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CAUSAL BOUNDARY FOR STABLY CAUSAL SPACE-TIMES

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I. Rácz: Causal boundary for stably causal space-times. KFKI-1987-77/B

ABSTRACT

An explicit identification rule is defined for stably causal space-times on the set of the ideal points of the space-time. Furthermore, tne properties of the extended Alexandrov topology are examined.

И. Рац: Граница причинности для стабильно причинного пространства-времени. KFKI-1987-77/B

АННОТАЦИЯ

Дано явное правило идентификации стабильно причинного пространства-времени во множестве идеальных точек пространства-времени. Кроме этого исследуется, какими свойствами обладает расширенная топология Александрова.

Rácz I.: Kauzális határkonstrukció stabilan kauzális téridőkre. KFKI-1987-77/B

KIVONAT

Stabilan kauzális téridőkre egy explicit azonositási szabályt adunk meg a téridő ideális pontjainak halmazán. Ezenfelül megvizsgáljuk milyen tulajdonságokkal rendelkezik a kiterjesztett Alexandrov-topológia.

1. INTRODUCTION:

It has been shown in [1] that a very wide class of the boundary constructions for space-times in general relativity yields an unsatisfactory boundary even for causally well -behaved space-times. Although this proceeding left the Geroch-Kronheimer-Penrose (GKP) construction [2] intact, the 'singular' portion of the causal boundary of Taub's plane-symmetric static vacuum space-time as found by [3], is a single point and not, as one would expect, a one-dimensional set. In fact, the authors of [3] state that it might not be fruitful to describe the structure of the singularities by using the GKP construction. Nevertheless, we should like to save the causal boundary construction partly because it has the advantage of being comparatively simple. Moreover as it is shown in [4] there exists a viable modification of the GKP construction which yields a satisfactory causal boundary wherever the GKP does it and is free of the defects in the Taub's space-time.

Let us first collect the standard definitions and results. The aim is to describe the structure of the singularities. So in practice an appropriate boundary construction assigns a topological space $\overline{M}$ with an embedding $\theta: M \rightarrow \overline{M}$ of the space-time manifold, M, into $\overline{M}$, such that $\theta[M]$ is an open, dense topological subspace in $\overline{M}$. Then the points of $\overline{M} \setminus \theta[M]$ represent the set of the 'boundary' points while the topology of $\overline{M}$ tells us when a point sequence in $\theta[M]$ approaches a boundary point [1].

The minimal requirements that one would expect are the following [5]:

(a) The set of the boundary points is to be determined by the geometry of the space-time.

(b) If the space-time is extendible through a part of its boundary this part should coincide with the boundary defined by the extension.

On the basis of singularity theorems we expect that the boundary points are represented by 'ideal' endpoints of b-incomplete, inextendible causal curves [6]. If we use only the conformally invariant causal structure for constructing 'ideal' endpoints to the inextendible causal curves the resulting set represents not only the singular portion of the boundary but the boundary at 'infinity' as well. Thus in order to describe the structure of the singularities by using the causal boundary of the space-time one has to find a distinguishing rule to divide the set of boundary points into two disjoint subsets. (We shall not consider this problem here, as it is not necessary at this stage at all.)

In the causal boundary constructions the ideal endpoints are represented by the chronological pasts or futures of the inextendible causal curves. Thus two past (resp. future) inextendible causal curves have the same past (resp. future) ideal endpoint if their chronological futures (resp. pasts) are equal. When the distinguishing causality condition holds on the space-time, the set $\hat{M}$ of indecomposable past sets

(IP's) (resp. the set $\check{M}$ of indecomposable future sets (IF's)) represents both the points of the space-time and the future boundary (resp. the past boundary). It is simple to construct $\hat{M}$ and $\check{M}$. Furthermore there exist past and future endpoints in $\hat{M} \cup \check{M}$ to every causal curve in the sense that $\hat{M}$ (resp. $\check{M}$) is future- (resp. past-) complete [2]. But we have to define some kind of identifications on $\hat{M} \cup \check{M}$. All of the space-time events are doubly represented in $\hat{M} \cup \check{M}$ by their chronological future and past. Moreover, some further identifications may also be required between the 'ideal' points to get an appropriate boundary [3,7]. The topology is an enlargement of the Alexandrov topology of the space-time, which coincides with the manifold topology if and only if the strong causality condition holds on M. (Unfortunately the Alexandrov topology has many widely different extensions.) The fact is that the topology and the identifications are obtained simultaneously in the GKP construction (when it does exist, see Ref. [7]). (For causally continuous space-times the Budic-Sachs (BS) construction [8] gives another conformally invariant way of attaching a boundary to the space-time. Although there first the identification rule is defined and then the topology, but the BS construction does not satisfy the condition (b). Thus it is not unexpected that topologies of the GKP and BS constructions do not coincide with each other, not even for causally continuous space-times. Furthermore as is shown in

[3], the identification maps of these two constructions also differ.)

In this paper a viable modification of the GKP construction is given. In Section 2. we give an explicit identification rule on the set of ideal points of the space-times.This identification coincides with the identification rule generated by the null pair equivalence, for causally continuous space-times. Then in Section 3. we analyse the properties of the extended Alexandrov topology for stably causal space-times. It is shown that each causal curve has a unique endpoint in the extended Alexandrov topology whenever the given equivalence relation is finite.

Unless otherwise noted the conventions and notations of this paper are the same as those used in [6].

2. IDENTIFICATION :

In this section we give an identification rule on $\hat{M} \cup \check{M}$, which seems to yield a satisfactory boundary point set structure for physically realistic (stably causal) space-times. The GKP construction and its modification (see in Ref. [4]) give an appropriate boundary structure for the well-known space-times. However, owing to the implicit identification rule, they do not solve all of the difficulties. Let, for example, the space-time M be the subset of the three-dimensional Minkowski space-time with the inextendible null geodesic γ cutting out. This space-time is causally continuous. Then the open subsets $I^{\pm}(p)$ of the three-dimensional Minkowski space, where $p \in \tau$, are TIF and TIP in our space-time and are T_2 separated (see for example the following open sets in $M^{\#}$ or $\hat{M} \cup \check{M}$: $\{I^-[\tau']\}^{ext}$ and $\{I^+[\tau']\}^{ext}$, where τ' is an inextendible null geodesic in our space-time which has the same past and future τ has in the original Minkowski space). Thus neither the GKP construction nor its modification identifies this TIP with this TIF, so they do not satisfy the condition (b).

Now we define our identification rule in two steps. First we give an identification rule between IP's (resp. IF's) and then between IP's and IF's. Some identification may be required between IP's (resp. IF's) which is indicated by Fig.2. in Ref. [7]. There might be two (or more) TIP or TIF which are different but one would expect in the point set structure of the causal boundary of the space-time that they represent the same point. (We shall omit the duals of the definitions and statements.)

Let P_1 and P_2 be IP's. Then P_1 and P_2 are *equivalent* to each other ($P_1 \sim P_2$) if there exist for arbitrary $p \in P_1 \cup P_2$ generators S_1 and S_2 of P_1 and P_2, resp., such that $I^+[S_1] \cup I^+[S_2] \subset I^+(p)$ ($S \subset M$ is a generator of P if $P = I^-[S]$). It seems to be reasonable to identify these IP's because for arbitrary timelike curve generators τ_1 and τ_2 of P_1 and P_2, resp., the observers belonging to these curves can communicate to each other after having slightly perturbed the space-time metric. Note that the relation "$\sim$" is reflexive and symmetrical but not an equivalence relation yet. The minimal requirement for an appropriate identification relation on $\hat{M}$ is that P and $I^-(p)$ ($p \in M$) be identified iff $P = I^-(p)$, so it is useful to prove the following proposition:

Proposition 2.1.: If the stable causality condition holds on the space-time then the past sets $I^-(p)$ ($p \in M$) and P ($\in \hat{M}$) are equivalent iff $P = I^-(p)$.

Proof: Suppose on the contrary that $P \in M$ and $I^-(p)$ are such that $P \sim I^-(p)$ but $P \neq I^-(p)$ and let $\{s_i\}$ be a future chronological sequence (i.e. $s_{i+1} \gg s_i$ for all $i \in N$) such that $P = I^-[\{s_i\}]$. Furthermore $\{p_i\}$ be a past chronological sequence (i.e. $p_{i+1} \ll p_i$ for all $i \in N$) with p being the limit point of $\{p_i\}$. Since $P \sim I^-(p)$ for arbitrary s_i we have $I^+(s_i) \supset I^+(p)$, so there exists for all $i \in N$ a future inextendible timelike curve γ_i from s_i through p_i. Let γ denote the limit curve of the sequence $\{\gamma_i\}$ through p and let $q \in \{J^-(p) \cap \gamma\} \setminus \{p\}$. (Such a $q \in M$ exists if p is not the limit point of the sequence $\{s_i\}$, i.e. when $P \neq I^-(p)$.) Now let $\bar{g}$ be an arbitrary but fixed Lorentzian metric on M larger than g ($\bar{g} > g$); (i.e. $g(V,V) \leq 0$ implies for every non-zero V ($\in T_p$) that $\bar{g}(V,V) < 0$ for any $p \in M$). Since $q \in J^-(p)$, there exists $r \in I^-(p)$: $r \in J^+(q,\bar{g})$. Owing to $P \sim I^-(p)$ so there exists a generator S_r of P such that $I^+[S_r] \subset I^+(r)$. Thus for arbitrary $s \in S_r$, $s \in J^+(r,\bar{g})$ and as S_r is a generator of P there exists $n \in N$: $s_i \in J^+(s,\bar{g})$ for $i > n$. Since q is a limit point of the sequence $\{\gamma_i\}$, we have $q \in$

$J^+(s,\bar{g})$. Thus there exist closed causal curves in M for $\bar{g} > g$ and because $\bar{g}$ was arbitrary the stable causality condition does not hold on M. ∎

When the stable causality condition does not hold on the space-time the relation $\sim$ might identify PIP's and TIP's , see for example the sets P and $I^-(q)$ (which are by our definition equivalent) of Fig.1. in Ref.[7].

Now we define the identification rule between IP's and IF's. Let $\hat{\mathcal{S}}$ be a subset in $\hat{M}$. Then $P \in \hat{\mathcal{S}}$ is called an *almost maximal element* in $\hat{\mathcal{S}}$ if there is no $P' \in \hat{\mathcal{S}}$ such that P' contains P as a proper subset and $P' \not\sim P$. ($P \in \hat{\mathcal{S}}$ is *maximal element* in $\hat{\mathcal{S}}$ if there is no $P' \in \hat{\mathcal{S}}$ such that P' contains P as a proper subset). Suppose that $F \in \check{M}$ such that $\downarrow F$ (the common past of F) is not empty. We should like to find the IP's which are close to F in some sense. Let S be an arbitrary generator of F and $\hat{\mathcal{S}}(\downarrow S)$ the subset of $\hat{M}$ whose elements are included in $\downarrow S$. We denote by $\mathfrak{P}(S_F)$ the set of IP's which are almost maximal elements of $\hat{\mathcal{S}}(\downarrow S_F)$ (where S_F is a generator of F). Now we define the identification rule as follows: Let $P \in \hat{M}$ and $F \in \check{M}$, then P and F are *equivalent* to each other ($P \approx F$) if $P \in \mathfrak{P}(S_F)$ and $F \in \mathfrak{F}(S_P)$ for some generators S_F and S_P of F and P, respectively, as well as $I^+(s_P) \cap I^-(s_F) \neq \emptyset$ for arbitrary $s_P \in S_P$ and $s_F \in S_F$. (The last part of the this definition is necessary for assuring that the identified elements, $F \in \check{M}$ and $P \in \hat{M}$, be really close to each other.) The minimal requirement for an appropriate identification is that $P = I^-(p)$ and $F \in \check{M}$ be identified iff $F = I^+(p)$.

Proposition 2.2.: If the stable causality condition holds on the space-time then $P \approx I^+(p)$ $(p \in M)$ if and only if $P = I^-(p)$.

Proof: As $S_F = \{r_i\}$ where $\{r_i\}$ is a past chronological sequence (i.e. $r_{i+1} \ll r_i$ for all $i \in \mathbf{N}$ such that p is the limit point of $\{r_i\}$) is a generator of $I^+(p)$, $S_P = \{q_i\}$ where $\{q_i\}$ is a future chronological sequence (i.e. $q_{i+1} \gg q_i$ for all $i \in \mathbf{N}$) such that p is the limit point of $\{q_i\}$ is a generator of $I^-(p)$ and $I^-(p)$ is a maximal element in $\hat{\mathcal{Y}}(\downarrow\{r_i\})$ and $I^+(p)$ is a maximal element in $\check{\mathcal{Y}}(\uparrow\{q_i\})$ when the strong causality condition holds on M, furthermore $I^+(s_P) \cap I^-(s_F) \neq 0$ for arbitrary $s_P \in S_P$ and $s_F \in S_F$, so $I^+(p)$ and $I^-(p)$ are equivalent for arbitrary $p \in M$.

How suppose, on the contrary, that $P \in \hat{M}$ such that $P \neq I^-(p)$ and $P \approx I^+(p)$. Then $I^-(p) \subset \downarrow S$ for every generator S of $I^+(p)$ and either $P \supset I^-(p)$ or $I^-(p) \not\subset P$ holds. Suppose first that $P \supset I^-(p)$. Then $p \in \bar{P}$ (where $\bar{P}$ denotes the closure of P in the manifold topology) so the set $\{\bar{P} \cap J^+(p)\} \setminus \{p\} \neq \emptyset$. Let $q \in \{\bar{P} \cap J^+(p)\} \setminus \{p\}$, then $I^-(p) \subset P \subset \downarrow S \subset \downarrow I^+(p)$ for some generator S of $I^+(p)$ since $P \approx I^+(p)$, so the strong causality condition does not hold on M.

Suppose now that $P \not\supset I^-(p)$. Let $\{p_i\}$ be a past chronological sequence (i.e. $p_{i+1} \ll p_i$ for all $i \in \mathbf{N}$) such that p is the limit point of $\{p_i\}$; furthermore let S be the generator of P for which $I^+(p)$ is almost maximal in $\check{\mathcal{F}}(\uparrow S)$ and $\{s_i\}$ be a point sequence in M such that $I^-[\{s_i\}] = P$ and $I^-(s_{i+1}) \supset I^-(s_i)$ for all $i \in \mathbb{N}$. As $I^+(p) \in \mathcal{F}(S)$ there exist, γ_i, past inextendible timelike curves from p_i through s_i. Denote by γ the past inextendible non-spacelike limit curve of the sequence $\{\gamma_i\}$ through p (see lemma 6.2.1. in Ref. [6]). Then for an arbitrary $q \in \gamma \cap J^-(p)\setminus\{p\}$ we have $I^+(p) \subset I^+(q)$ which by (proposition 2.1.) implies that $I^+(p)$ is not in $\mathcal{F}(S)$. ∎

Although the relation $\approx$ is not an equivalence relation we still should like to unify the above two relations in a single equivalence relation. To do this, let R be *the smallest equivalence relation* on $\hat{M} \cup \check{M}$ such that R includes all of the ordered pairs the elements of which are identified by any one of the relations "$\sim$" or "$\approx$". Note that, by the proposition 2.1. and 2.2., R is for stably causal space-times a disjoint union of two equivalence relations R_i and R_b ($R = R_i \cup R_b$), where R_i is the equivalence relation which identifies each PIP with the PIF generated by the same point of the space-time. Also, R_b is the equivalence relation that identifies the TIP's and TIF's which are close to each other.

It is not too hard to see that our identification of IP's and IF's is a generalization of the "maximal naked counterpart" identification (see in Ref. [7]). When we use the word "maximal" instead of "almost maximal" and only one generator $S = P$ (or $S = F$) in our definition it coincides with the maximal naked counterpart identification.

In a certain sense the equivalence relation R is a generalization of the "hull pair" equivalence (see in Ref. [8]). To see this we prove the following simple lemma:

Lemma 2.1.: If the space-time is causally continuous and $F \in \check{M}$, then $\downarrow F = \downarrow S$ holds for every generator S of F.

Proof: First we prove that for causally continuous space-times $\downarrow F = \downarrow \bar{F}$. Let $p \in \bar{F}$. Then $I^+(p) \subset F$, which implies that $\downarrow F \subset \downarrow I^+(p)$. Because of $\downarrow \bar{F} := \mathrm{int}\{\cap_{p \in \bar{F}} I^-(p)\}$ and $I^-(p) = \downarrow I^+(p)$ for causally continuous space-times $\downarrow \bar{F} \supset \downarrow F$ holds on M. As $F \subset \bar{F}$; the relation $\downarrow F \supset \downarrow \bar{F}$ holds as well. Hence $\downarrow F = \downarrow \bar{F}$.

Now let S be an arbitr generator of F. Then $S \subset \bar{F}$ holds and therefore by causal continuity $\downarrow S \supset \downarrow \bar{F} = \downarrow F$. Conversely, let $p \in \downarrow S$. Then for all $s \in S$, $p \ll s$ and because of $F = I^+[S]$ for all $q \in F$ there exist $s^* \in S$ such that $s^* \ll q$. Thus $p \ll q$ holds for all $q \in F$, which implies that $p \in \downarrow F$. ∎

Using this lemma one can prove the following proposition :

Proposition 2.3.: Let the space-time be causally continuous and (P, F) be a hull pair, where $P \in \hat{M}$ and $F \in \check{M}$. Then P and F are identified by the relation $\approx$.

Proof: By lemma 2.1. for causally continuous space-times $\downarrow S = \downarrow F$ (resp. $\uparrow S = \uparrow P$) for arbitrary S generator of F (resp. P), while (P,F) is a hull pair, i.e. $\downarrow F = P$ and $\uparrow P = F$. Trivially there exist generators S_F and S_P of F and P, respectively, (for example $S_F = F$ and $S_P = P$) such that $I^+(s_P) \cap I^-(s_F) \neq \emptyset$ for arbitrary $s_P \in S_P$ and $s_F \in S_F$, whence by using our definition $P \approx F$ holds. ∎

Since there exist causally continuous space-times for which the elements of a hull pair are not indecomposable, the converse of the previous statement fails even for causally continuous space-times.

3. TOPOLOGY :

Assuming that the reader is familiar with the GKP construction [2,6,7], in this Section first we give a brief review of the standard definitions and results we shall need, then we analyse the properties of the extended Alexandrov topology for stably causal space-times. (We assume throughout this section that the space-time is stably causal.)

The intermediate space $M^{\#}$ is introduced by taking the the union $\hat{M} \cup \check{M}$ and identifying each PIP with the PIF generated by the same point of the space-time, i.e. $I^+(p) \in \check{M}$ with $I^-(p) \in \hat{M}$. In other words, $M^{\#}$ is the quotient space $(\hat{M} \cup \check{M})/R_1$. Thus the points of $M^{\#}$ are subsets in $\hat{M} \cup \check{M}$ and include one or two point(s) depending on whether their element(s) is(are) terminal or proper set(s) in $\hat{M} \cup \check{M}$. Then the open sets of $M^{\#}$ are by definition the unions and the finite intersections of the sets of the form A^{int}, A^{ext}, B^{int} and B^{ext}, where $A \in \check{M}$ and $B \in \hat{M}$, while:

$$A^{int} := \{ P^{*} \in M^{\#} \mid P \in \hat{M} \text{ and } P \cap A \neq \emptyset \}$$

$$A^{ext} := \{ P^{*} \in M^{\#} \mid P \in \hat{M} \text{ and for all } S \text{ generator of } P \;\; I^{+}[S] \not\subseteq A \},$$

(where P^{*} denotes the equivalence class which includes $P \in \hat{M}$). B^{int} and B^{ext} are defined in a similar way, with past and future interchanged. Using this topology on $M^{\#}$ the map $i: M \to M^{\#}$, where $i(p) = \{I^{+}(p), I^{-}(p)\}$, is an open dense embedding of M into $M^{\#}$ (see Ref.[2,7]). However, our aim is to construct a space in which each PIP and PIF generated by the same point of the space-time are identified, just as the TIP's and TIF's which are close to each other are done. This is just the quotient space $\bar{M} = M^{\#}/R_b$. Denote by $\pi: M^{\#} \to \bar{M}$ the natural projection generated by R_b. The equivalence relation R_b is trivial on $i(M)$ (the space-time is stably causal). Thus, using the proposition 2.1. of Ref.[7], the map $\delta: M \to \bar{M}$, where $\delta = \pi \circ i$, is an open dense embedding of M into $\bar{M}$.

As to the well-known examples this construction gives, with the exception of Taub space-time [3], the usual causal completions, with the topology of $\bar{M}$ being of the Hausdorff type for these space-times. Nevertheless, on physical grounds it appears unnecessary to require the Hausdorff property for the extended Alexandrov topology. In fact, one can construct further special causal spaces for which the topology is not Hausdorff on $\bar{M}$ (see figure 1), but there is no reason why to make further identifications to get Hausdorff spaces. Then the following possible expectation immediately come to mind: those non-Hausdorff topologies are preferred in which each curve has no more than one endpoint. (Let $\gamma: [t_1, t_2\rangle \to M$ be a curve in M (where $[t_1, t_2\rangle \subset R$ and "$\rangle$" denote "]" or ")" according to the curve γ being extendible or not in M. Then $p \in \bar{M}$ is said to be an endpoint of the curve $\bar{\gamma} = \delta \circ \gamma$ if for arbitrary neighbourhood U_p of p there exist a $t_* \in [t_1, t_2\rangle$ such that $\bar{\gamma}([t_*, t_2\rangle) \subset U_p$.) However, for the topology of the extended Taub space-time this property does not hold either. It is clear that the curve $\bar{\gamma}$ has more than one endpoint in $\bar{M}$ if the endpoints of the curve $i \circ \gamma$ are not R-equivalent in $M^{\#}$. Now, using the notations of Ref.[3], we shall show that there exist curves in the Taub space-time such that their endpoints are not R-equivalent. Let, for example, the curve $\lambda_\tau^{c,\tilde{c}} (\subset M)$ be such that:

$$\lambda_\tau^{c,\tilde{c}} := \{ (t, z, x, y) \mid t = 1/2(c+\tilde{c}),\ z = 1/2(c-\tilde{c})+O(\tau),\ x = \tau,\ y = \tau \},$$

where $\tau \in [1, \infty)$, $O(\tau)\cdot\tau^2 \to 0$ as $\tau \to \infty$ and $O(\tau) > 0$.

Lemma 3.1.: Let $I^+(\gamma_-^c) \in \check{M}$ and $I^-(\gamma_+^{\tilde{c}}) \in \hat{M}$ such that $\tilde{c} < c$ and let $\lambda_\tau^{c,\tilde{c}}$ the curve defined above. Then $I^+(\gamma_-^c)^*$ and $I^-(\gamma_+^{\tilde{c}})^*$ are endpoints of the curve $i \circ \lambda_\tau^{c,\tilde{c}}$.

Proof: Using lemma 3. of Ref.[3], we have to show that there exists $\tau^{*}\in[1,\infty)$ such that $i\circ\lambda_{\tau}^{c,\tilde{c}}([\tau^{*},\infty)) \subset \{O_1 \cap O_2\} \cap \{O_3 \cap O_4\}$, where $O_1 = I^{-}(\gamma_{+}^{c+\alpha})^{int} \cap I^{-}(\gamma_{+}^{c})^{ext}$; $O_2 = \bigcap_{i=1}^{k} I^{-}(\gamma_{-}^{c-\alpha,a_i,b_i})^{ext}$; $O_3 = I^{+}(\gamma_{-}^{\tilde{c}-\tilde{\alpha}})^{int} \cap I^{+}(\gamma_{-}^{\tilde{c}})^{ext}$; $O_4 = \bigcap_{j=1}^{l} I^{+}(\gamma_{+}^{\tilde{c}+\tilde{\alpha},\tilde{a}_j,\tilde{b}_j})^{ext}$ and $\alpha > 0$; $a_1, \ldots, a_k$; $b_1, \ldots, b_k$; $\tilde{\alpha} > 0$; $\tilde{a}_1, \ldots, \tilde{a}_l$; $\tilde{b}_1, \ldots, \tilde{b}_l$ are arbitrary real numbers. By the definition of $\lambda_{\tau}^{c,\tilde{c}}$ one can easily see that for arbitrary α and $\tilde{\alpha}$ there exists a $\tau_1\in[1,\infty)$ such that $i\circ\lambda_{\tau}^{c,\tilde{c}}([\tau_1,\infty)) \subset O_1 \cap O_3$. Furthermore, all of the functions $r^{(i)}(\tau) := [(\tau - a_i)^2 + (\tau - b_i)^2]^{1/2}$ and $r^{(j)}(\tau) := [(\tau - \tilde{a}_j)^2 + (\tau - \tilde{b}_j)^2]^{1/2}$ increase for arbitrary a_i, b_i $(i=1,\ldots,k)$ and $\tilde{a}_j, \tilde{b}_j$ $(j=1,\ldots,l)$ if $\tau > \max\{a_i, b_i; \tilde{a}_j, \tilde{b}_j\}$ and are without bound. Hence, according to the proof of lemma 5. of Ref.[3], there exists $\tau_2\in[1,\infty)$ such that $i\circ\lambda_{\tau}^{c,\tilde{c}}([\tau_2,\infty)) \subset O_2 \cap O_4$. Setting τ^{*} to be $\max\{\tau_1,\tau_2\}$ one finally obtains that $i\circ\lambda_{\tau}^{c,\tilde{c}}([\tau^{*},\infty)) \subset \{O_1 \cap O_2\} \cap \{O_3 \cap O_4\}$. ∎

However, when we work with a causal boundary construction, where the boundary points are represented by equivalence classes of TIP's and TIF's, the minimal requirement, which is expected to hold, is that for each causal curve γ in M there be a unique endpoint of the curve $\bar{\gamma} = \delta\circ\gamma$ in $\bar{M}$, as it was proposed in Ref. [7]. It is also shown there that for each causal curve γ the curve $\delta\circ\gamma$ has a unique endpoint in $\bar{M}$ if and only if the equivalence classes are closed in $M^{\#}$ and the endpoints of $i\circ\gamma$ are R-equivalent.

Now we shall show that for stably causal space-times the endpoints of the curve $i\circ\gamma$ are R-equivalent.

Proposition 3.1.: Let K^* be an element in $M^\#$ such that there exists a future causal curve $\gamma: [t_1, t_2\rangle \to M$ in M for which K^* is an endpoint of the curve $i\circ\gamma$ in the topology of $M^\#$. Then K and $I^-[\gamma]$ are R-equivalent if the space -time is stably causal.

Proof: First suppose on the contrary that K and $I^-[\gamma]$ are not R-equivalent and $K \in \hat{M}$. Then K and $I^-[\gamma]$ are not identified by the relation $\sim$ either, whence by the definition of the relation $\sim$ there exist a $p \in K \cup I^-[\gamma]$ such that $K^* \in [I^+(p)]^{int}$ and $I^-[\gamma]^* \in [I^+(p)]^{ext}$ (or $I^-[\gamma]^* \in [I^+(p)]^{int}$ and $K^* \in [I^+(p)]^{ext}$). Consequently $I^-[\gamma]^*$ and K^* are T_2 separated in $M^\#$. This means that K^* cannot be an endpoint of the curve $i\circ\gamma$.

Now suppose that $K^* \in M^\# \setminus i(M)$ is such that $K \in \check{M}$. We shall show that K^* cannot be an endpoint of $i\circ\gamma$ if the space-time is stably causal. Suppose on the contrary that K^* is an endpoint of $i\circ\gamma$. Then for each $p \in K$ there exists a t_* $\in [t_1, t_2\rangle$ such that $i\circ\gamma([t_*, t_2\rangle) \subset [I^-(p)]^{int}$. Consequently $I^-[\gamma] \subset \downarrow F$ as the map $i: M \to M^\#$ is embedding. Furthermore K^* $\in [I^-[\gamma]]^{ext}$ because otherwise K could not be an endpoint of $i\circ\gamma$. Thus there exists a generator S (which is not a single point) of K such that $I^-[S] \subset I^-[\gamma]$. Since $I^-[S] \subset I^-[\gamma] \subset$ $\downarrow K$ we have that there do exist closed causal curves in M through arbitrary $s \in S$ and for arbitrary $\tilde{g} > g$. Therefore the stable causality condition does not hold on M.

Finally, if $K^* \in i(M)$ such that $K = I^+(p)$, then according to the condition of the prop.3.1. p ($\in M$) is the endpoint of the curve γ, whence $I^-[\gamma] = I^-(p)$. Thus $I^-[\gamma]$ and K are R-equivalent since the space-time is stably causal. ∎

Therefore, each causal curve has unique endpoint in the extended Alexandrov topology if the equivalence classes are closed. However, there exists a rather annoying example of a stably causal space-time (see Fig.1.) where the equivalence classes are not closed subsets in $M^\#$. For example, $P \in \hat{M}$ is a limit point of the subset $\pi^{-1}(\pi(F))$, but $P \not\approx F$. Nevertheless we have even a rather large class of space-times for which the equivalence classes are closed. The idea of this class was called into my mind by L.B.Szabados. This class consists of those space-times for which the equivalence relation R is finite; i.e. the equivalence classes include only finite number of elements. To see just how, it is worth proving the following proposition:

Proposition 3.2.: If the equivalence relation R is finite then the equivalence classes are closed in the topology of $M^\#$.

Proof: We have to show that for an arbitrary subset, $\{K_1^*, \ldots, K_n^* \mid (K_i, K_j) \in R$ for arbitrary $i, j = 1, \ldots, n\}$, of $M^\#$ and $L^* \in M^\# \setminus \{K_1^*, \ldots, K_n^*\}$ there exists an open set O such that $L^* \in O$ and $\{K_1^*, \ldots, K_n^*\} \subset M^\# \setminus O$. When there exists $p \in$

M such that $K_1^* = I^+(p)^*$ or $L^* = I^+(p)^*$ the proof is trivial because then the arbitrary K_i^*'s $(i=1,\dots,n)$ and L^*'s are T_1 separated in $M^\#$ (see proposition 3.2. in Ref.[7]) and the set $\mathcal{O} := \bigcap_{i=1}^{n} \mathcal{O}_i$, where $K_i^* \notin \mathcal{O}_i$ and $L^* \in \mathcal{O}_i$, is thus open and satisfies the above requirement. Now suppose that K_i and L are terminal sets such that $K_1,\dots,K_l; L \in \hat{M}$ and $K_{l+1},\dots,K_n \in \check{M}$. Then, by the definition of the topology on $M^\#$, K_j^* cannot be for arbitrary $j \in \{l+1,\dots,n\}$ in any open neighbourhood of L^*. Furthermore $L \notin \{K_1,\dots,K_n\}$, so by the definition of the relation $\sim$ L^* and K_i^* $(i=1,\dots,l)$ are T_2 related in $M^\#$; i.e. there exist open neighbourhoods $\mathcal{O}_i$ $(i=1,\dots,l)$ such that $L^* \in \mathcal{O}_i$ and $K_i^* \notin \mathcal{O}_i$. Consequently for the open set $\mathcal{O} = \bigcap_{i=1}^{l} \mathcal{O}_i$ we have $L^* \in \mathcal{O}$ and $\{K_1^*,\dots,K_n^*\} \not\subset \mathcal{O}$. ∎

For example, the equivalence classes for the Taub space-time consist of one or two elements (see Ref.[3]), so the causal curves have a unique endpoint in the extended Alexandrov topology.

4. CONCLUSION:

An explicit identification rule has been given on the set of the ideal points of the space-times, which seems to yield a satisfactory boundary point set structure for stably causal space-times. We have examined the topological properties of the resulting causal boundary construction. It has been shown that for stably causal space-times the causal completion defined here has the feature of each causal curve

γ in M the curve $\phi\circ\gamma$ having a unique endpoint in $\overline{M}$ if the equivalence relation R is finite. Since all of the well-known inextendible space-times have the very same property the causal curves are *expected to have for the physically* interesting space-times a unique endpoint in the extended Alexandrov topology.

We have described a construction by which one may attach boundary points to space-times. But, as it was noted in the first section, one has to separate these boundary points into two disjoint *sets to describe the structure of* the *singularities*. This exciting problem also deserves further investigation.

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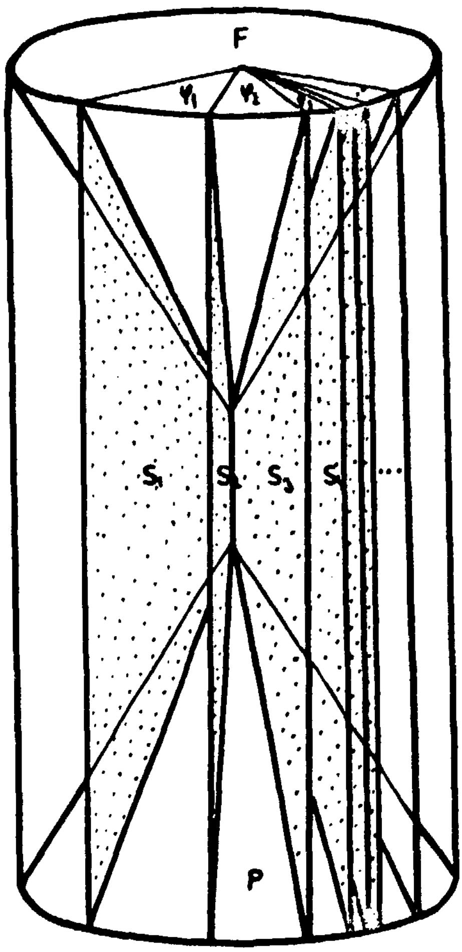

Figure 1.

M is represented by a subset of the three-dimensional Minkowski space-time. The dotted two-surfaces S_i ($i \in N$) and their limit surface are removed and the infinite sum of the angels φ_i ($i \in N$) is required to be smaller than $2\cdot\pi$. Then $(P,F) \notin R$ and they are not T_2 related in the topology of the intermediated space $M^\#$.

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