies the following property:

$$\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U) \quad \text{and} \quad \mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U),$$

and it is abundant.

To begin the process of describing the largest full subsemigroup of S which is superabundant or a band of cancellative monoids, let us see the effect of λ on superabundant full subsemigroups of S through the following results.

Lemma 4.5 If M is a superabundant full subsemigroup of S, then for any $t \in M$, $(t, t^2) \in \lambda$.

Proof Let M be a superabundant full subsemigroup of S. Then for any element t in M, there exists f in B such that $t \in H_f^*$ and thus $t^2 \in H_f^*$ (Corollary 1.3). For any element e in B, let g and h be in B where $et \in H_g^*$, $et^2 \in H_h^*$. It follows that; $g\ \mathcal{R}^*et\ \mathcal{R}^*ef$, $h\ \mathcal{R}^*et^2\ \mathcal{R}^*ef$ and thus

$$(et)^*\delta = g\delta \qquad (et\ \mathcal{L}^*g)$$

$$= (ef)\delta \qquad (g\ \mathcal{R}\ ef)$$

$$= h\delta \qquad (ef\ \mathcal{R}\ h)$$

$$= (et^2)^*\delta \qquad (h\ \mathcal{L}^*et^2)$$

Similarly, $(te)^+\delta = (t^2e)^+\delta$.

Therefore $(t, t^2) \in \lambda$.

Corollary 4.6 If M is a superabundant full subsemigroup of S, then the restriction of λ on M is a semilattice congruence on M.

Proof By Lemma 4.5 all the elements of M/λ are idempotents and the idempotents commute by Corollary 4.2.

In fact the restriction of λ on M is the minimum semilattice good congruence on M as a direct consequence of the following Lemma:

Lemma 4.7 If η is any semilattice (band) good congruence on S, then $\lambda \subseteq \eta (\mu \subseteq \eta)$.

Proof Let η be a semilattice good congruence on S, and a, b be elements in S. Then

$$(a, b) \in \lambda \implies (ea)^*\delta = (eb)^*\delta \qquad \forall e \in B$$

$$\implies (ea)^*\eta = (eb)^*\eta \qquad (\delta \subseteq \eta)$$

$$\implies (ea)\eta = (eb)\eta \qquad (\eta \text{ is good, } S/\eta \text{ is a semilattice})$$

$$\implies ea^+\eta = eb^+\eta \qquad (ea\ \mathcal{R}^*\ ea^+,\ eb\ \mathcal{R}^*\ eb^+)$$

$$\implies a^+\eta = a^+b^+\eta,\ b^+a^+\eta = b^+\eta \qquad (\text{taking in turn } e = a^+,\ e = b^+)$$

$$\implies a^+\eta = b^+\eta \qquad (a^+b^+\eta = b^+a^+\eta)$$

$$\implies a\eta = b\eta$$

Hence $\lambda \subseteq \eta$.

The easy proof of the other part of the Lemma is omitted.

The following Lemma is needed for further investigation.

Lemma 4.8 If $x\lambda$ is an idempotent in S/λ, then there exists an idempotent g in S such that $x \in H^*_g$.

Proof Let $x\lambda$ be an idempotent in S/λ. Choose e in B such that $(x, e) \in \lambda$, that is, $(x\rho, e\rho) \in \mu(S/\delta)$ (Lemma 4.1), in particular $(x\rho, e\rho) \in H^*(S/\delta)$. Therefore, $x^+\rho = e\rho$, $x^*\rho = e\rho$ and $E(x^+) = E(e) = E(x^*)$. Since $x x^+ x^* = x x^* x^+ x^* = xx^* = x$ and for any s, t in S;

$$xs = xt \Longrightarrow x^*s = x^*t \Longrightarrow x^+x^*s = x^+x^*t$$

then $x \, L^* \, x^+x^*$ by Corollary 1.2.

A similar argument leads to $x \, \mathcal{R}^* \, x^+x^*$. Hence $(x, x^+x^*) \in H^*$. We may take g to be x^+x^*.

We combine Lemma 4.8 with Corollary 1.3 to get

Corollary 4.9 If $x\lambda$ is an idempotent in S/λ, then H^*_x is a cancellative monoid.

The following Lemma is mentioned in [1] and is needed for the final result. It is included here for the sake of completeness.

Lemma 4.10 A semigroup P is a band of cancellative monoids if and only if P is superabundant and H^* is a congruence.

Proof Let $P = \bigcup_{\alpha \in X} T_\alpha$ be a band of cancellative monoids $T_\alpha (\alpha \in X)$ and let e_α be the identity of T_α. Let $a \in T_\alpha$, $s \in T_\beta$, $t \in T_\gamma$ be such that $as = at$. Then $\alpha\beta = \alpha\gamma = \delta$, say, and $e_\alpha s, e_\alpha t \in T_\delta$. Hence $ae_\delta e_\alpha s = ae_\delta e_\alpha t$ and since T_δ is cancellative, we have $e_\alpha s = e_\alpha t$. Thus $a \, L^* e_\alpha$. Similarly $a \, \mathcal{R}^* e_\alpha$ and so P is superabundant. It is easy to see that $T_\alpha = H^*_a$ for any $a \in T_\alpha$ and it follows that H^* is a congruence.

On the other hand, if P is superabundant then each H^*-class is a cancellative monoid and $x \, H^* \, x^2$ for any $x \in P$. If H^* is a congruence, then P/H^* is a band and the natural homomorphism $\varphi : P \rightarrow P/H^*$ defined by $x\varphi = H^*_x$ shows that P is a band of the H^*-classes.

Theorem 4.11 Let $A = \{x \in S;\ (x, x^2) \in \lambda\}$ $(M = \{x \in S;\ (x, x^2) \in \mu\})$. Then $A(M)$ is the largest full subsemigroup of S which is superabundant (a band of cancellative monoids).

Proof Since for any idempotent $x\lambda$ in S/λ, there exists an idempotent e in S such that $x \in H^*_e$ (Lemma 4.8), then clearly $A = \{x \in S;\ (x, x^2) \in \lambda\}$ is a superabundant full subsemigroup of S. By Lemma 4.5, any superabundant full subsemigroup of S is a subset of A.

To show the other part of the Theorem, it is readily $M = \{x \in S;\ (x, x^2) \in \mu\}$ is a superabundant full subsemigroup of S. Since for any $x \in M$; $x\mu$ is an idempotent in S/μ and there exists an idempotent e in S with $(x, e) \in \mu$ so it is easy to see that the H^*-classes of M are the μ-classes and H^* is a congruence. Hence by Lemma 4.10, M is a band of cancellative monoids. Let N be a full subsemigroup of S which is a band of cancellative monoids. Then $H^*(N)$ is a congruence and N is superabundant (Lemma 4.10). Notice that

$n \in N \Longrightarrow (n, e) \in H^*(N)$	for some e in B
$\Longrightarrow (n, e) \in \mu(N)$	$(H^*(N) = \mu(N))$
$\Longrightarrow (n, e) \in \mu(S)$	$(L^*(N) = L^*(S) \cap (N \times N)$ and $R^*(N) = R^*(S) \cap (N \times N))$
$\Longrightarrow n \in M$	

proving N is a subset of M.

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