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NEW ANALYTICALLY SOLVABLE MODELS OF RELATIVISTIC POINT INTERACTIONS

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1. Finitely many point interactions

Let D_0 denote the one-dimensional free Dirac operator in the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$

$D_0 = D, \quad \mathcal{D}(D_0) = H^{2,1}(\mathbb{R}) \otimes \mathbb{C}^2,$

$$D = -ic\frac{d}{dx} \otimes \sigma_1 + (c^2/2) \otimes \sigma_3 = \begin{pmatrix} c^2/2 & -ic\frac{d}{dx} \\ -ic\frac{d}{dx} & -c^2/2 \end{pmatrix}, \tag{1.1}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2}$$

are Pauli matrices in $\mathbb{C}^2$, $c > 0$ abbreviates the velocity of light, and $H^{2,1}(\mathbb{R})$ denotes the standard Sobolev space. The corresponding free resolvent is then given by

$$R_k = (D_0 - z)^{-1}, \quad z \in \mathbb{C} \setminus \{(-\infty, -c^2/2] \cup [c^2/2, \infty)\} \tag{1.3}$$

with integral kernel

$$R_k(x-x') = (i/2c) \begin{pmatrix} \zeta & \operatorname{sgn}(x-x') \\ \operatorname{sgn}(x-x') & \zeta^{-1} \end{pmatrix} e^{ik|x-x'|}, \tag{1.4}$$

$$\zeta(z) \equiv \zeta(k) = [z + (c^2/2)]/ck(z), \quad ck(z) = [z^2 - (c^4/4)]^{1/2}, \quad \operatorname{Im} k(z) > 0. \tag{1.5}$$

Relativistic point interactions concentrated on the set $Y = \{y_1, \ldots, y_N\} \subset \mathbb{R}$, $N \in \mathbb{N}$ can now be constructed as follows. Define in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ the closed, symmetric operator

$$\dot{D}_Y = D, \quad \mathcal{D}(\dot{D}_Y) = \{g \in H^{2,1}(\mathbb{R}) \otimes \mathbb{C}^2 \,|\, g(y_j) = 0,\ j = 1, \ldots, N\}. \tag{1.6}$$

Here $g(y) = 0$ abbreviates $g_1(y) = g_2(y) = 0$ where

$$g(y) = \begin{pmatrix} g_1(y) \\ g_2(y) \end{pmatrix} \in H^{2,1}(\mathbb{R}) \otimes \mathbb{C}^2. \tag{1.7}$$

By inspection $\dot{D}_Y$ has deficiency indices $(2N, 2N)$ and hence a $4N^2$-parameter family of self-adjoint extensions. In this letter we shall study two particular N-parameter families of self-adjoint extensions of $\dot{D}_Y$ which will turn out to be closely related to their nonrelativistic counterparts, viz. δ- and δ'-interactions. (For recent extensive treatments of the latter cf. [3], [16], [18].) The two families are defined by

$D_{\alpha,Y} = D,$

$$\mathcal{D}(D_{\alpha,Y}) = \{g \in L^2(\mathbb{R}) \otimes \mathbb{C}^2 \,|\, g_1 \in AC_{loc}(\mathbb{R}),\ g_2 \in AC_{loc}(\mathbb{R} \setminus Y); \\ g_2(y_j+) - g_2(y_j-) = -(i\alpha_j/c) g_1(y_j),\ j = 1, \ldots, N\}, \\ \alpha = (\alpha_1, \ldots, \alpha_N),\ -\infty < \alpha_j \leq \infty,\ j = 1, \ldots, N \tag{1.8}$$

and by

$T_{\beta,Y} = D,$

$$\mathcal{D}(T_{\beta,Y})=\{g\in L^2(\mathbb{R})\otimes\mathbb{C}^2 \mid g_1\in AC_{loc}(\mathbb{R}\setminus Y),\ g_2\in AC_{loc}(\mathbb{R});\ g_1(y_j+)-g_1(y_j-)=i\beta_j c g_2(y_j),\ j=1,\ldots,N\},\ \beta=(\beta_1,\ldots,\beta_N),\ -\infty<\beta_j\le\infty,\ j=1,\ldots,N \tag{1.9}$$

where $AC_{loc}(\Omega)$ denotes the set of locally absolutely continuous functions on $\Omega\subseteq\mathbb{R}$. By Krein's formula [1] their respective resolvents explicitly read

$$(D_{\alpha,Y}-z)^{-1}=R_k+\sum_{j,j'=1}^{N}[M_{\alpha,Y}(k)]^{-1}_{jj'}\,(\overline{\tilde f_k(\cdot-y_{j'})},\cdot)f_k(\cdot-y_j),\quad z\in\rho(D_{\alpha,Y}),\ \mathrm{Im}k>0 \tag{1.10}$$

where

$$M_{\alpha,Y}(k)=-[(4c^2/\alpha_j)\delta_{jj'}+2ic\zeta e^{ik|y_j-y_{j'}|}]_{j,j'=1}^{N}\ , \tag{1.11}$$

$$f_k(x)=\begin{pmatrix}\zeta\\ \mathrm{sgn}(x)\end{pmatrix}e^{ik|x|}\ ,\qquad \tilde f_k(x)=\begin{pmatrix}-\zeta\\ \mathrm{sgn}(x)\end{pmatrix}e^{ik|x|}\ , \tag{1.12}$$

$$z\in\mathbb{C}\setminus\{(-\infty,-c^2/2]\cup[c^2/2,\infty)\},\ \mathrm{Im}k>0$$

and

$$(T_{\beta,Y}-z)^{-1}=R_k+\sum_{j,j'=1}^{N}[\tilde M_{\beta,Y}(k)]^{-1}_{jj'}\,(\overline{\tilde g_k(\cdot-y_{j'})},\cdot)g_k(\cdot-y_j),\quad z\in\rho(T_{\beta,Y}),\ \mathrm{Im}k>0, \tag{1.13}$$

where

$$\tilde M_{\beta,Y}(k)=[(4/\beta_j)\delta_{jj'}-2ic\zeta^{-1}e^{ik|y_j-y_{j'}|}]_{j,j'=1}^{N}\ , \tag{1.14}$$

$$g_k(x)=\begin{pmatrix}\mathrm{sgn}(x)\\ \zeta^{-1}\end{pmatrix}e^{ik|x|}\ ,\qquad \tilde g_k(x)=\begin{pmatrix}\mathrm{sgn}(x)\\ -\zeta^{-1}\end{pmatrix}e^{ik|x|}\ , \tag{1.15}$$

$$z\in\mathbb{C}\setminus\{(-\infty,-c^2/2]\cup[c^2/2,\infty)\},\quad \mathrm{Im}k>0.$$

Obviously the above result completely determines all spectral properties of $D_{\alpha,Y}$ and $T_{\beta,Y}$. E.g. for $\alpha,\beta\in\mathbb{R}$ their essential spectra are purely absolutely continuous and coincide with $(-\infty,-c^2/2]\cup[c^2/2,\infty)$ whereas their bound states in the gap $(-c^2/2,c^2/2)$ are determined by the respective zeros of the determinants of $M_{\alpha,Y}$ and $M_{\beta,Y}$. These results and the corresponding ones for the on-shell scattering matrix can now be obtained in complete analogy to their nonrelativistic counterparts (i.e. δ- and δ'-interactions [3]).

At this point we would like to emphasize that to the best of our knowledge the above models are new. However, another N-parameter family of relativistic point interactions has been discussed extensively in the literature. The corresponding boundary condition reads (cf. e.g. [20], [26])

$$g_1(y_j+)=\cos(\gamma_j)g_1(y_j-)-i\sin(\gamma_j)g_2(y_j-),$$
$$g_2(y_j+)=\cos(\gamma_j)g_2(y_j-)-i\sin(\gamma_j)g_1(y_j-),\quad \gamma_j\in\mathbb{R},\ j=1,\ldots N. \tag{1.16}$$

Moreover we note that the fact that $\dot{D}_Y$ in general has a $4N^2$-parameter family of self-adjoint extensions obviously is responsible for the puzzles discussed e.g. in [7], [42], [43].

As will become clear in the next sections the boundary conditions in (1.8) and (1.9), in contrast to (1.16), give rise to a different formulation of $D_{\alpha,Y}$, $T_{\beta,Y}$ in terms of a second order difference operator in $\ell^2(\mathbb{Z})$. This connection will turn out to be extremely usefull when studying the case $N=\infty$. In particular it will enable us to carry over all known results on nonrelativistic random δ- and δ'-interactions to the present case in Sect. 3.

We finally end up with a few more remarks in the special case N = 1. The point spectra of $D_{\alpha,y}$ and $T_{\beta,y}$ are then given by

$$\sigma_p(D_{\alpha,y})=\begin{cases}\frac{c^2}{2}\ \frac{4c^2-\alpha^2}{4c^2+\alpha^2}, & \alpha<0\\ \emptyset, & \alpha>0,\ \alpha=\infty,\end{cases} \tag{1.17}$$

$$\sigma_p(T_{\beta,y})=\begin{cases}\frac{c^2}{2}\ \frac{\beta^2c^2-4}{\beta^2c^2+4}, & \beta<0\\ \emptyset, & \beta>0,\ \beta=\infty\ .\end{cases} \tag{1.18}$$

Moreover, applying the strategy of [14], [15] one proves that $(D_{\alpha,y}-(c^2/2)-z)^{-1}$ and $(T_{\beta,y}-(c^2/2)-z)^{-1}$, $z\in\mathbb{C}\setminus\mathbb{R}$ are holomorphic in norm with respect to c^{-1} around $c^{-1}=0$ and that

$$\text{n-}\lim_{c\to\infty}(D_{\alpha,y}-(c^2/2)-z)^{-1}=(-\Delta_{\alpha,y}-z)^{-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad -\infty<\alpha\le\infty, \tag{1.19}$$

$$\text{n-}\lim_{c\to\infty}(T_{\beta,y}-(c^2/2)-z)^{-1}=(\Xi_{\beta,y}-z)^{-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad -\infty<\beta\le\infty. \tag{1.20}$$

where $-\Delta_{\alpha,y}$ and $\Xi_{\beta,y}$ denote the *nonrelativistic* δ- *and* δ'-*interaction centered* at $y\in\mathbb{R}$ respectively [3], i.e.

$$-\Delta_{\alpha,y}=-\frac{d^2}{dx^2}\ ,$$

$$\mathcal{D}(-\Delta_{\alpha,y})=\{g\in H^{2,2}(\mathbb{R}\setminus\{y\})\mid g(y+)=g(y-),g'(y+)-g'(y-)=\alpha g(y)\},\quad -\infty<\alpha\le\infty, \tag{1.21}$$

$$\Xi_{\beta,y}=-\frac{d^2}{dx^2}\ ,$$

$$\mathcal{D}(\Xi_{\beta,y})=\{g\in H^{2,2}(\mathbb{R}\setminus\{y\})\mid g'(y+)=g'(y-),g(y+)-g(y-)=\beta g'(y)\},\quad -\infty<\beta\le\infty. \tag{1.22}$$

Thus $D_{\alpha,y}$ and $T_{\beta,y}$ are natural relativistic generalizations of $-\Delta_{\alpha,y}$ and $\Xi_{\beta,y}$. In particular the bound state energies $E_{\alpha,y}$ of $D_{\alpha,y}$, $\alpha<0$ and $\tilde{E}_{\beta,y}$ of $T_{\beta,y}$, $\beta<0$ with the rest energy $c^2/2$ subtracted, turn out to be holomorphic, with respect to c^{-2} around their respective nonrelativistic limits

$$E_{\alpha,y}-c^2/2=-(\alpha^2/4)[1+(\alpha/2c)^2]^{-1},\quad \alpha<0, \tag{1.23}$$

$$\tilde{E}_{\beta,y}-c^2/2=-(4/\beta^2)[1+(2/\beta c)^2]^{-1},\quad \beta<0. \tag{1.24}$$

Trivially the above remarks extend to the case $N\in\mathbb{N}$.

2. Infinitely many point interactions on a lattice

Let $Y=\{y_j\in\mathbb{R}\mid j\in\mathbb{Z}\}$ be a discrete subset of $\mathbb{R}$ satisfying

$$\inf_{\substack{j,j'\in\mathbb{Z}\\ j\ne j'}}|y_j-y_{j'}|=d>0,$$
$$y_j<y_{j+1},\quad j\in\mathbb{Z},\qquad \bigcup_{j\in\mathbb{Z}}[y_j,y_{j+1}]=\mathbb{R}. \tag{2.1}$$

Clearly the analogs of (1.8) and (1.9) are then defined by

$$D_{\alpha,Y}=D,$$
$$\mathcal{D}(D_{\alpha,Y})=\{g\in L^2(\mathbb{R})\otimes \mathbb{C}^2 \,|\, g_1\in AC_{loc}(\mathbb{R}),\ g_2\in AC_{loc}(\mathbb{R}\backslash Y);$$
$$g_2(y_j+)-g_2(y_j-)=-(i\alpha_j/c)g_1(y_j),\ j\in\mathbb{Z}\},$$
$$\alpha=\{\alpha_j\}_{j\in\mathbb{Z}},\ -\infty<\alpha_j\leq\infty,\ j\in\mathbb{Z} \tag{2.2}$$

and

$$T_{\beta,Y}=D,$$
$$\mathcal{D}(T_{\beta,Y})=\{g\in L^2(\mathbb{R})\otimes \mathbb{C}^2 \,|\, g_1\in AC_{loc}(\mathbb{R}\backslash Y),\ g_2\in AC_{loc}(\mathbb{R});$$
$$g_1(y_j+)-g_1(y_j-)=i\beta_j c g_2(y_j),\ j\in\mathbb{Z}\},$$
$$\beta=\{\beta_j\}_{j\in\mathbb{Z}},\ -\infty<\beta_j\leq\infty,\ j\in\mathbb{Z}. \tag{2.3}$$

Self-adjointness of $D_{\alpha,Y}$ and $T_{\beta,Y}$ can be shown in analogy to the treatment in [17] (in connection with Schrödinger operators).

At this point we would like to mention that the spectra of $D_{\alpha,Y}$ and $T_{\beta,Y}$ are closely related since one trivially infers that

$$[1\otimes\sigma_2]D_{\alpha,Y}[1\otimes\sigma_2]^{-1}=-T_{\alpha/c^2,Y}\ ,\ \alpha=\{\alpha_j\}_{j\in\mathbb{Z}},\ -\infty<\alpha_j\leq\infty,\ j\in\mathbb{Z}, \tag{2.4}$$

where

$$\sigma_2=\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\ . \tag{2.5}$$

Next we turn to the difference equation approach mentioned in Sect. 1. At the same time we specialize to the lattice $Y = a\mathbb{Z}$, $a > 0$. (In principle the difference equations to be derived below exist as well for the general situation (2.1). However the resulting equations in general cannot be analyzed in detail.)

Assume that $\psi(k,x)$ satisfies

$$(D_{\alpha,a\mathbb{Z}}\psi)(k,x)=E\psi(k,x),\ x\in\mathbb{R}\backslash a\mathbb{Z},\quad \psi(k,x)=\begin{pmatrix}\psi_1(k,x)\\ \psi_2(k,x)\end{pmatrix},$$
$$\psi_1(k,aj+)=\psi_1(k,aj-),\ \psi_2(k,aj+)-\psi_2(k,aj-)=-(i\alpha_j/c)\psi_1(k,aj),\ j\in\mathbb{Z},\ E\in\mathbb{R},\ \mathrm{Im}k>0 \tag{2.6}$$

and $\phi(k,x)$ satisfies

$$(T_{\beta,a\mathbb{Z}}\phi)(k,x)=E\phi(k,x),\ x\in\mathbb{R}\backslash a\mathbb{Z},\quad \phi(k,x)=\begin{pmatrix}\phi_1(k,x)\\ \phi_2(k,x)\end{pmatrix},$$
$$\phi_2(k,aj+)=\phi_2(k,aj-),\ \phi_1(k,aj+)-\phi_1(k,aj-)=i\beta_j c\phi_2(k,aj),\ j\in\mathbb{Z},\ E\in\mathbb{R},\ \mathrm{Im}k>0. \tag{2.7}$$

Then a completely analogous derivation to the corresponding nonrelativistic situations in connection with $-\Delta_{\alpha,a\mathbb{Z}}$, $\Xi_{\beta,a\mathbb{Z}}$ [3], [18], [37] yields the difference equations

$$\psi_{j+1}(k)+\psi_{j-1}(k)=\{2\cos(ka)+(\alpha_j/c)\xi(k)\sin(ka)\}\psi_j(k),$$
$$\psi_j(k)=\psi_1(k,aj),\ \mathrm{Im}k>0,\ k\neq m\pi/a,\ j,m\in\mathbb{Z} \tag{2.8}$$

in the case of $D_{\alpha,a\mathbb{Z}}$ and

$$\phi_{j+1}(k)+\phi_{j-1}(k)=\{2\cos(ka)-\beta_j c\xi(k)^{-1}\sin(ka)\}\phi_j(k),$$
$$\phi_j(k)=\phi_2(k,aj),\ \mathrm{Im}k>0,\ k\neq m\pi/a,\ j,m\in\mathbb{Z}. \tag{2.9}$$

in the case of $T_{\beta,a\mathbb{Z}}$. Conversely any solution of (2.8) (resp. of (2.9)) defines a unique solution $\psi(k,x)$ (resp. $\phi(k,x)$) of (2.6) (resp. (2.7)). In addition $\psi(k)\in L^p(\mathbb{R})$ (resp. $\phi(k)\in L^p(\mathbb{R})$) is equivalent to $\{\psi_j(k)\}_{j\in\mathbb{Z}}\in \ell^p(\mathbb{Z})$ (resp. $\{\phi_j(k)\}_{j\in\mathbb{Z}}\in \ell^p(\mathbb{Z})$) for $p=\infty$ or $p=2$ and also exponential growth respectively decay of $\psi(k,x)$ (resp. $\phi(k,x)$) is equivalent to that of $\{\psi_j(k)\}_{j\in\mathbb{Z}}$ (resp. $\{\phi_j(k)\}_{j\in\mathbb{Z}}$) and at the same rate.

As a consequence, the difference equations for $D_{\alpha,a\mathbb{Z}}$, $T_{\beta,a\mathbb{Z}}$, $-\Delta_{\alpha,a\mathbb{Z}}$, $\tilde{\Delta}_{\beta,a\mathbb{Z}}$ are all of the identical type. This fact will heavily be used in the next section. Since no linear combination of $g_1(k,x)$ and $g_2(k,x)$ will be continuous at all $y_j\in Y$ if the boundary conditions *(1.16)* are considered, no such difference equation exists in this case.

Historically, the use of a difference equation in connection with $-\Delta_{\alpha,Y}$ seems to go back to [29] cf. also [31]. This approach was rediscovered in [6] and considrably simplified in [37].

Finally, in the rest of this section, we analyze the energy band spectra of $D_{\alpha,a\mathbb{Z}}$ and $T_{\beta,a\mathbb{Z}}$ in the special case of periodicity i.e. where $\alpha_j=\alpha$ respectively $\beta_j=\beta$, $j\in\mathbb{Z}$. By standard methods this case can be tackled by a direct integral decomposition approach (cf. e.g. [30]). We first introduce the reduced operators $D_\alpha(\theta)$, $T_\beta(\theta)$ in $L^2((-a/2,\ a/2))\otimes\mathbb{C}^2$ by

$$D_\alpha(\theta)=D,\ \mathcal{D}(D_\alpha(\theta))=\{g(\theta)\in H^{2,1}((-a/2,a/2)\setminus\{0\})\otimes\mathbb{C}^2 \mid g_n(\theta,-a/2+)=e^{i\theta a}g_n(\theta,a/2-),\ n=1,2;\ g_1(\theta,0+)=g_1(\theta,0-),\ g_2(\theta,0+)-g_2(\theta,0-)=-(i\alpha/c)g_1(\theta,0)\},\quad -\infty<\alpha\leq\infty,\quad \theta\in[-\pi/a,\ \pi/a) \tag{2.10}$$

and

$$T_\beta(\theta)=D,\ \mathcal{D}(T_\beta(\theta))=\{g(\theta)\in H^{2,1}((-a/2,a/2)\setminus\{0\})\otimes\mathbb{C}^2 \mid g_n(\theta,-a/2+)=e^{i\theta a}g_n(\theta,a/2-),\ n=1,2;\ g_2(\theta,0+)=g_2(\theta,0-),\ g_1(\theta,0+)-g_1(\theta,0-)=i\beta c\ g_2(\theta,0)\},\quad -\infty<\beta\leq\infty,\ \theta\in[-\pi/a,\ \pi/a). \tag{2.11}$$

Then

$$D_{\alpha,a\mathbb{Z}} \cong \frac{a}{2\pi}\int_{[-\pi/a,\pi/a)}^{\oplus} d\theta D_\alpha(\theta),\qquad T_{\beta,a\mathbb{Z}} \cong \frac{a}{2\pi}\int_{[-\pi/a,\pi/a)}^{\oplus} d\theta T_\beta(\theta) \tag{2.12}$$

where $\cong$ abbreviates unitary equivalence.

We first summarize the spectral properties of $D_\alpha(\theta)$.

<u>Theorem 2.1.</u> Let $-\infty<\alpha\leq\infty$, $\theta\in[-\pi/a,\pi/a)$. Then the essential spectrum of $D_\alpha(\theta)$ is empty and thus the spectrum of $D_\alpha(\theta)$ is purely discrete. In particular its eigenvalues $E_m^\alpha(\theta)$, $m\in\mathbb{Z}\setminus\{0\}$ are given by

$$E_m^\alpha(\theta)=\mathrm{sgn}(m)\{[k_m^\alpha(\theta)]^2c^2+(c^4/4)\}^{1/2},\ m\in\mathbb{Z}\setminus\{0,1\},$$
$$E_1^\alpha(\theta)=\{[k_1^\alpha(\theta)]^2c^2+(c^4/4)\}^{1/2} \tag{2.13}$$

where for $m=1$ the branch for the square root to be chosen in (2.13) depends on α and θ and where $k_m^\alpha(\theta)$, $m\in\mathbb{Z}\setminus\{0\}$, ordered with respect to their absolute values are solutions of

$$\cos(\theta a)=\cos(ka)+[\alpha\xi(k)\backslash 2c]\sin(ka),\ \mathrm{Re}k>0,\ \mathrm{Im}k>0 \tag{2.14}$$

with

$$\xi(k)=\zeta(E)=(\mathrm{sgn}(E)[k^2c^2+(c^4/4)]^{1/2}+(c^2/2))/ck,$$

$$k(E)=\begin{cases}|E^2-(c^4/4)|^{1/2}, & |E|>c^2/2\\ i|(c^4/4)-E|^{1/2}, & |E|<c^2/2.\end{cases} \tag{2.15}$$

For $\alpha\in\mathbb{R}\setminus\{0\}$, except for $\alpha=-ac^2$, $m=1$, $\theta=0$, the eigenvalues $E_m^\alpha(\theta)$ are simple. If $\alpha=-ac^2$, then $E_1^{-ac^2}(0)$ has multiplicity two.

Let $\alpha>0$. For $E>0$ we obtain

$$c^2/2<E_1^\alpha(0)<E_1^\alpha(-\pi/a)=[(\pi^2c^2/a^2)+(c^4/4)]^{1/2}<E_2^\alpha(-\pi/a)<$$
$$<E_2(0)=[(4\pi^2c^2/a^2)+(c^4/4)]^{1/2}<E_3(0)<\dots\ . \tag{2.16}$$

For $E<0$ we get

$$E_{-1}^\alpha(0)=-c^2/2>E_{-1}^\alpha(-\pi/a)>E_2^\alpha(-\pi/a)=-[(\pi^2c^2/a^2)+(c^4/4)]^{1/2}>$$
$$>E_{-2}^\alpha(0)>E_{-3}^\alpha(0)=-[(4\pi^2c^2/a^2)+(c^4/4)]^{1/2}>E_{-3}^\alpha(-\pi/a)>\dots\ . \tag{2.17}$$

In addition

$$\sigma(D_\alpha(\theta))\cap(-c^2/2,c^2/2)=\emptyset,\quad \alpha>0,\quad \theta\in[-\pi/a,\pi/a]. \tag{2.18}$$

Next let $\alpha<0$. For $E>-c^2/2$ we obtain

$$E_1^\alpha(0)<E_1^\alpha(-\pi/a)<E_2^\alpha(-\pi/a)=[(\pi^2c^2/a^2)+(c^4/4)]^{1/2}<E_2^\alpha(0)<$$
$$E_3^\alpha(0)=[(4\pi^2c^2/a^2)+(c^4/4)]^{1/2}<E_3^\alpha(-\pi/a)<\dots,$$
$$-c^2/2<E_1^\alpha(0)<c^2/2,\quad \alpha\in\mathbb{R}, \tag{2.19}$$
$$E_1^{-(2c)\tanh(ac/4)}(0)=0,\quad E_1^\alpha(0)=-c^2/2,\quad \alpha<-ac^2,$$
$$-c^2/2<E_1^\alpha(-\pi/a)<[(\pi^2c^2/a^2)+(c^4/4)]^{1/2},\quad \alpha\in\mathbb{R}$$
$$E_1^{-4/a}(-\pi/a)=c^2/2,\quad E_1^{-(2c)\coth(ac/4)}(-\pi/a)=0,$$
$$E_1^\alpha(-\pi/a)\ \xrightarrow[\alpha\to-\infty]{}\ -c^2/2\ .$$

For $E<-c^2/2$ we get

$$E_{-1}^\alpha(0)>E_{-1}^\alpha(-\pi/a)=-[(\pi^2c^2/a^2)+(c^4/4)]^{1/2}>E_{-2}^\alpha(-\pi/a)>$$
$$>E_{-2}^\alpha(0)=-[(4\pi^2c^2/a^2)+(c^4/4)]^{1/2}>E_{-3}^\alpha(0)>\dots\ , \tag{2.20}$$
$$E_1^\alpha(0)\begin{cases}=-c^2/2, & -ac^2<\alpha<0\\ <-c^2/2, & \alpha<-ac^2.\end{cases}$$

All non constant eigenvalues $E_m^\alpha(\theta)$, $\theta\in[-\pi/a,\pi/a]$, $m\in\mathbb{Z}\setminus\{0\}$ are strictly increasing w.r. to α.

For $\alpha=0$ the eigenvalues are given by

$$E_{m\pm}^0(\theta)=\pm([\theta+2\mathrm{sgn}(m)||m|-1|\pi a^{-1}]^2c^2+(c^4/4))^{1/2},\ m\in\mathbb{Z}\setminus\{0\},\ \theta\in[-\pi/a,\pi/a]. \tag{2.21}$$

They are only degenerate for $\theta=-b/2$, $m\in\mathbb{Z}\setminus\{0\}$ and for $\Theta=0$, $|m|\geq 2$.

See [3] for a detailed proof of this theorem.

Since

$$[1\otimes\sigma_2]D_\alpha(\theta)[1\otimes\sigma_2]^{-1}=-T_{\alpha/c^2}(\theta),\quad -\infty<\alpha<\infty,\quad \theta\in[-\pi/a,\pi/a] \tag{2.22}$$

(cf. (2.4)) the corresponding spectral properties of $T_\beta(\theta)$ also follow from Theorem 2.1. Applying (2.12) we then get

<u>Theorem 2.2.</u> Let $\alpha\in\mathbb{R}$. Then $D_{\alpha,a\mathbb{Z}}$ has purely absolutely continuous spectrum

$$\sigma(D_{\alpha,a\mathbb{Z}})=\sigma_{ac}(D_{\alpha,a\mathbb{Z}})=\bigcup_{m\in\mathbb{Z}\setminus\{0\}}[a_m^\alpha,\ b_m^\alpha],\quad a_m^\alpha<b_m^\alpha<a_{m+1}^\alpha,\ a_{-m}^\alpha>b_{-m}^\alpha>a_{-(m+1)}^\alpha,\ m\in\mathbb{N},$$

$$\sigma_{sc}(D_{\alpha,a\mathbb{Z}})=\sigma_p(D_{\alpha,a\mathbb{Z}})=\emptyset. \tag{2.23}$$

Here for $\alpha>0$

$$a_m^\alpha=\begin{cases}E_m^\alpha(0), & m\text{ odd}\\ E_m^\alpha(-\pi/a), & m\text{ even},\end{cases}$$

$$a_m^\alpha>[(m-1)^2\pi^2c^2a^{-2}+(c^4/4)]^{1/2},\quad m\in\mathbb{N}, \tag{2.24}$$

$$b_m^\alpha=\begin{cases}E_m^\alpha(-\pi/a)=[(m^2\pi^2c^2/a^2)+(c^4/4)]^{1/2}, & m\text{ odd}\\ E_m^\alpha(0)=[(m^2\pi^2c^2/a^2)+(c^4/4)]^{1/2}, & m\text{ even},\ m\in\mathbb{N},\end{cases}$$

$$a_{-m}^\alpha=\begin{cases}E_{-m}^\alpha(0)=-[(m-1)^2\pi^2c^2a^{-2}+(c^4/4)]^{1/2}, & m\text{ odd}\\ E_{-m}^\alpha(-\pi/a)=-[(m-1)^2\pi^2c^2a^{-2}+(c^4/4)]^{1/2}, & m\text{ even},\ m\in\mathbb{N},\end{cases}$$

$$b_{-m}^\alpha=\begin{cases}E_{-m}^\alpha(-\pi/a), & m\text{ odd}\\ E_{-m}^\alpha(0), & m\text{ even},\end{cases} \tag{2.25}$$

$$b_{-m}^\alpha<-[(m-1)^2\pi^2c^2a^{-2}+(c^4/4)]^{1/2},\quad m\in\mathbb{N}.$$

For $\alpha<0$

$$-c^2/2<a_1^\alpha<c^2/2,\ \alpha<0,\quad a_1^\alpha=-c^2/2,\ \alpha<-ac^2,\quad a_1^{-(2c)\tanh(ac/4)}=0,$$

$$a_m^\alpha=\begin{cases}E_m^\alpha(0)=[(m-1)^2\pi^2c^2a^{-2}+(c^4/4)]^{1/2}, & m\text{ odd}\\ E_m^\alpha(-\pi/a)=[(m-1)^2\pi^2c^2a^{-2}+(c^4/4)]^{1/2}, & m\text{ even},\ m=2,3,4,\ldots,\end{cases}$$

$$-c^2/2<b_1^\alpha<[(\pi^2c^2/a^2)+(c^4/4)]^{1/2}, \tag{2.26}$$

$$b_1^{-4/a}=c^2/2,\quad b_1^{-(2c)\coth(ac/4)}=0,\quad b_1^\alpha \xrightarrow[\alpha\to-\infty]{} -c^2/2,$$

$$b_m^\alpha=\begin{cases}E_m^\alpha(-\pi/a), & m\text{ odd}\\ E_m^\alpha(0) & m\text{ even},\quad m\in\mathbb{N},\end{cases}$$

$$b_m^\alpha>[(m-1)^2\pi^2c^2a^{-2}+(c^4/4)]^{1/2},\quad m=2,3,4,\ldots,$$

$$a_{-1}^\alpha\begin{cases}=-c^2/2, & -ac^2<\alpha<0\\ <-c^2/2, & \alpha<-ac^2,\end{cases}$$

$$a_{-m}^\alpha=\begin{cases}E_{-m}^\alpha(0), & m\text{ odd}\\ E_{-m}^\alpha(-\pi/a), & m\text{ even},\end{cases} \tag{2.27}$$

$$a_{-m}^\alpha>-[(m^2\pi^2c^2/a^2)+(c^4/4)]^{1/2},\quad m\in\mathbb{N},$$

$$b_{-m}^\alpha=\begin{cases}E_{-m}^\alpha(-\pi/a)=-[(m^2\pi^2c^2/a^2)+(c^4/4)]^{1/2}, & m\text{ odd}\\ E_{-m}^\alpha(0)=-[(m^2\pi^2c^2/a^2)+(c^4/4)]^{1/2}, & m\text{ even},\ m\in\mathbb{N}.\end{cases}$$

As $|m|\to\infty$, the length of the m-th gap respectively the width of the m-th band asymptotically fulfills

$$|a_{m+1}^\alpha-b_m^\alpha|\underset{m\to\infty}{=}|b_{-m}^\alpha-a_{-(m+1)}^\alpha|\underset{m\to\infty}{=}(2c/a)\arctan(|\alpha|/2c)+O(m^{-1}),$$
$$|b_m^\alpha-a_m^\alpha|\underset{m\to\infty}{=}|a_{-m}^\alpha-b_{-m}^\alpha|\underset{m\to\infty}{=}(c/a)[\pi-2\arctan(|\alpha|/2c)]+O(m^{-1}). \tag{2.28}$$

For $\alpha\in\mathbb{R}\setminus\{0\}$, $D_{\alpha,a\mathbb{Z}}$ has infinitely many gaps in its spectrum. Except for $\alpha=-ac^2$, all possible gaps do actually occur. Only for $\alpha=-ac^2$ one gap closes at $E=-c^2/2$ since $D_{-ac^2}(0)$ has $-c^2/2$ as an eigenvalue of multiplicity two. For $\alpha=0$, $D_{0,a\mathbb{Z}}$ equals the free Dirac operator D_0 with spectrum

$$\sigma(D_0)=(-\infty,-c^2/2]\cup[c^2/2,\infty) \tag{2.29}$$

and due to the degeneracy of $E_m^0(0)$, $|m|\geq 2$ and $E_m^0(-\pi/a)$, $m\in\mathbb{Z}\setminus\{0\}$ all gaps in $\mathbb{R}\setminus(-c^2/2,c^2/2)$ close. Furthermore

$$\begin{aligned}&\sigma(D_{\alpha,a\mathbb{Z}})\subset\sigma(D_{\alpha',a\mathbb{Z}}),\quad 0<\alpha'<\alpha,\\ &\sigma(D_{\alpha,a\mathbb{Z}})\supset\sigma(D_{\alpha',a\mathbb{Z}}),\quad -\infty<\alpha'<\alpha<-ac^2.\end{aligned} \tag{2.30}$$

Since (2.4) obviously applies in the present special case of periodicity, the spectrum of $T_{\beta,a\mathbb{Z}}$ immediately follows from Theorem 2.2.

Following [8] one easily proves that $(D_{\alpha,a\mathbb{Z}}-(c^2/2)-z)^{-1}$ and $(T_{\beta,a\mathbb{Z}}-(c^2/2)-z)^{-1}$, $z\in\mathbb{C}\setminus\mathbb{R}$ are holomorphic with respect to c^{-1} in norm and that

$$\begin{aligned}&\text{n-}\lim_{c\to\infty}(D_{\alpha,a\mathbb{Z}}-(c^2/2)-z)^{-1}=(-\Delta_{\alpha,a\mathbb{Z}}-z)^{-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \alpha\in\mathbb{R},\\ &\text{n-}\lim_{c\to\infty}(T_{\beta,a\mathbb{Z}}-(c^2/2)-z)^{-1}=(\Xi_{\beta,a\mathbb{Z}}-z)^{-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \beta\in\mathbb{R}.\end{aligned} \tag{2.31}$$

Moreover, first order relativistic corrections of energy bands with respect to c^{-2} can be explicitly computed since $E_m^\alpha(\theta)-(c^2/2)$, $\tilde{E}_m^\beta(\theta)-(c^2/2)$, $\theta\in[-\pi/a,\pi/a)$, $m\in\mathbb{N}$ turn out to be holomorphic in c^{-2} around their nonrelativistic limits. In particular the discriminant (2.14) and its analog for $T_\beta(\theta)$

$$\cos(\theta a)=\cos(ka)-[\beta c/2\xi(k)]\sin(ka),\ \mathrm{Re}k>0,\ \mathrm{Im}k>0 \tag{2.32}$$

are easily seen to reproduce their nonrelativistic counterparts [3], [18]

$$\cos(\theta a)=\cos(ka)+(\alpha/2k)\sin(ka),\ \mathrm{Re}k>0,\ \mathrm{Im}k>0 \tag{2.33}$$

for $-\Delta_\alpha(\theta)$ and

$$\cos(\theta a)=\cos(ka)-(\beta k/2)\sin(ka),\ \mathrm{Re}k>0,\ \mathrm{Im}k>0 \tag{2.34}$$

for $\Xi_\beta(\theta)$ as $c\to\infty$.

We end up with a few more remarks. First of all, following [3], [18], one can easily extend the above case to more complex periodic situations where, e.g., $\gamma_{j+p}=\gamma_j$ for some $p\in\mathbb{N}$ (here γ denotes α or β) and compute the density of states explicitly. Also half-crystals ($\gamma_j=\gamma_+$, $j>0$, $\gamma_j=\gamma_-$, $j<-1$), surface states and scattering off defects or impurities can be developed in analogy to [2], [18], [32], [33], [3] (and the extensive reference list therein). In all of these cases one can use the underlying difference equation most efficiently.

For analogous results for the model with boundary conditions (1.16) (with $Y=a\mathbb{Z}$) we refer to [6], [7], [9], [10], [13], [19], [34]-[36], [38]-[43].

3. Relativistic random point interactions

Let γ_j, $j\in\mathbb{Z}$ be independent, identically distributed (i.i.d.) real-valued random variables on the canonical probability space $(\Omega,\mathcal{F},P)$ where $\mathrm{supp}(P_{\gamma_0})$ is assumed to be compact and $\Omega=[\mathrm{supp}(P_{\gamma_0})]^{\mathbb{Z}}$. Thus any $\omega\in\Omega$ is given by $\omega=\prod_{j\in\mathbb{Z}}\omega_j$ and $\omega_j=\gamma_j(\omega)$, $j\in\mathbb{Z}$ in this representation. In all the following results $D_{\alpha(\omega),a\mathbb{Z}}$ and $T_{\beta(\omega),a\mathbb{Z}}$ with $\alpha(\omega)=\{\alpha_j(\omega)\}_{j\in\mathbb{Z}}$, $\beta(\omega)=\{\beta_j(\omega)\}_{j\in\mathbb{Z}}$ and α_j,β_j, $j\in\mathbb{Z}$ i.i.d. real-valued random variables on $(\Omega,\mathcal{F},P)$ can be treated on exactly the same footing. Thus we introduce the following unifying notation: The operator H_ω in $L^2(\mathbb{R})$ by definition represents $D_{\alpha(\omega),a\mathbb{Z}}$ or $T_{\beta(\omega),a\mathbb{Z}}$ in particular $\omega_j=\gamma_j(\omega)$ now plays the role of $\alpha_j(\omega)$ or $\beta_j(\omega)$, $j\in\mathbb{Z}$. The associated deterministic operators are abbreviated by the symbol $H_{\xi,a\mathbb{Z}}$.

Next let $\{T_j\}_{j\in\mathbb{Z}}$ be the shift operator in Ω defined by

$$(T_\ell\omega)_j=\omega_{j-\ell},\quad \omega\in\Omega,\quad j,\ell\in\mathbb{Z} \tag{3.1}$$

such that

$$\gamma_j(T_\ell\omega)=\gamma_{j-\ell}(\omega)=\omega_{j-\ell}\quad \omega\in\Omega,\ j,\ell\in\mathbb{Z}. \tag{3.2}$$

Then $\{T_j\}_{j\in\mathbb{Z}}$ is a family of measure preserving ergodic transformations [23]. Moreover let $\{U_j\}_{j\in\mathbb{Z}}$ denote the family of unitary translation operators in $L^2(\mathbb{R})$

$$(U_jg)(x)=g(x-ja),\quad g\in L^2(\mathbb{R}),\ j\in\mathbb{Z}. \tag{3.3}$$

Then

$$U_jH_\omega U_j^{-1}=H_{T_j\omega},\quad \omega\in\Omega,\ j\in\mathbb{Z} \tag{3.4}$$

and hence we get [22], [23]

Theorem 3.1. Let $\{H_\omega\}_{\omega\in\Omega}$ be defined as above. Then $\sigma(H_\omega)$, $\sigma_{ess}(H_\omega)$, $\sigma_c(H_\omega)$, *$\sigma_{ac}(H_\omega)$, $\sigma_{sc}(H_\omega)$ and $\overline{\sigma_p(H_\omega)}$ all equal certain non random sets Σ, Σ_{ess}, Σ_c,* Σ_{ac}, Σ_{sc} and Σ_p for P-a.e. $\omega\in\Omega$. Moreover $\sigma_d(H_\omega)=\emptyset$ for P-a.e. $\omega\in\Omega$. For any $\lambda\in\mathbb{R}$ there exists a subset $\Omega_\lambda\subseteq\Omega$ with $P(\Omega_\lambda)=1$ such that λ is no eigenvalue of H_ω, $\omega\in\Omega_\lambda$.

Next we introduce the concept of stochastic (resp. admissible) potentials: A sequence $\Phi(\omega)=\{\gamma_j(\omega)\}_{j\in\mathbb{Z}}$, $\omega\in\Omega$ with γ_j, $j\in\mathbb{Z}$ i.i.d. real-valued random variables and $\mathrm{supp}(P_{\gamma_0})$ compact is called a stochastic potential. The corresponding Hamiltonian is then defined by $H(\Phi(\omega))$. The class A of admissible potentials then consists of all $\Phi=\{\xi_j\in\mathrm{supp}(P_{\gamma_0})\}_{j\in\mathbb{Z}}$. The class P of all periodic admissible potentials is then given by all $\Phi\in A$ such that the corresponding sequence $\{\xi_j\}_{j\in\mathbb{Z}}$ satisfies

$$\xi_{j+L}=\xi_j,\quad j\in\mathbb{Z} \tag{3.5}$$

for some $L\in\mathbb{Z}\setminus\{0\}$. For $\Phi\in A$, $H(\Phi)$ denotes the Hamiltonian H_ω with $\gamma_j(\omega)$ replaced by ξ_j, $j\in\mathbb{Z}$ (i.e. $H(\Phi)=H_{\xi,a\mathbb{Z}}$). Then we have

Theorem 3.2. Let $\Phi(\omega)$ be the stochastic potential as defined above. Then

(i) $\sigma(H(\phi)) \subset \Sigma, \quad \phi \in A$. (3.6)

(ii) $\Sigma = \bigcup_{\phi \in A} \sigma(H(\phi)) = \overline{\bigcup_{\phi \in P} \sigma(H(\phi))}$. (3.7)

(iii) $\Sigma = \bigcup_{\xi \in \mathrm{supp}(P_{\gamma_0})} \sigma(H_{\xi, a\mathbb{Z}})$ (3.8)

where $\Sigma = \sigma(H(\Phi(\omega))$ for P-a.e. $\omega \in \Omega$.

For a proof of the above theorem one can follow [18] (cf. also [3], [4]) step by step. The main ideas involved originally appeared in [24], [25].

As a consequence of the above and of Theorem 2.2 we get .

Corollary 3.3. Assume the hypotheses of Theorem 3.2 and denote by $\mu = \inf[\mathrm{supp}(P_{\gamma_0})]$, $\nu = \sup[\mathrm{supp}(P_{\gamma_0})]$. Then we have

(i) $\Sigma = \bigcup_{m \in \mathbb{Z} \setminus \{0\}} [a_m, b_m]$ (3.9)

$a_m < b_m < a_{m+1}, a_{-m} > b_{-m} > a_{-(m+1)}, \; m \in \mathbb{N}, \quad a_m, b_m \xrightarrow[m \to \pm\infty]{} \pm\infty.$

(ii) $\sigma(H_\omega)$ has infinitely many open gaps for P-a.e. $\omega \in \Omega$ unless $0 \in \mathrm{supp}(P_{\gamma_0})$. If $0 \in \mathrm{supp}(P_{\gamma_0})$ then

$(-\infty, -c^2/2] \cup [c^2/2, \infty) \subseteq \Sigma.$ (3.10)

ii) If $\mu > 0$ then

$\Sigma = \sigma(D_{\mu, a\mathbb{Z}})$ (3.11)

and if $\nu < -ac^2$ then

$\Sigma = \sigma(D_{\nu, a\mathbb{Z}})$

in the case where $H_\omega = D_{\omega, a\mathbb{Z}}$ represents relativistic δ-interactions.

By (2.4) a similar result holds for $T_{\omega, a\mathbb{Z}}$.

On the basis of the results in [11], [12] it is natural to conjecture that expontential localization of the spectrum occurs (and hence $\Sigma_c = \emptyset$) under mild assumptions on the density of γ_0. For conditions implying $\Sigma_{ac} = \emptyset$ cf. [21], [28].

Finally we show that the well-known Saxon and Hutner conjecture [32] concerning gaps in the spectra of (nonrelativistic) random δ-interactions extends to the relativistic case. We start with a deterministic result.

Lemma 3.4. Let $\xi = \{\xi_j \in \mathbb{R}\}_{j \in \mathbb{Z}}$ be a bounded sequence. Let

$\Gamma \subseteq \bigcap_{j \in \mathbb{Z}} \rho(H_{\xi, a\mathbb{Z}})$ be open. Then $\Gamma \subseteq \rho(H_{\xi, a\mathbb{Z}})$.

Since the difference equation for $H_{\xi,a\mathbb{Z}}$ is of the same type as that of $-\Delta_{\xi,a\mathbb{Z}}$ (or $\Xi_{\xi,a\mathbb{Z}}$) the proof of Theorems 3.3 and 3.6 of [18] apply in the present case.

We also note that in the special case where $\xi_{j+2}=\xi_j$, $j\in\mathbb{Z}$ the analog of Lemma 3.4 for the model with boundary conditions (1.16) ($Y = a\mathbb{Z}$) has been derived in [41] by following the original proof in connection with the Schrödinger equation in [27].

Given the above lemma, Theorem 3.2 (iii) implies a proof of the relativistic Saxon and Hutner conjecture

Theorem 3.5. Assume the hypotheses of Theorem 3.2. Let
$\Gamma\subseteq \bigcap_{\lambda\in \text{supp}(P_{\gamma_0})} \rho(H_{\lambda,a\mathbb{Z}})$ be open. Then $\Gamma\cap\Sigma=\emptyset$.

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Новые аналитически рещаемые модели релятивистского взаимодействия нулевого радиуса

Предлагаются две новые аналитически решаемые релятивистские модели взаимодействия нулевого радиуса. Изучается их спектральное поведение в случае конечного и бесконечного числа центров.

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New Analytically Solvable Models of Relativistic Point Interactions

Two new analytically solvable models of relativistic point interactions in one dimension (being natural extensions of the nonrelativistic δ-resp, δ'-interaction) are considered. Their spectral properties in the case of finitely many point interactions as well as in the periodic case are fully analyzed. Moreover we explicitly determine the spectrum in the case of independent, identically distributed random coupling constants and derive the analog of the Saxon and Hutner conjecture concerning gaps in the energy spectrum of such systems.

The investigation has beem performed at the Laboratory of Theoretical Physics, JINR.

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