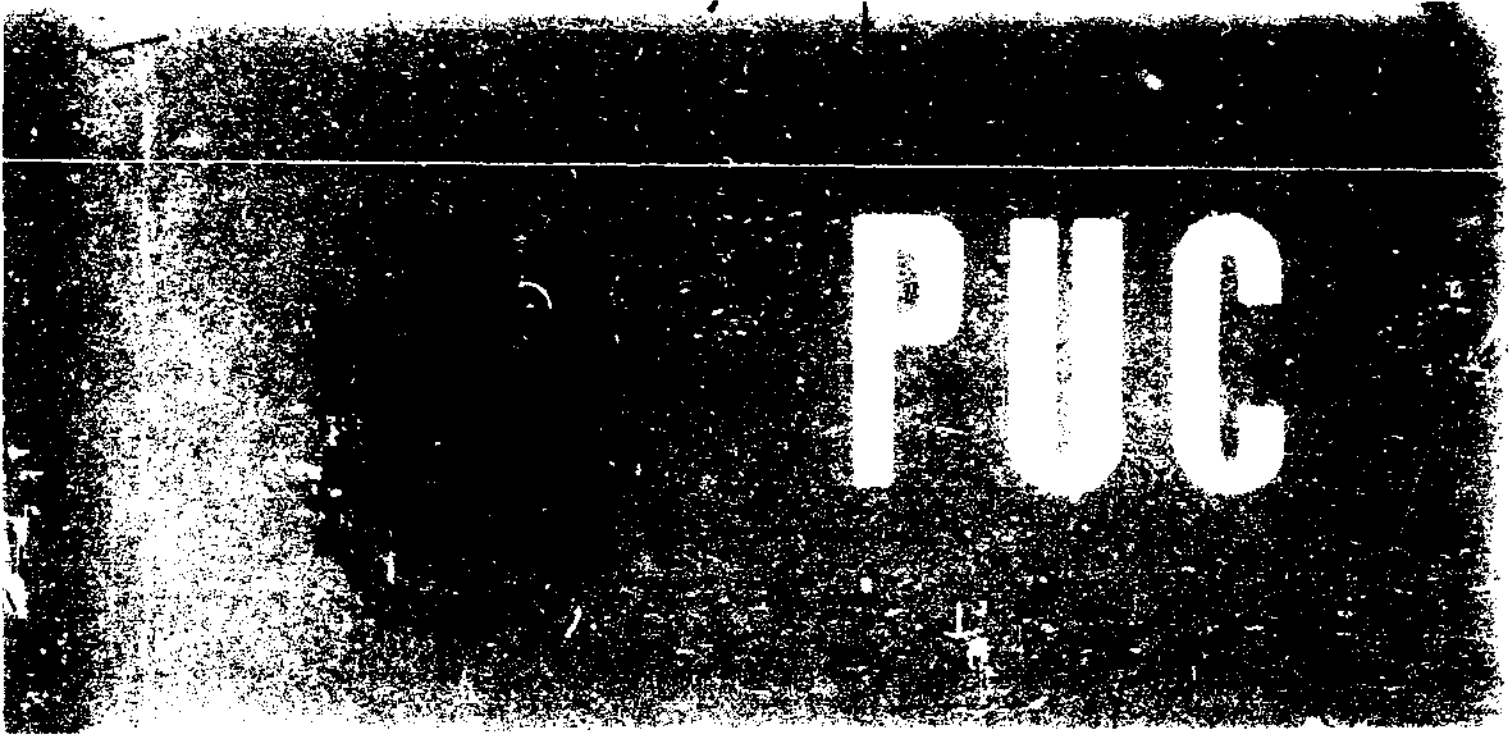

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BOOTSTRAP PERCOLATION: A RENORMALISATION GROUP APPROACH*

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ABSTRACT:. In bootstrap percolation, sites are occupied at random with probability p, but each site is considered active only if at least m of its neighbours are also active. Within an approximate position-space renormalization group framework on a square lattice we obtain the behaviour of the critical concentration p_c and of the critical exponents ν and β for m = 0 (ordinary percolation), 1,2 and 3. We find that the bootstrap percolation problem can be cast into different universality classes, characterized by the values of m.

RESUMO: Na percolação de "bootstrap", sítios são aleatoriamente ocupados com probabilidade p, mas cada sítio é considerado ativo somente se, pelo menos, m de seus primeiros viziuhos também estiverem ativos. Através de um esquema aproximado do Grupo de Renormalização no espaço de posições, para a rêde quadrada obtém-se o comportamento da concentração crítica p_c e dos expoentes críticos ν e β para m=0 (percolação ordinária), 1, 2 e 3. Concluímos que o problema de percolação de "bootstrap" pode ser classificado em diferentes classes de universalidade, caracterizadas pelos valores de m.

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The problem of bootstrap percolation was proposed by Chalupa, Leath and Reich (1979, hereafter referred to as CLR), as a model for dilute magnetic systems in which competition between crystal-field and exchange interactions plays a crucial role in the ultimate determination of magnetic properties. Consider a lattice built up of atoms characterised by Blume-Capel energy levels (Blume 1966, Capel 1966): a non-magnetic singlet ground state and a low-lying magnetic triplet first excited site. A site may become magnetically active due to Zeeman splitting, if acted upon by a sufficiently strong local (exchange) field created by its neighbouring magnetic sites (Kogut and Leath 1981). In a dilute system, however, such activation may not occur if some of the neighbouring sites are occupied by perfectly non-magnetic atoms.

This physical situation was modelled in a simplified way by CLR through a correlated percolation problem (for a review on percolation see e.g. Essam 1980): sites on a lattice are randomly occupied with probability p; occupied sites with less than m ($0 \leq m < z$, z = coordination number) occupied nearest neighbours are then "culled", i.e., considered inactive. This "culling" process is successively repeated until a stable configuration is achieved. This bootstrap percolation problem was solved exactly by CLR on Bethe lattices and very interesting new features were found, namely: (i) the percolation transition was found to be first order (discontinuity of the order parameter at the percolation threshold) for $m \geq 3$; and (ii) for $m=2$ the order parameter critical exponents is $\beta=2$, whereas it is 1 both for $m=0$ (ordinary percolation) and 1. Similar results were also obtained for certain branching media (Yaag and Zhang 1983). Recent computer simulations on real lattices (Kogut and Leath 1981) support the overall features found for Bethe lattices. First order transitions occur for $m \geq 3$ (square lattice) or $m \geq 4$ (triangular and simple

cubic lattices). When the transition is second-order, the exponent β is also found to be m-dependent.

The purpose of the present work is to discuss how the particular features of bootstrap percolation on a square lattice arise within a position-space renormalisation-group (PSRG) framework (Young and Stinchcombe 1975, Reynolds et al. 1980).

Let us first consider a b=3 scaling factor: the cell of Fig.1(a) (periodic boundary conditions assumed), in which each site is occupied at random with probability p, is renormalised into a single site with occupation probability p'. For m=0 (ordinary percolation) the cell configurations that contribute to p' are those in which a path of occupied sites "gets across" the cell along a given direction (say, vertically) and returns to the starting point (Fig.2). This criterion is chosen due to the small cells we use; it is an attempt to allow for the correlations present when m≠0. For a fixed non-zero m, the configurations that contribute to p' are those in which each site on the "percolating" path has at least m active neighbours. In this way, the recursion relations for b=3 are the following:

m = 0, 1, 2:

$$p' = p^9+9p^8q+36p^7q^2+63p^6q^3+45p^5q^4+18p^4q^5+3p^3q^6 \qquad (1)$$

m = 3:

$$p' = p^9+9p^8q+9p^7q^2+3p^6q^3 \qquad (2)$$

$$(q = 1 - p)$$

For b=4 (16 sites) the configurations are generated by a computer program; note that the cases m=0, 1 and 2 cannot be distinguished as far as percolating across the cell is concerned, for any cell size. This is because each site on a "percolating" path must have at least two active neighbours.

As in real (Wilson and Kogut 1974) the fixed point p^* of the recursion relation (1) or (2) is an approximation for the critical concentration p_c, and the eigenvalue $\lambda_p = (dp'/dp)_{p^*}$ of the linearised recursion relation is related to the correlation length exponent ν through

$$\nu = \ln b / \ln \lambda_p \tag{3}$$

In Table 1 we show our results for p_c and ν, obtained from cells of sizes $b=3$ and 4. Note that p_c and ν are the same for $m=0,1$ and 2 since the recursion relation for p' is the same.

In order to calculate "magnetic" exponents, we introduce a "ghost" site (Kasteleyn and Fortuin 1969) connected to each lattice site with probability h. Paths getting across the cell through the ghost site contribute to p' with terms of order h^2, thus not contributing to critical exponents, since criticality is at $h=0$. On the other hand, the probability p'h' of the renormalised site being connected to the ghost site gives the renormalisation of h (Harland and Stinchcombe 1977, Reynolds et al. 1980): To first order in h, we have for $b=3$:

$m=0$:

$$p'h' = h[9p^9+72p^8q+252p^7q^2+498p^6q^3+582p^5q^4+366p^4q^5+138p^3q^6+30p^2q^7+3pq^8] \tag{4}$$

$m=1$:

$$p'h' = h[9p^9+72p^8q+252p^7q^2+498p^6q^3+579p^5q^4+354p^4q^5+120p^3q^6+18p^2q^7] \tag{5}$$

$m=2$:

$$p'h' = h[9p^9+72p^8q+252p^7q^2+432p^6q^3+291p^5q^4+96p^4q^5+12p^3q^6] \tag{6}$$

$m=3$:

$$p'h' = h[9p^9+72p^8q+90p^7q^2+30p^6q^3] \tag{7}$$

$$(q \equiv 1 - p)$$

In two dimensions, the eigenvalue $\lambda_h = dh'/dh|_{h=0,p=p^*}$ is related to the exponent d through

$$\beta = \nu\,(2-y_h),\quad y_h = \ln \lambda_h / \ln b \tag{8}$$

The results for β and y_h are displayed in Table 1.

Let us now comment on our results. First, the behaviour of p_c against m shows the trend found by Kogut and Leath (1981), although we did not recover the result $p_c=1$ for m=3. This is due to the fact that in the small cells we use rectangular voids are statistically stable; on larger lattices large rectangular voids appear which are completely unstable (see the discussion in Kogut and Leath 1981). The fact that, for m=3, p_c (b=4) is slightly smaller than p_c(b=3) is also a small-cell effect, though of a different nature: it is related to the approximations implicit in the definition of the recursion relations; similar oscillations arise in small-cell PSRG for ordinary percolation (Reynolds et al. 1980). Second, the values we obtain for β increase with m, a tendency found by Kogut and Leath (1981). The slight difference between $\beta(m=0)$ and $\beta(m=1)$ is due to isolated entry sites on the cell; this, being essentially a surface effect, is believed to become negligible for larger b. On the other hand, the estimate for $\beta(m=2)$ differs appreciably from those for m=0 and 1; although we cannot pursue numerical accuracy with our small-cell approximations, this is consistent with the results of Kogut and Leath (1981) that the case m=2 is in a different universality class from both m=0 and 1.

Finally, the fact that λ_h is quite close to b^d for m=3 suggests that we are close to a first-order transition (Nienhuis and Nauenberg 1975, Fisher and Berker 1982). Actually, at the fixed point p=1 one has $\partial h'/\partial h = b^d$ (see eq.(7) for b=3); however, as pointed out above, the small-cell approximation gives a non-trivial fixed point close to p=1; it will be only at large b that this fixed point will tend to merge with the one at p=1. It is in this context that our results should be considered; in particular the result for ν at m=3 is quoted for comparison with those for m=0 through 2.

In summary, our approximate PSRG calculations support the view that the bootstrap percolation problem on a square lattice can be cast into different universality classes, according to the value of m. The critical exponent ν is the same for m=0, 1 and 2, while β varies with m. For m>3 we did not recover a first order transition as argued by CLR and Kogut and Leath (1981), due to the small cells used. We are currently investigating bootstrap percolation on other lattices, as well as a related problem, namely "high density" percolation (Reich and Leath 1978) in which sites with less than m active neighbours do not take part in the percolating cluster, but contribute as active sites all the time during the culling process.

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TABLE CAPTION

Table 1: Critical parameters obtained from our PSRG calculations

[a] Kogut and Leath (1981)

[b] Conjectured by den Nijs (1979) assumed for m=0 through 2.

FIGURE CAPTIONS

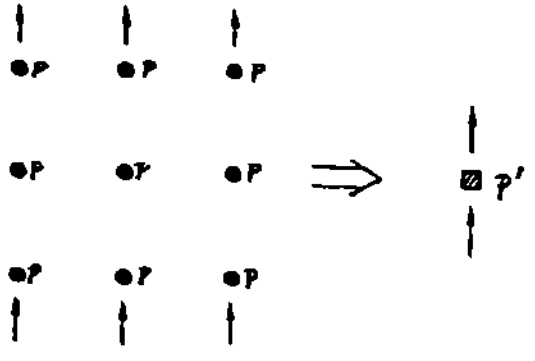
Figure 1. (a) A 3x3 cell in the original lattice; each site as a probability p of being occupied initially. The arrows indicate entries and exits from the cell. Under the PSRG it is transformed into a single site in (b) with probability p' of being active.

Figure 2. (a) A typical percolating configuration: full (empty) circles denote occupied (vacant) sites. The percolating path includes links due to periodic boundary conditions (dashed lines).

(b) A configuration that does not percolate because it does not come back to the starting point (same convention as in (a)).

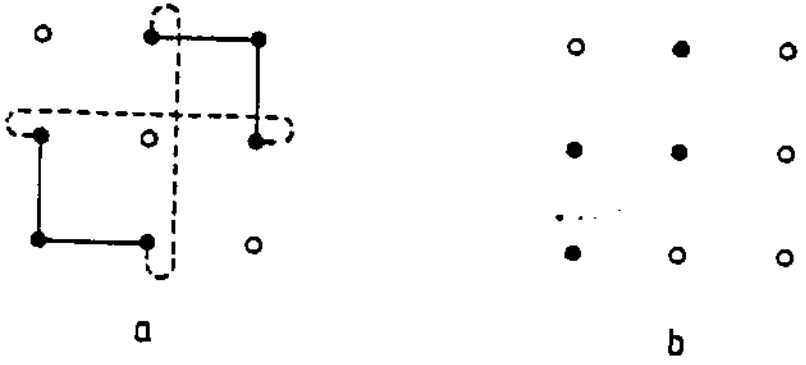
TABLE I

m	P_c			ν			β			λ_h	
	b=3	b=4	KL[a]	b=3	b=4	dN[b]	b=3	b=4	KL[a]	b=3	b=4
0	0.662	0.643	0.593	1.596	1.555	4/3	0.069	0.048	0.19	8.59	15.3
1	0.662	0.643	0.593	1.596	1.555	4/3	0.075	0.053	0.19	8.55	15.3
2	0.662	0.643	0.593	1.596	1.555	4/3	0.345	0.315	0.22	7.10	12.1
3	0.956	0.931	1	1.831	1.802	-	-	-	-	8.72	15.0



a b

Fig. 1



a b

Fig. 2