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REMARKS ON NON-COMPACT SIGMA MODELS

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REMARKS ON NON COMPACT SIGMA MODELS

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Abstract

We consider O(N,1) noncompact sigma models quantized in an indefinite metric space. In two dimensions, by a direct computation we prove the existence of a positive metric subspace where the S matrix is unitary. The properties of the model as function of the temperature are investigated in various space time dimensions.

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I. INTRODUCTION

Non linear sigma models defined on symmetric spaces provide a very powerful tool for the investigation of structural properties of field theory. In particular in two dimensions the models simulate various aspects of the physical reality as dynamical mass generation, asymptotic freedom and so on. In our opinion this motivates and justifies the study "in extenso" of such models. With this in mind, we would like to present some results on the behavior of non compact sigma models at finite temperature. Non compact non linear sigma models have some peculiar features which deserve mentioning. Firstly, at the classical level some of the models are ruled out on the grounds of the positivity of the energy. To see where this comes about, let us consider a generalized model defined on a coset space G/H where G is a noncompact Lie group and H is a maximal subgroup of G, invariant under some involutive automorphism of G. To be specif let G = O(N,1). In that case

$$q\eta q^t = q^t \eta q = \eta \qquad ;\forall q \in O(N,1) \tag{I.1}$$

where $\eta = \begin{pmatrix} -1 & 0 \\ 0 & I_N \end{pmatrix}$ is the metric tensor. The basic field g, taking values in the coset space G/H can be written as

$$g = q^{-1} g_o q \tag{I.2}$$

where g_o is a distinguished element of G, satisfying $g_o^2 = I$ and leaving H invariant, i.e.,

$$g_o h g_o = h \qquad h \in H \tag{I.3}$$

If $g_o = \eta$ then H is the O(N) subgroup of G and the space G/H is a symmetric space. If $g_o \neq \eta$ the subgroup H will be of the noncompact type. In what follows we will restrict ourselves to the case in which g_o has just one -1 eigenvalue occurrying at its i^{th} diagonal place. Denoting $q_{i\alpha}$ by φ_α it is easy to see that

$$\eta_{ii} = \varphi\eta\varphi = -\varphi_1^2 + \vec{\varphi}^2; \qquad \vec{\varphi}^2 = \sum_{\alpha\neq 1} \varphi_\alpha^2 \tag{I.4}$$

In both cases of noncompact and compact H the Lagrangian density will be taken as

$$\mathcal{L} = \frac{\eta_{ii}}{4} Tr[\partial_\mu g \partial^\mu g^{-1}] \qquad (I.5)$$

Now from (I.2) and (I.1) results

$$g_{\alpha\beta} = \delta_{\alpha\beta} - 2\eta_{\alpha\alpha}\eta_{ii}\varphi_\alpha\varphi_\beta \qquad (I.6)$$

which gives

$$\mathcal{L} = \frac{\eta_{\alpha\beta}}{2}\partial_\mu\varphi_\alpha\partial^\mu\varphi_\beta \qquad (I.7)$$

In the compact case (H is compact, i = 1), using the constraint (I.4) it can be rapidly verified that the Hamiltonian is positive definite. Indeed (summation over latin indices from 2 to N+1 is understood)

$$(\varphi_1\partial_o\varphi_1)^2 = \varphi_i\partial_o\varphi_i\varphi_j\partial_o\varphi_j \leq (\varphi_i)^2(\partial_o\varphi_j)^2 \qquad (I.8)$$

implying that

$$(\partial_o\varphi_1)^2 \leq (\partial_o\varphi_j)^2 \qquad (I.9)$$

and so

$$\mathcal{H} = \frac{1}{2}[(\partial_o\varphi_i)^2 + (\partial_1\varphi_i)^2 - (\partial_o\varphi_1)^2 - (\partial_1\varphi_1)^2] \geq 0 \qquad (I.10)$$

Similarly, in the noncompact case (H is noncompact) the Hamiltonian turns out to be unbounded both from above and from below. This type of model is therefore classically inconsistent.

Due to the noncompact nature of the group G it is natural to use an indefinite metric space for the quantization of the models. In two dimensions if the symmetry is not spontaneously broken, the model exhibits dynamical mass generation and asymptotic freedom. The bare coupling constant tends to zero as the cutoff is removed so that all the above classical geometric caracterization becomes meaningless.[1]

As in the usual model,[2] it turns out that the model has an infinite number of conserved charges which survive quantization. The non local ones are given recursively by

$$\begin{aligned} Q^{(n)} &= \int_{-\infty}^{\infty} j_o^{(n)}\,dx \\ j_\mu^{(n)}(x,t) &= j_\mu^{(o)}(x,t)\int_{-\infty}^{\infty}\frac{1}{2}\epsilon(x-y)\,j_o^{(n-1)}(y,t)\,dy \qquad (I.11) \\ &\quad - \epsilon_\mu^\nu j_\nu^{(n-1)}(x,t) \end{aligned}$$

The zeroth order currents are just the local "isospin" currents

$$j_\mu^{(o)} = i(\varphi_a \partial_\mu \varphi_b) \tag{I.12}$$

Analogously to reference [2], it is then possible to write an explicit S matrix which in our case is pseudo-unitary. In the positive metric subspace made of states with an even number of φ_1 particles this S matrix is unitary. In this paper by a direct computation this result is explicitly verified.

Noncompact sigma models may be important to the understanding of some phenomenological aspects of supergravity. Indeed, through the process of dimensional reduction, high dimensional supergravity theories present scalar sectors which are noncompact sigma models in various coset spaces.[3] Not surprisingly, this aroused a great interest on the subject.[4-8]

Our work is organised as follows. In section II we discuss the unitarity problem for the S matrix of noncompact sigma models. In section III we analyse the possibility for the existence of phases of broken and unbroken symmetry as function of the temperature. This is done for both compact and noncompact models in D≤4 space time dimensions. Although perturbatively nonrenormalizable in four dimensions, there is an interesting analogy with ordinary φ^4 models which justifies the study of such extreme situation.

II. 1/N EXPANSION. UNITARITY.

The 1/N expansion for the noncompact sigma model can be derived in a standard way by introducing an auxiliary collective field to enforce the constraint which the basic field must satisfy. The Lagrangian for the model is (from now on the φ_1 field is denoted by σ)

$$\begin{aligned}\mathcal{L} = \frac{1}{2}[(\partial_\mu\varphi_\alpha\partial^\mu\varphi_\alpha - m^2\varphi_\alpha\varphi_\alpha - (\partial_\mu\sigma)^2 + m^2\sigma^2] \\ + \frac{\lambda}{(N+1)^{1/2}}[-\sigma^2 + \varphi_\alpha\varphi_\alpha + \frac{N+1}{f}]\end{aligned} \tag{II.1}$$

The minus sign in the terms associated to the sigma field suggest a negative metric quantization for this field, a procedure that we will adopt henceforth. The mass parameter m is determined from the tadpole diagram of fig. I and actually is nonvanishing only in the phase where asymptotic freedom holds (see next section for more details). The λ propagator can be got by first computing the two point vertex function, which is given in the leading aproximation by the bubble diagram, fig. II.

$$\pi(p) = \int \frac{d^D k}{(2\pi)^D} \frac{i}{(k+p)^2 - m^2 + i\epsilon} \frac{i}{k^2 - m^2 + i\epsilon} \qquad (II.2)$$

In two dimensions, for example

$$\begin{aligned} \Delta_\lambda(p) &= [\pi(p)]^{-1} = 2\pi i [p^2(p^2 - 4m^2)]^{1/2} [\ln \frac{1-x}{1+x}]^{-1} \\ x &= \sqrt{1 - \frac{4m^2}{p^2}}; \qquad \text{for} \quad p^2 > 4m^2 \end{aligned} \qquad (II.3)$$

Summing up we have the correspondence

$$\begin{aligned} &\sigma \quad \text{propagator} \Longleftrightarrow -i[p^2 - m^2 + i\epsilon]^{-1} \\ &\varphi \quad \text{propagator} \Longleftrightarrow i[p^2 - m^2 + i\epsilon]^{-1} \\ &\lambda \quad \text{propagator} \Longleftrightarrow \Delta_\lambda(p) \\ &\lambda\varphi\varphi \quad \text{vertex} \Longleftrightarrow i(N+1)^{-1/2} \\ &\lambda\sigma\sigma \quad \text{vertex} \Longleftrightarrow -i(N+1)^{\frac{1}{2}} \end{aligned} \qquad (II.4)$$

These Feynman rules must be compared with the ones for the compact case. The differences are just the minus sign in the σ field propagator and in the trilinear vertex $\lambda\sigma^2$. The overall effect of these extra minus sign can be easily evaluated using that in a generic graph G

$$N_\sigma + 2n_\sigma = 2V_\sigma \qquad (II.5)$$

where

$N_\sigma =$ number of external σ lines
$n_\sigma =$ number of external σ lines
$V_\sigma =$ number of $\lambda\ \sigma^2$ vertices

Therefore, denoting the $(N_\sigma, N\varphi)$ Green functions for the compact and noncompact models by G_{com} and G_{ncom}, respectively, we have

$$G_{ncom} = (-1)^{\frac{N_\sigma}{2}} G_{com} \qquad (II.6)$$

Thus the compact and noncompact versions are related in a very simple way. The above relation can be used to prove that, in the positive metric subspace made of states containing an even number of σ, the S matrix for the noncompact model is unitary. For, let $|i>$ and $|f>$ be two arbitrary initial and final states, each containing an even number of sigma particles. Writing $S = I + iT$, the unitarity condition $S^\dagger S = I$, restricted to the subspace, amounts to

$$i[\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle] = -\langle f|T^\dagger T|i\rangle \qquad (II.7)$$

Thus in the noncompact case we must prove that

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum_n \langle f|T^\dagger|n\rangle\langle n|T|i\rangle \qquad (II.8)$$

where a complete set of intermediary states have been introduced. Now, due to (II.6), this is the same as

$$\langle f|T|i\rangle_{com} - \langle f|T^\dagger\rangle_{com} = i\sum_n (-1)^n \langle f|T^\dagger|n\rangle_{com}\langle n|T|i\rangle_{com} \qquad (II.9)$$

Furthermore, the intermediary states must have an even number of sigma particles, since the Lagrangian is even in σ. But then (II.8) is identical to the unitary relation for the compact sigma model. We are then led naturally to the question: are unitary the usual sigma models? The answer depends on the number of dimensions of the underlying Minkowksi space. In two dimensions the answer is affirmative and this can be seen in a variety of ways. One possibility, for example, is to use the fact that the model has an infinite number of conservation laws, which survive quantization. This permits an explicit construction of an exact unitary S matrix. Another possibility, more in the spirit of the perturbative expansions, is a direct check via Cutkosky[9] cutting identities, as we will show now.

Cutkosky cutting relations are very useful identities valid under the following conditions. Consider a generic Feynman graph G and its associated amplitude, made of a product of propagators, each having a decomposition into a sum of a positive and a negative frequency parts

$$\Delta(x) = \Delta^{+}(x) + \Delta^{-}(x) \qquad (II.10)$$

where Δ^{+} (resp. Δ^{-}) have support inside the future (resp. past) light cone. The following identity holds

$$\sum_{P} J_{+} J_{-} \prod \Delta_{con}(z_a - z_b) = 0 \qquad (II.11)$$

where

$$\Delta_{con}(z_a - z_b) = \begin{cases} \Delta^{+}(z_a - z_b) & \text{if } z_a^o > z_b^o \\ \Delta^{-}(z_a - z_b) & \text{if } z_b^o > z_a^o \end{cases}$$

and the sum is over all partitions P of the set of vertices of G into two disjoints subsets V_{+} and V_{-}, one of which can be empty (i.e., $V_{+} \cap V_{-} = \emptyset$ and $V_{+} \cup V_{-} = V$ = set of vertices of G)

$$J_{+} = i^{|V_{+}|} \prod_{V_i,\, V_j \in V_{+}} \Delta(z_i - z_j) \qquad (II.12a)$$

$$J_{-} = (-i)^{|V_{-}|} \prod_{V_i,\, V_j \in V_{-}} (\Delta(z_i - z_j))^{*} \qquad (II.12b)$$

and the productory in (II.11) runs over the set of lines which link vertices from V_{+} to vertices in V_{-}.

The proof of (II.11)[10] amounts to just combinatorics and will be ommited here. If all lines of G correspond to propagators of physical particles then (II.11) can be identified with the unitarity relation, restricted to G. Our case is more complicated because the auxiliary field is not associated to a physical particle. Nevertheless, in two dimensions the propagator of the λ field is a pure imaginary (see II.3) and the unitarity relation restricted to G can be written as

$$\prod_{V_\alpha, V_\beta \in G} \Delta_{\lambda}(x_\alpha - x_\beta) [\sum_{P} J_{+} J_{-} \prod \Delta_{con}(z_a - z_b)] = 0 \qquad (II.13)$$

This relation is satisfied in view of (II.11). In more than two dimensions the λ propagator is complex and the validity of the unitarity relation is unclear. We could give arguments in favour of the unitarity relation (for example, the non linear sigma model can be considered as a limit of the linear sigma model for which the unitarity relation probably holds) but a rigorous proof is not available; a direct verification is also unfeasible. In any case, as we shown, the unitarity problem is independent of the compactness property of the underlying symmetry group.

III. Non compact sigma models at finite temperature

The properties of non linear sigma models of the compact type at finite temperature have been analysed in various places in the literature.[11] Here we want to investigate the behavior of the noncompact models at finite temperature. To facilitate the comparison between the two versions of the models we introduce a parameter a so that its values 1 and -1 correspond to the compact and noncompact versions respectively. Using this notation the Lagrangian interpolating the two possibilities can be written as

$$\mathcal{L} = \frac{a}{2}[(\partial_\mu \sigma)^2 - m^2\sigma] + \frac{1}{2}(\partial_\mu \varphi_\alpha \partial^\mu \varphi_\alpha - m^2\vec{\varphi}^2) + \frac{\lambda}{(N+1)^{1/2}}[\vec{\varphi}^2 + a\sigma^2 + \frac{N+1}{af}] \qquad (III.1)$$

We assume that $\langle\lambda\rangle = 0$ because any non zero value could be absorved into a redefinition of the mass parameter m^2. This condition, on the other hand, implies that

$$\langle\vec{\varphi}^2\rangle + a\langle\sigma^2\rangle = \frac{N+1}{af(\Lambda)} \qquad (III.2)$$

where a (Pauli-Villars) regulator Λ is implicit. To calculate the left hand side of this equation, let us indicate by σ_o and $\varphi_o e$ the vacuum expectation values of σ and of one of the components of $\vec{\varphi}$ and redefine the fields so that $\sigma \rightarrow \sigma + \sigma_o$ and $\vec{\varphi} \rightarrow \vec{\varphi} + (0, ..., \varphi_o, ..., 0)$. From (III.1) we see immediately that the condition $\langle\sigma\rangle = \langle\varphi_i\rangle = 0$ implies that

$$m^2\sigma_o = m^2\varphi_o = 0$$

In the leading 1/N order the left hand side of (III.2) receives contribution from the graph of fig. I. At finite temperature[12] it is given by

$$a\sigma_o^2+\varphi_o^2=\frac{N+1}{af(\Lambda)}-(N+1)\int\frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-m^2+i\epsilon} -(N+1)\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{2w_k}\frac{1}{e^{\beta w_k}-1} \qquad (III.3)$$

where $w_k=\sqrt{k^2+m^2}$. From this equation we see the following possibilities.

1 .$m^2\neq 0$; $\sigma_o=\varphi_o=0$. O(N,1) is unbroken.

2. $m^2=0$. In this case we can have

2.a $\sigma_o\neq 0$; $\varphi_o=0$. O(N,1) is broken to O(N).

2.b $\varphi_o\neq 0$; $\sigma_o=0$. O(N,1) is broken to O(N-1.1).

since any other configuration can be reached by a transformation of O(N,1). For D $\leq$ 3, defining a renormalized coupling constant by

$$\frac{1}{af(\Lambda)}=\frac{1}{af_R}-\int\frac{d^Dk}{(2\pi)^D}\left[\frac{i}{k^2-\Lambda^2+i\epsilon}-\frac{i}{k^2-\mu^2+i\epsilon}\right] \qquad (III.4)$$

we found:

1. In two dimensions. Because of infrared divergences, only the massive phase can be realized in the two dimensional case.

$$\frac{1}{af_R}=\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}-\int\frac{dk}{\pi}\frac{1}{2w_k}\frac{1}{e^{\beta w_k}-1} \qquad (III.5)$$

which, given f_R always has one solution for m for any value of T. At high temperatures, for example

$$\frac{1}{af_R}=\frac{1}{4\pi}\ln\frac{m^2}{\mu^2}-\frac{1}{2\beta m} \qquad (III.6)$$

2. Three dimensions. Here too, due to infrared divergences, only the massive phase can be realized at non zero temperature. In this situation the mass gap equation can be explicitly solved for m

$$\frac{m}{4\pi}-\frac{1}{2\pi\beta}\ln(e^{m\beta}-1)=-\left(\frac{1}{af_R}+\frac{\mu}{4\pi}\right) \qquad (III.7)$$

where the renormalized coupling constant is given by (III.4), with D = 3. Thus

$$m = \frac{1}{2\beta}\ln[\frac{e^{\frac{A\beta}{2}} + \sqrt{e^{A\beta}+4}}{2}] \qquad (III.8)$$

where

$$A = 4\pi(\frac{1}{af_R} + \frac{\mu}{4\pi}) \qquad (III.9)$$

and m increases linearly with the temperature T, for high temperatures.

Other phases of broken symmetries and $m = 0$ are in principle possible at zero temperature. These phases are however unstable, due to the presence of tachyonic excitations in the λ field propagator.[5]

Let us now consider the four dimensional case.If we insist on renormalizability the model will reduce itself to a free model. This is so because the dominant contribution to the two point vertex function of the auxiliary field

$$\Pi(p) = \lim_{\Lambda\to\infty}\int\frac{d^4p}{(2\pi)^D}[\frac{i}{(k+p)^2 - m^2 + i\epsilon}\frac{i}{k^2 - m^2 + i\epsilon} - (m^2 \Rightarrow \Lambda^2)] \qquad (III.10)$$

is logarithicamlly divergent and therefore the propagator will vanish when $\Lambda \to \infty$ (we do not subtract this divergence because this would induce a λ^2 counterterm, turning the model indistinguishable from a φ^4 theory). So, as it is easily seen, this implies that, with the exception of the φ_a or σ field two point functions, all connected Green functions vanish. Defining a renormalized coupling constant, in a similar way to (III.4), by

$$\frac{1}{af(\Lambda)} = \frac{1}{af_R} - \frac{m^2}{16\pi^2}\ln\frac{\Lambda^2}{\mu^2} + \frac{\Lambda^2 - \mu^2}{16\pi^2} \qquad (III.11)$$

we obtain a mass depending on the temperature in the following way

$$-m^2\ln\frac{m^2}{e\mu^2} - \frac{4\pi^2}{3}T^2F(m^2\beta^2) = 16\pi^2[\frac{1}{af_R} + \frac{\mu^2}{16\pi^2}] \qquad (III.12)$$

where

$$F(y) = \int_0^\infty \frac{x^2\,dx}{\sqrt{x^2+y}(\exp\sqrt{x^2+y} - 1)}$$

is a monotonically decreasing function $(F(0) = 1,\ F(\infty) = 0)$.

It must be noticed that, since (III.11) contains m^2, the renormalization is temperature dependent. This assumption implies that m^2 is a free parameter and that all the burden of the elimination of the divergence is played solely by the coupling constant f.

The left hand side of (III.12) is double valued (see fig. III). For small T, depending on the values of f_R and μ^2 there will be two, one or no solutions for $m^2(T)$. As T increases, the maximum of the left hand side of (III.12) decreases and, for high enough T no real solution exists.

Figure Captions

Fig. I Lowest order contribution to the mass m. The continuous line represents either the σ or the φ_i propagagtor.

Fig. II Dominant contribution to the λ field two point vertex function.

Fig. III Graphical determination of the mass parameter m. The solid and dashed lines correspond to the left and right hand sides of (III.12).

FIGURES

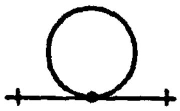

Figure I

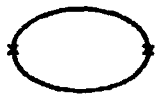

Figure II

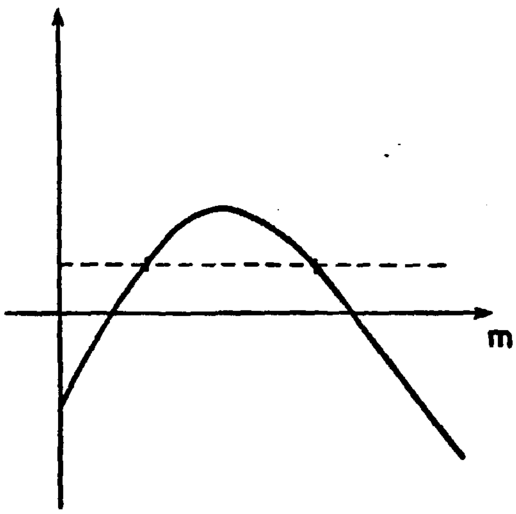

Figure III

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