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ON 2-BANACH ALGEBRAS *

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1. INTRODUCTION

Let X be a vector space of dimension greater than 1 and $\|\cdot,\cdot\|$ a real function on $X \times X$ satisfying the following conditions:

1. $\|a,b\| = 0$ if and only if a and b are linearly dependent;
2. $\|a,b\| = \|b,a\|$;
3. $\|\alpha a,b\| = |\alpha| \cdot \|a,b\|$, for any real α;
4. $\|a+b,c\| \leq \|a,c\| + \|b,c\|$, for all $a, b, c \in X$.

$\|\cdot,\cdot\|$ is called a 2-norm and X equipped with $\|\cdot,\cdot\|$ is a 2-normed space [5]. Gähler [5] has proved that $\|\cdot,\cdot\|$ is a non-negative function. The concept of 2-normed space was introduced by Gähler and various aspects of this structure have been studied (see for instance [7], [10], [11]). In [3] the concept of 2-metric space and its topological properties were investigated. In [4] the notion of 2-inner product and its relation with 2-norm was studied.

The purpose of this paper is to give the notion of a 2-Banach algebra and to study its structure. A 2-Banach algebra is a real algebra (of dimension greater than 2) which is a 2-Banach space (with respect to 2-norm topology) and in addition, the following condition:

5. $\|a,bc\| \leq K\ \|a,b\| \cdot \|a,c\|$, $a, b, c \in A$, K some positive constant, being satisfied. It is a 2-dimensional analogue of the concept of Banach algebra.

In Sec. 2 we give some basic properties to develop the subject.

Some fundamental properties of bivectors (see [6]) and tensor product are discussed in Sec. 3.

In Sec. 4 several classical results of Banach algebras have been extended to the 2-Banach algebra case.

In Sec. 5 a condition has been obtained under which a 2-Banach algebra becomes a Banach algebra.

Sec. 6 deals with the relation between algebra of bivectors and 2-normed algebra.

2. BASIC PROPERTIES OF 2-BANACH SPACES

The concept of 2-metric, 2-norm and 2-inner product are 2-dimensional analogues of the concepts of metric, norm and inner product. It is shown that $\sigma(a,b,c) = \|b-a,\ c-a\|$ and $\|a,b\| = (a,a|b)^{1/2}$, where $\sigma(a,b,c)$ and $(\cdot,\cdot|\cdot)$ are 2-metric and 2-inner product respectively.

A sequence $\{x_n\}$ in a 2-normed space X is called a <u>Cauchy sequence</u> if there exists $y, z \in X$ such that y and z are linearly independent, the $\lim \|x_n - x_m,\ y\| = 0$ and the $\lim \|x_n - x_m,\ z\| = 0$.

A sequence $\{x_n\}$ in a 2-normed space X is said to be a <u>convergent sequence</u> if there is an $x \in X$ such that the $\lim \|x_n - x,\ y\| = 0$, for all $y \in X$.

In [4] a "natural" topology was introduced for 2-metric and in particular, for 2-normed spaces. For a 2-normed space this topology is the locally convex T_2-topology which is generated by a family $\{\mu_b\}_{b \in X}$ of all semi-norms defined in X by the relation $\mu_b(a) = \|a,b\|$. For further details see ([4],[6]).

A 2-functional F is a real-valued mapping defined on $X \times X$, where X is a 2-normed space.

A 2-operator T is a mapping on $X \times X$ into a normed space Y (Y be a semi-normed space). Note that every 2-operator is a 2-functional ($Y = \mathbb{R}$). Throughout this section X stands for a 2-normed space unless otherwise stated.

A 2-operator T is called <u>bilinear</u> if

1. $T(a+c,\ b+d) = T(a,b) + T(a,d) + T(c,b) + T(c,d)$;
2. $T(\alpha a,\ \beta b) = \alpha\beta T(a,b)$, where α, β are in $\mathbb{R}$.

A 2-operator T is called <u>2-bounded</u> if there exists a real constant $\alpha \geq 0$ such that $\|T(a,b)\| \leq \alpha \|a,b\|$ for all (a,b) $X \times X$. If T is 2-bounded then we define the norm of T, $\|T\|$, by

$$\|T\| = g\ell b\ \{\alpha: \|T(a,b)\| \leq \alpha \|a,b\|,\ \text{for all } (a,b) \in X \times X\}.$$

If T is not 2-bounded we write $\|T\| = \infty$.

A 2-operator T is <u>2-continuous</u> at (a,b) $X \times X$, if given $\epsilon > 0$ there is a $\delta > 0$ such that $\|T(a,b) - T(c,d)\| < \epsilon$, whenever $\|a-c,\ b\| < \delta$ and $\|c,\ b-d\| < \delta$ or $\|a-c,\ d\| < \delta$ and $\|a,\ b-d\| < \delta$. T is 2-continuous if it is 2-continuous at each point of $X \times X$.

It is well-known that

1. every 2-norm is a 2-continuous functional;
2. every bilinear 2-operator which is 2-continuous at $(0,0)$ is 2-continuous;
3. every bilinear 2-operator is 2-continuous if and only if it is 2-bounded;
4. for every bilinear 2-bounded 2-operator T and for every a and $b \in X$ linearly dependent, $T(a,b) = 0$;
5. for a bilinear 2-bounded 2-operator T, the norm is given by

$$\|T\| = \sup\{\|T(x,y)\| : \|x,y\| = 1,\ (x,y) \in X \times X\}$$
$$= \sup\left\{\frac{\|T(x,y)\|}{\|x,y\|} : \|x,y\| \neq 0,\ (x,y) \in X \times X\right\}.$$

3. BIVECTORS

The concept of bivectors plays a very vital role in the study of 2-normed and 2-inner product spaces. The structure of 2-normed algebra is also closely related with the algebra of bivectors. Let B'_X be the set of all formal expressions $\sum_{i}^{n} a_i \times b_i$, where a_i, b_i $(i = 1, 2, \ldots, n)$ are elements of X, X being a vector space. Define an equivalence relation "$\sim$" on B'_X by $\sum_{i=1}^{n} a_i \times b_i \sim \sum_{i=1}^{m} a'_i \times b'_i$ if for arbitrary linear functions f and g on X,

$$\sum_{i=1}^{n} \begin{vmatrix} f(a_i) & g(a_i) \\ f(b_i) & g(b_i) \end{vmatrix} = \sum_{i=1}^{m} \begin{vmatrix} f(a'_i) & g(a'_i) \\ f(b'_i) & g(b'_i) \end{vmatrix} .$$

Let B_X be the quotient space $B'_X/\sim$. The elements of B_X are called <u>bivector over</u> X ([5], [6]) and the elements of B'_X belonging to a bivector called <u>representatives</u> of this bivector. The bivector with the representative $\sum_{i=1}^{n} a_i \times b_i$ will be denoted by $\mathfrak{b}(\sum_{i=1}^{n} a_i \times b_i)$. If a bivector has a representative of the form $a \times b$, then it is said to be simple. Only in the case where $\dim X \leq 3$ does every bivector over X turn out to be simple. The space B_X becomes a vector space with $\alpha\mathfrak{b}(\sum_{i=1}^{n} a_i \times b_i) = \mathfrak{b}(\sum_{i=1}^{n} \alpha a_i \times b_i)$, when α is real, and $\mathfrak{b}(\sum_{i=1}^{n} a_i \times b_i) + \mathfrak{b}(\sum_{i=1}^{m} a_{i+n} \times b_{i+n}) = \mathfrak{b}(\sum_{i=1}^{n+m} a_i \times b_i)$.

If $\|\cdot\|$ is a norm on B_X, then $\|a,b\| = \|\mathfrak{b}(a \times b)\|$ defines a 2-norm $\|\cdot,\cdot\|$ on X ([5], Theorem 12). There is an example in [2], p. 52, which shows that for every 2-norm $\|\cdot,\cdot\|$ on X, there need not exist a norm $\|\cdot\|$ on B_X which satisfies $\|\mathfrak{b}(a \times b)\| = \|a,b\|$, for all $a, b \in X$. If all bivectors over X are simple, i.e. $\dim X \leq 3$, then every 2-norm on X has a corresponding norm on B_X for which $\|a,b\| = \|\mathfrak{b}(a\ b)\|$ for all $a,b \in X$ ([5], p. 21). If $(\cdot,\cdot)$ is an inner product on B_X, then $(a,b|c) = (\mathfrak{b}(a \times c), \mathfrak{b}(b \times c))$ defines a 2-inner product $(\cdot,\cdot|\cdot)$ on X. In case $\dim X \leq 3$, every 2-inner product on X has a corresponding inner product on B_X for which $(a,b|c) = (\mathfrak{b}(a \times c), \mathfrak{b}(b \times c))$ for all $a,b,c \in X$ (see [3]).

Now recall the definition of tensor product of algebras. Let A and D be any real algebras. A tensor product of A and D is an algebra denoted by $A \otimes D$, together with a bilinear mapping $A \times D \to A \otimes D$, denoted by $(u,v) \to u \otimes v$ such that

1. $A \otimes D$ is generated as an algebra by $\{u \otimes v : u \in A,\ v \in D\}$;

2. If $\Phi : A \times D \to M$ is a bilinear mapping (that is, $\Phi(u,*) : D \to M$ and $\Phi(*,v) : A \to M$ are homomorphisms for all $u \in A$ and $v \in D$), then there is a homomorphism $\psi : A \otimes D \to M$ such that $\psi(u \otimes v) = \Phi(u,v)$ for all $u \in A$ and $v \in D$.

The hypothesis that $(u,v) \to u \otimes v$ is bilinear implies:

a) $u \otimes (v_1 a + v_2 b) = (u \otimes v_1)a + (u \otimes v_2)b$;

b) $(u_1 a + u_2 b) \otimes v = (u_1 \otimes v)a + (u_2 \otimes v)b$;

c) $u \otimes 0 = 0 \otimes v = 0$;

d) $ua \otimes v = (u \otimes v)a = u \otimes va$.

Note that the multiplication in $A \otimes D$ satisfies the relation $(u_1 \otimes v_1)(u_2 \otimes v_2) = u_1 u_2 \otimes v_1 v_2$. For further details see [8].

4. 2-BANACH ALGEBRAS

<u>Definition 4.1</u> - A 2-normed algebra A is a real algebra, of dimension greater than 2, which is a 2-normed space and in addition, the following condition

$$\|a, bc\| \leqslant K \|a, b\| \cdot \|a, c\| \quad ,$$

where all a,b,c are mutually linearly independent elements of A, is satisfied for some positive constant K.

If also A is a 2-Banach space, then it is called a 2-Banach algebra.

<u>Definition 4.2</u> - We say that a 2-Banach algebra A has an identity element e if for every $a \in A$, $e \cdot a = a \cdot e = a$ and $\|a,e\| \neq 0$.

<u>Example 4.1</u> - Let $\mathbb{R}^3$ be a vector space of dimension 3. Let $a = a_1e_1 + a_2e_2 + a_3e_3$ and $b = b_1e_1 + b_2e_2 + b_3e_3$, where $\{e_1,e_2,e_3\}$ denotes a basis in $\mathbb{R}^3$. Define

$$\|a, b\| = \left[(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \right]^{1/2}.$$

Then $(\mathbb{R}^3, \|\cdot,\cdot\|)$ is a 2-Banach algebra.

Example 4.2 - Let X be a commutative Banach algebra with an involution and S be the set all positive multiplicative functionals f on A with $\|f\| \leq 1$. For each $x, y \in X$ define

$$\|x, y\| = \sup_{f,g \in S} \left| f(x)g(y) - g(x)f(y) \right| .$$

Then $(X, \|\cdot,\cdot\|)$ is a 2-Banach algebra.

Theorem 4.1 - Let A be a 2-normed algebra. Then multiplication is a bilinear 2-continuous mapping $A \times A \to A$ (here A is considered as a semi-normed space).

Proof - Let z be any element of A. Consider

$$\|xy - ab, z\| = \|(x - a)(y - b) + a(y - b) + (x - a)b, z\|$$

$$\leq K\|x - a, z\| \|y - b, z\| + K\|a, z\|\cdot\|y - b, z\| + K\|x - a, z\|\cdot\|b, z\| .$$

for some positive constant K. *This shows that for* $\epsilon > 0$, *there exists* δ such that $\|xy - ab, z\| < \epsilon$ whenever $\|x-a, z\| < \delta$ and $\|y-b, z\| < \delta$. This proves the continuity.

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Theorem 4.2 - Let A be a 2-Banach algebra with identity e and a positive constant K, as involved in the definition (4.1), such that $0 < K \leq 1$. If $\|e-x, z\| < 1$ for all $z \in A$, then x^{-1} exists and is given by $x^{-1} = e + \sum_{n=1}^{\infty} (e-x)^n$.

In case, if $K > 1$ and $\|e-x, z\| < \frac{1}{K}$, then x^{-1} also exists and is given by the same expression as above.

Proof - If we put $r = \|e-x, z\| < 1$, then

$$\|(e - x)^n, z\| \leq K^{n-1}[\|e - x, z\|] = K^{n-1}r^n ,$$

for all sufficiently large n. This shows that the n-th partial sums of the series $\sum_{n=1}^{\infty} (e-x)^n$ forms a Cauchy sequence in A (with respect to 2-norm $\|\cdot,\cdot\|$). Since A is a 2-Banach space, the partial sums converge to an element of A, which we denote by $\sum_{n=1}^{\infty} (e-x)^n$. If we let $y = e + \sum_{n=1}^{\infty} (e-x)^n$, then, in view of Theorem (4.1), the joint continuity of multiplication implies that

$$y - xy = (e - x)y = (e - x) + \sum_{n=2}^{\infty} (e - x)^n = \sum_{n=1}^{\infty} (e - x)^n = y - e ,$$

so $xy = e$. Similarly, $yx = 1$. Hence the proof.

One can also prove the second half of the theorem on the similar lines.

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Corollary 4.1 - Let A be a 2-Banach algebra, $x \in A$ and λ be any scalar. If $|\lambda| > \|x,z\|$, for every $z \in A$, then $(\lambda e-x)$ is invertible and $(\lambda e-x)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n}x^{n-1}$.

Theorem 4.3 - Let A be a 2-Banach algebra with identity e and $G(A)$ the set of all invertible elements of A. Then $G(A)$ is an open set of A with respect to 2-norm topology.

Proof - Consider the case when the constant K (as involved in definition (4.1)) lies between $0 < K \leq 1$. Let $x_0 \in G(A)$ and $x \in A$ such that

$$\|x_0 - x, z\| < \frac{K}{\|x_0^{-1}, z\|}, \text{ where } \|x_0^{-1}, z\| \neq 0 \text{ and } z \in A .$$

Then

$$\|e - x_0^{-1}x, z\| = \|x_0^{-1}(x_0 - x), z\| < K^2 < 1 .$$

By theorem (4.2), $x_0^{-1}x \in G(A)$. Since $x = x_0(x_0^{-1}x)$, it follows that $x \in G(A)$. Hence $G(A)$ is open in A with respect to 2-norm topology.

Now consider the other case when the constant K is greater than 1. The result follows proceeding on the above lines and assuming that

$$\|x_0 - x, z\| \leq \frac{1}{K^2\|x_0^{-1}, z\|} .$$

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As a consequence of Theorem 4 [5, p. 7] and Theorem (4.1) we get the following result:

Theorem 4.4 - Every 2-normed algebra A is a locally convex topological algebra.

Definition 4.3 - Let A be a 2-Banach algebra with identity e. A non-zero bilinear 2-functional F defined on $A \times A$ is called 2-multiplicative 2-functional if

1. $F(ab, c) = F(a,c)\, F(b,c)$;

2. $F(a, bc) = F(a,b)\, F(a,c)$, for every a, b, c mutually linearly independent elements of A.

Theorem 4.5 - Let F be a 2-multiplicative 2-functional. If x and y are linearly independent and either of them is invertible then

$$F(x, y) \neq 0 \text{ and } F(x, e) = F(e, y) = 1, \; x, y \in A .$$

Note that if x and y are linearly dependent then $F(x,y) = 0$.

Proof - Since F is not identically zero, there exists some element $y \in A$ such that $F(x,y) \neq 0$. Thus $F(x,y) = F(xe,y) = F(x,y)F(e,y)$, it follows that $F(e,y) = 1$ and similarly $F(x,e) = 1$, for all x,y linearly independent with e.

If either x or y is invertible and both linearly independent then $F(x,y) \neq 0$. In fact,

$$1 = F(e, y) = F(xx^{-1}, y) = F(x, y)\, F(x^{-1}, y)$$

which implies that $F(x,y) \neq 0$.

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Theorem 4.6 - Let A be a 2-Banach algebra with identity e. Then every 2-multiplicative 2-functional F is 2-continuous.

Proof - Let $x \in A$ with $\|x,z\| < 1$, $z \in A$. Let λ be a scalar such that $|\lambda| \geq 1$, then $\|\frac{x}{\lambda},z\| = \frac{1}{|\lambda|}\|x,z\| < 1$. By Theorem (4.2), $(e-\lambda^{-1}x)$ is invertible, and therefore in view of Theorem (4.5), $F(e-\lambda^{-1}x, z) \neq 0$, for z linearly independent with $e-\lambda^{-1}x$. But then $F(e,z) - \lambda^{-1}F(x,z) = 1 - \lambda^{-1}F(x,z) \neq 0$. Hence $F(x,z) \neq \lambda$, or $|F(x,z)| < 1$, which proves the continuity of F.

With slight modification we get the proof in case $K > 1$.

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Definition 4.4 - A non-zero bilinear 2-functional F on $A \times A$ is called a Jordan 2-functional if $F(a^2,b) = F(a,b)^2$ and $F(a,b^2) = F(a,b)^2$, for all a, b, linearly independent elements of A.

Theorem 4.7 - Every Jordan 2-functional F on a commutative 2-Banach algebra A is 2-multiplicative.

Proof - Since F is a Jordan 2-functional,

$$F((a+c)^2, b) = [F(a+c, b)]^2 ,$$

for all a,b,c linearly independent elements of A. Keeping in view, the definition of Jordan 2-functional we get

$$F(a^2 + c^2 + 2ac, b) = [F(a, b) + F(c, b)]^2 .$$

This implies that

$$F(2ac, b) = 2F(a, b) F(c, b) ,$$

or

$$F(ac, b) = F(a, b) F(c, b) .$$

Similarly,

$$F(a, bc) = F(a, b) F(a, c) .$$

Hence the proof.

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Definition 4.5 - A subset I of a commutative 2-Banach algebra A is called an ideal if

1. I is a subspace of A;
2. $xy \in I$ whenever $x \in A$ and $y \in I$.

If $I \neq A$, I is called a proper ideal. Maximal ideals are proper ideals which are not contained in any proper ideal.

In view of the fact that every 2-normed algebra is a locally convex topological algebra, the following results hold:

1. A proper ideal of a commutative 2-Banach algebra A with identity e does not contain any one of its invertible elements;
2. The closure $\bar{I}$ of an ideal (with respect to 2-norm topology) is also an ideal;
3. The maximal ideals of A are closed.

Theorem 4.8 - Let F be a 2-multiplicative 2-functional on A×A. Then the null space of F, N(F), is a maximal ideal.

Proof - Let $N(F) = \{x \in A : F(x,y) = 0$, for every y from A$\}$. Then $F(ax,y) = F(a,y)F(x,y) = 0$, for $x \in N(F)$, $a \in A$. Hence N(F) is an ideal and it is maximal because it has codimension 1.

One may observe that the results of this section are 2-dimensional analogues of the well-known results concerning Banach algebras (see for instance, [1], [9]).

5. 2-BANACH ALGEBRA AND BANACH ALGEBRA

In view of the remarks made in Sec. 3 concerning relation between 2-norm and norm, one may observe that a 2-Banach algebra need not be a Banach algebra. However, the following theorem provides a condition under which a 2-Banach algebra can be made into a Banach algebra.

Theorem 5.1 - Let $(A, \|\cdot,\cdot\|)$ be a 2-Banach algebra and $a_1, a_2, \ldots a_m$ be any elements of A such that any set of points $b_1, b_2, \ldots b_n, \ldots$ with the condition $\lim_{i\to\infty} \|b_i, a_j\| = 0$, $(j = 1,2,\ldots,m)$ also satisfy the condition $\lim_{i\to\infty} \|b_i, a\| = 0$, for any $a \in A$. Then

$$\|a\| = \sum_{j=1}^{m} \|a, a_j\| ,$$

is an algebra norm. Moreover, $(A, \|\cdot\|)$ is a Banach algebra and the topologies generated by $\|\cdot\|$ and $\|\cdot,\cdot\|$ are equivalent.

Proof - In view of Theorem 1 and Theorem 2 [6, pp. 338, 340-341], it is sufficient to prove that

$$\|xy\| \leq \|x\| \, \|y\|$$

for every x, y from A. We can write

$$\|x\| = K \sum_{j=1}^{m} \|x, \tfrac{1}{K} a_j\| ,$$

by using properties of 2-norm $\|\cdot,\cdot\|$ (where K is a positive constant as given in the definition (4.1)). Therefore,

$$\|xy\| = K \sum_{j=1}^{m} \|xy, \tfrac{1}{K} a_j\| , \quad x, y \in A ,$$

$$\leq K^2 \sum_{j=1}^{m} (\|x, \tfrac{1}{K} a_j\| \cdot \|y, \tfrac{1}{K} a_j\|)$$

$$\leq \sum_{j=1}^{m} \|x, a_j\| (\|y, a_1\| + \ldots + \|y, a_m\|)$$

$$= \sum_{j=1}^{m} \|x, a_j\| \cdot \sum_{j=1}^{m} \|y, a_j\|$$

$$= \|x\| \cdot \|y\| .$$

This proves the Theorem.

6. BIVECTORS AND 2-BANACH ALGEBRAS

Let A and D be any two real algebras and let U be an algebra of tensor product $A \otimes D$ (see [1], [8]). In the space of bivectors $B_{\mathcal{U}}$ let us introduce the operation of multiplication in such a way that the algebra norm on $B_{\mathcal{U}}$ satisfies the relation

$$\|b \left[\left(\sum_{i=1}^{m} a_i \otimes b_i \right) \left(\sum_{k=1}^{q} f_k \otimes g_k \right) x \left(\sum_{j=1}^{n} c_j \otimes d_j \right) \right] \|$$

$$= \|b \left[\sum_{i=1}^{m} \sum_{k=1}^{q} (a_i f_k \otimes b_i g_k) \; x \left(\sum_{j=1}^{n} c_j \otimes d_j \right) \right] \|$$

$$< K \|b \left[\left(\sum_{i=1}^{m} a_i \otimes b_i \right) x \left(\sum_{j=1}^{n} c_j \otimes d_j \right) \right] \| \cdot \|b \left[\left(\sum_{k=1}^{q} f_k \otimes g_k \right) x \left(\sum_{j=1}^{n} d_j \otimes d_j \right) \right] \| ,$$

where

$$u = \sum_{i=1}^{m} a_i \otimes b_i , \quad v = \sum_{j=1}^{n} c_j \otimes d_j ,$$

$w = \sum_{k=1}^{q} f_k \otimes g_k$ are elements of $A \otimes D = \mathcal{U}$ and K is a positive constant.

Theorem 6.1 - Let $B_{\mathcal{U}}$ be a normed algebra with an algebra norm $\|\cdot\|$ having the property described above. Then $\mathcal{U} = A \otimes D$ is a 2-normed algebra with respect to algebra 2-norm $\|u,v\| = \|b(u \; x \; v)\|$.

Proof - In view of the results mentioned in Sec. 3, it is sufficient to show that

$$\|uw, v\| < K \|u, v\| \, \|w, v\| ,$$

K some positive constant. We have

$$\|uw, v\| = \|\daleth(uw \times v)\|$$

$$= \left\|\daleth\left(\sum_{i=1}^{m}\sum_{k=1}^{q} a_i f_k \otimes b_i g_k\right) \times \left(\sum_{j=1}^{n} c_j \otimes d_j\right)\right\|$$

$$< K\left\|\daleth\left[\left(\sum_{i=1}^{m} a_i \otimes b_i\right) \times \left(\sum_{j=1}^{n} c_j \otimes d_j\right)\right]\right\| \cdot$$

$$\left\|\daleth\left[\left(\sum_{k=1}^{q} f_k \otimes g_k\right) \times \left(\sum_{j=1}^{n} c_j \otimes d_j\right)\right]\right\|$$

$$= K\,\|u, v\| \cdot \|w, v\| \quad .$$

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This proves the theorem.

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