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Lie algebras under constraints and nonbijective canonical transformations

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Abstract. The concept of a Lie algebra under constraints is developed in connection with the theory of nonbijective canonical transformations. A finite dimensional vector space M, carrying a faithful linear representation of a Lie algebra L, is mapped into a lower dimensional space $\tilde{M}$ in such a manner that a subalgebra L_0 of L is mapped into $D(L_0) = 0$. The Lie algebra L under the constraint $D(L_0) = 0$ is the largest subalgebra L_1 of L that can be represented faithfully on $\tilde{M}$. If L_0 is semi-simple, then L_1 is shown to be the centraliser $cent_L L_0$. If L is semi-simple and L_0 is a one-dimensional diagonal subalgebra of a Cartan subalgebra of L, then L_1 is shown to be the factor algebra $cent_L L_0 / L_0$. The latter two results are applied to nonbijective canonical transformations generalising the Kustaanheimo-Stiefel transformation.

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124

1. Introduction

In the recent years, the LC transformation (Levi-Civita 1956), an $R^2 \to R^2$ map with discrete kernel, and the KS transformation (Kustaanheimo and Stiefel 1965), an $R^4 \to R^3$ map with continuous kernel, have been investigated and used in various domains of theoretical physics. The LC transformation is closely related to the usual conformal map and is therefore connected to the algebra of ordinary complex numbers. The KS transformation may be considered as a by-product of the theory of spinors and thus turns out to be connected to the algebra of ordinary quaternions (Kustaanheimo and Stiefel 1965, Blanchard and Sirugue 1981, Vivarelli 1983, Cornish 1984, Kibler and Négadi 1984b). The LC and KS transformations have been employed in classical and quantum mechanics and the reader is referred to the paper by Lambert and Kibler (1987b) for an extensive bibliography. Let us just mention that the KS transformation is of interest in the study of dynamical systems either in a partial-differential-equation approach (Ikeda and Miyachi 1970, Boiteux 1972, Barut, Schneider and Wilson 1979, Kibler and Négadi 1984b) or in a path-integral approach (Duru and Kleinert 1979, Blanchard and Sirugue 1981, Young and DeWitt-Morette 1986) or in a phase-space approach (Gracia-Bondia 1984). In this direction, the KS transformation has been very recently applied to a quantum-mechanical investigation of the Hartmann potential (Kibler and Winternitz 1987) and of a Aharonov-Bohm-like potential (Kibler and Négadi 1987).

There exist several nonbijective quadratic transformations generalising the LC and KS transformations. In particular, Kibler and Négadi (1984b, see also Lambert and Kibler 1987b) have studied a compact $R^4 \to R^4$ transformation and a compact $R^2 \to R^+$ transformation that parallel the LC and KS transformations, respectively. Furthermore, Iwai (1985) and Iwai and Rew (1985) have defined and used in symmetry reduction problems an $R^4 \to R^3$ transformation which may be thought of as a noncompact extension of the KS transformation. General attempts of introducing nonbijective quadratic transformations have been achieved by Boiteux (1982), Polubarinov (1984) and Lambert, Kibler and Ronveaux (1986). Finally, Lambert and Kibler (1987a, b) have recently introduced and studied from both an algebraic and geometric viewpoint (i) compact and noncompact $R^{2m} \to R^{2m}$ transformations with $2m = 2, 4, 8, \ldots$ extending the LC transformation and referred to as quasi Hurwitz transformations and (ii) compact and noncompact $R^{2m} \to R^{2m-n}$ transformations with $(2m, 2m - n) = (2,1)$, (4,1), (4,3), (8,1) and (8,5) extending the KS transformation and referred to as Hurwitz transformations. Such a study is based on the use of anti-involutions of Cayley-Dickson algebras, the latter algebras being generalisations of the algebras of complex numbers, quaternions and octonions.

It is the aim of this paper to develop a group theoretical approach to the Hurwitz transformations $R^{2m} \to R^{2m-n}$, with $(2m, 2m - n) = (2, 1)$, (4,3) and (8,5), which comprize and extend the KS transformation. The whole philosophy of this approach may be summed up as follows. In view of the nonbijectivity of the $R^{2m} \to R^{2m-n}$ map, we may introduce, for $2m$ fixed, $n = m - 1 + \delta(m, 1)$ 1-forms which are not total differentials and equate them to zero. We can then associate a vector field to each of the n 1-forms arising in the $R^{2m} \to R^{2m-n}$ transformation. For 2m fixed, each of the vector fields X_i with $i = 1, 2, \ldots, n$ is defined in the real symplectic Lie algebra $sp(4m, R)$ and the n vector fields together span a subalgebra L_0 of $sp(4m, R)$. Indeed, the algebra L_0 may be considered as a specific realisation of the Lie algebra of the ambiguity group discussed by Mello and Moshinsky (1975) and Moshinsky and Seligman (1978, 1979) in connection with geneneral $R^p \to R^{p'}$ (nonbijective) transformations with $p < p'$. The algebra L_0 will be called a *constraint Lie algebra* since its n generators X_i satisfy $X_i \psi = 0$ for any function ψ of class $C(R^{2m-n})$. [In this vein, it is to be noted that the constraints $X_i = 0$ $(i = 1, 2, \ldots, n)$ written in the phase space $R^{2m} \times R^{2m}$ are nothing but primary constraints of the generalised Hamiltonian formalism developed by Dirac (1964).] At this stage, one may ask the question: what is the group theoretical significance of the constraint conditions (also called superposition rules by Boiteux (1982)) $X_i \psi = 0$, $i = 1, 2, \ldots, n$? In other words, what is the subalgebra of $sp(4m, R)$ which survives when one forces the generators of $L_0 \subset sp(4m, R)$ to vanish ? These questions lead to studying *Lie algebras under constraints* and this is done in the present paper by introducing various constraints in $sp(4m, R)$ for $2m = 2$, 4 and 8.

This article constitutes a nontrivial extension of a series of papers by Kibler and Négadi (1983a, b, 1984a). In the latter works a unique constraint $X = 0$, corresponding to a constraint Lie algebra L_0 of type $so(2)$ for the KS transformation, is introduced into $sp(8, R)$. This leads to a Lie algebra under constraints isomorphic to $so(4,2)$. As a physical application, the noninvariance dynamical algebra $so(4,2)$ of the R^3 hydrogen atom may be obtained from the noninvariance dynamical algebra $sp(8, R)$ of the R^4 isotropic harmonic oscillator. This important result is a group theoretical complement of the well-known result that the KS transformation allows to convert, in a Schrödinger, Feynmann or Weyl-Wigner-Moyal formulation, the R^3 hydrogen atom problem into the R^4 isotropic harmonic oscillator problem. The $sp(8, R) - so(4,2)$ connection has been further worked out (i) by Quesne (1986) in relation with the independent-electron dynamical group of intrashell many-electron states as well as with the correlated electron dynamical group of intrashell doubly excited states and (ii) by Georgieva, Ivanov, Raychev and Roussev (1986) in relation with boson representations of symplectic algebras and their application to the theory of nuclear structure.

The Hurwitz transformations generalising the KS transformation are described in §2 in a unified and original way. Although the material contained in §2 turns out to be a by-product of the work by Lambert and Kibler (1987a, b), the adopted presentation is self-consistent and constitutes an alternative to the derivation of the Hurwitz transformations. Some general results on Lie algebras under constraints are presented as theorems in §3, where they are also applied to the Hurwitz transformations of §2. Constraint subalgeras L_0 of symplectic Lie algebras L are investigated in §4 for cases where L_0 and L are more general than for the cases corresponding to the Hurwitz transformations of §2. The final §5 is devoted to some concluding remarks.

2. Hurwitz transformations

We start from the (generalised) Hurwitz matrix

$$A(u) = \begin{bmatrix} -u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3 & c_3u_4 & -c_1c_3u_5 & -c_2c_3u_6 & c_1c_2c_3u_7 \\ u_1 & -u_0 & c_2u_3 & -c_2u_2 & c_3u_5 & -c_3u_4 & c_2c_3u_7 & -c_2c_3u_6 \\ u_2 & -c_1u_3 & -u_0 & c_1u_1 & c_3u_6 & -c_1c_3u_7 & -c_3u_4 & c_1c_3u_5 \\ u_3 & -u_2 & u_1 & -u_0 & c_3u_7 & -c_3u_6 & c_3u_5 & -c_3u_4 \\ u_4 & -c_1u_5 & -c_2u_6 & c_1c_2u_7 & -u_0 & c_1u_1 & c_2u_2 & -c_1c_2u_3 \\ u_5 & -u_4 & -c_2u_7 & c_2u_6 & u_1 & -u_0 & -c_2u_3 & c_2u_2 \\ u_6 & c_1u_7 & -u_4 & -c_1u_5 & u_2 & c_1u_3 & -u_0 & -c_1u_1 \\ u_7 & u_6 & -u_5 & -u_4 & u_3 & u_2 & -u_1 & -u_0 \end{bmatrix} \quad (1)$$

in dimension $2m = 8$, where u_α $(\alpha = 0, 1, \ldots, 7)$ are real numbers and $c_k = \pm 1$ $(k = 1, 2, 3)$. We also consider the column-vector $\mathbf{u}$, the metric matrix η and the conjugation matrix ε defined by (the sign ~ indicates matrix transposition)

$$\tilde{\mathbf{u}} = (-u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7)$$

$$\eta = \mathrm{diag}\,(1, -c_1, -c_2, c_1c_2, -c_3, c_1c_3, c_2c_3, -c_1c_2c_3) \quad (2)$$

$$\varepsilon = \mathrm{diag}\,(1, -1, 1, 1, 1, 1, -1, -1).$$

Let us finally introduce the matrices $A(u)$, $\mathbf{u}$, η and ε in dimensions $2m = 4$ and 2 in the following manner : $A(u)$, η and ε are the $2m \times 2m$ matrices consisting of the first $2m$ rows and columns of the corresponding matrices defined by equations (1) and (2), whereas $\mathbf{u}$ is the column-vector consisting of the first $2m$ rows of the corresponding column-vector defined by equation (2).

It can be verified that the matrices $A(u)$ for $2m = 2$, 4 and 8 satisfy the properties

$$\tilde{A}\,(u)\,\eta\,A(\mathbf{u}) = (\tilde{\mathbf{u}}\,\eta\,\mathbf{u})\,\eta$$

$$\tilde{A}\,(u) = \eta\,[-\,A\,(u) - 2u_0\,I_{2m}\,]\,\eta \quad (3)$$

where 1_{2m} stands for the $2m \times 2m$ unit matrix. The matrices $A(u)$ are of central importance in the celebrated Hurwitz (1898) theorem of arithmetics and its noncompact extension (Lambert and Kibler 1987b). [The compact cases treated by Hurwitz (1898) correspond to $c_1 = c_2 = c_3 = -1$, $c_1 = c_2 = -1$ and $c_1 = -1$ for $2m = 8$, 4 and 2, respectively.] The matrices $A(\mathbf{u})$ in dimensions $2m = 2$, 4 and 8 are related to the Cayley-Dickson algebras $A(c_1)$, $A(c_1, c_2)$ and $A(c_1, c_2, c_3)$ of dimensions $2m = 2$, 4 and 8 and they may be written in terms of elements of Clifford algebras of degrees $2m-1 = 1$, 3 and 7, respectively (Lambert and Kibler 1987b). In this respect, in the compact case $c_1 = c_2 = c_3 = -1$ for $2m = 8$, the Clifford algebra of degree $2m-1 = 7$ has been recently considered by Shaw (1987) in connection with a new view of $d = 7$ Dirac algebra.

We are now in a position to define nonbijective quadratic transformations. We shall deal in turn with the cases $2m = 8$, 4 and 2.

2.1. The case $2m = 8$

Let us define the $R^8 \rightarrow R^5$ map through

$$\mathbf{x} = A(\mathbf{u})\, \varepsilon\, \mathbf{u}. \tag{4}$$

In detail, we have

$$\begin{aligned}
x_0 &= u_0^2 - c_1 u_1^2 + c_2 u_2^2 - c_1 c_2 u_3^2 \\
&\quad + c_3 u_4^2 - c_1 c_3 u_5^2 + c_2 c_3 u_6^2 - c_1 c_2 c_3 u_7^2 \\
x_2 &= 2\,(-u_0 u_2 + c_1 u_1 u_3 + c_3 u_4 u_6 - c_1 c_3 u_5 u_7) \\
x_3 &= 2\,(-u_0 u_3 + u_1 u_2 - c_3 u_5 u_6 + c_3 u_4 u_7) \\
x_4 &= 2\,(-u_0 u_4 + c_1 u_1 u_5 - c_2 u_2 u_6 + c_1 c_2 u_3 u_7) \\
x_5 &= 2\,(-u_0 u_5 + u_1 u_4 + c_2 u_3 u_6 - c_2 u_2 u_7).
\end{aligned} \tag{5}$$

[In the general algebraic framework developed by Lambert and Kibler (1987b), the $R^8 \to R^5$ map given by (5) corresponds to the right Hurwitz transformation $\mathcal{H}_R^{(7)}$ associated to the anti-involution j_7 of $A(c_1, c_2, c_3)$.] Equation (5) may equally well be seen to result from the integration of $2\, A(u)\, \varepsilon\, du$. Indeed, the column-vector $2A(u)\, \varepsilon\, du$ is the transpose of the row-vector $(dx_0, \omega_1, dx_2, dx_3, dx_4, dx_5, \omega_6, \omega_7)$, where the 1-forms

$$\begin{aligned}
\omega_1 &= 2\,(-u_1du_0 + u_0du_1 + c_2u_3du_2 - c_2u_2du_3 \\
&\quad + c_3u_5du_4 - c_3u_4du_5 - c_2c_3u_7du_6 + c_2c_3u_6du_7) \\
\omega_6 &= 2\,(-u_6du_0 - c_1u_7du_1 - u_4du_2 - c_1u_5du_3 \\
&\quad + u_2du_4 + c_1u_3du_5 + u_0du_6 + c_1u_1du_7) \\
\omega_7 &= 2\,(-u_7du_0 - u_6du_1 - u_5du_2 - u_4du_3 \\
&\quad + u_3du_4 + u_2du_5 + u_1du_6 + u_0du_7)
\end{aligned} \tag{6}$$

which are not total differentials, can be taken to be equal to zero in view of the nonbijectivity of the $R^8 \to R^5$ map. The constraints $\omega_1 = \omega_6 = \omega_7 = 0$ make it possible to obtain

$$dx_0^{\,2} - c_2dx_2^{\,2} + c_1c_2dx_3^{\,2} - c_3dx_4^{\,2} + c_1c_3dx_5^{\,2} = 4\, r\, (d\tilde{u}\, \eta\, du) \tag{7}$$

where the "distance" $r = \tilde{u}\, \eta\, u$ satisfies

$$r^2 = x_0^{\,2} - c_2\, x_2^{\,2} + c_1c_2x_3^{\,2} - c_3x_4^{\,2} + c_1c_3x_5^{\,2}. \tag{8}$$

The basic property to be used in § 3 is

$$
\begin{bmatrix} \frac{\partial}{\partial x_0} \\ \frac{1}{2r} X_1 \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_4} \\ \frac{\partial}{\partial x_5} \\ \frac{1}{2r} X_6 \\ \frac{1}{2r} X_7 \end{bmatrix} = \frac{1}{2r} \eta\, A(u)\, \epsilon\, \eta \begin{bmatrix} -\frac{\partial}{\partial u_0} \\ \frac{\partial}{\partial u_1} \\ \frac{\partial}{\partial u_2} \\ \frac{\partial}{\partial u_3} \\ \frac{\partial}{\partial u_4} \\ \frac{\partial}{\partial u_5} \\ \frac{\partial}{\partial u_6} \\ \frac{\partial}{\partial u_7} \end{bmatrix} \tag{9}
$$

where the vector fields X_1, X_6 and X_7 associated to the 1-forms ω_1, ω_6 and ω_7, respectively, read

$$
\begin{aligned}
X_1 &= c_1 u_1 \frac{\partial}{\partial u_0} + u_0 \frac{\partial}{\partial u_1} + c_1 u_3 \frac{\partial}{\partial u_2} + u_2 \frac{\partial}{\partial u_3} \\
&\quad + c_1 u_5 \frac{\partial}{\partial u_4} + u_4 \frac{\partial}{\partial u_5} + c_1 u_7 \frac{\partial}{\partial u_6} + u_6 \frac{\partial}{\partial u_7} \\
X_6 &= -c_2 c_3 u_6 \frac{\partial}{\partial u_0} + c_2 c_3 u_7 \frac{\partial}{\partial u_1} + c_3 u_4 \frac{\partial}{\partial u_2} - c_3 u_5 \frac{\partial}{\partial u_3} \\
&\quad - c_2 u_2 \frac{\partial}{\partial u_4} + c_2 u_3 \frac{\partial}{\partial u_5} + u_0 \frac{\partial}{\partial u_6} - u_1 \frac{\partial}{\partial u_7} \\
X_7 &= c_1 c_2 c_3 u_7 \frac{\partial}{\partial u_0} - c_2 c_3 u_6 \frac{\partial}{\partial u_1} - c_1 c_3 u_5 \frac{\partial}{\partial u_2} + c_3 u_4 \frac{\partial}{\partial u_3} \\
&\quad + c_1 c_2 u_3 \frac{\partial}{\partial u_4} - c_2 u_2 \frac{\partial}{\partial u_5} - c_1 u_1 \frac{\partial}{\partial u_6} + u_0 \frac{\partial}{\partial u_7} .
\end{aligned} \tag{10}
$$

The operators X_1, X_6 and X_7 vanish when acting on functions $\psi(x_0, x_2, x_3, x_4, x_5)$ of class $C^1(R^5)$ and satisfy the commutation relations

$$[X_1, X_6] = -2X_7$$

$$[X_6, X_7] = -2c_2c_3X_1 \tag{11}$$

$$[X_7, X_1] = 2c_1X_6.$$

They therefore generate the Lie algebra su(2) or su(1, 1) according to whether as $(c_1, c_2, c_3) = (-1, \pm 1, \pm 1)$ or $(c_1, c_2, c_3) \neq (-1, \pm 1, \pm 1)$. Note that in view of (10), X_1, X_6 and X_7 are defined in the Lie algebra sp(16,R).

Following the geometrical analysis developed by Lambert and Kibler (1987b), and adapting it to the anti-involution j_7 inherent to the present work, the Hurwitz transformations characterised by equations (1) - (11) may be classified (up to homeomorphisms) into three types.

Type (c') : For $(c_1, c_2, c_3) = (-1, -1, -1)$, the map $R^8 \to R^5$ corresponds to the well-known Hopf fibration on spheres $S^7 \to S^4$ of *compact* fiber S^3.

Type (c") : For $(c_1, c_2, c_3) = (-1, 1, 1)$, the map $R^8 \to R^+ \times R^4 \subset R^5$ corresponds to a fibration on hyperboloids, viz $R^4 \times S^3 \to R^4$ of *compact* fiber S^3.

Type (nc) : For $(c_1, c_2, c_3) \neq (-1, \pm 1, \pm 1)$, the map $R^8 \to R^5$ corresponds to another fibration on hyperboloids, viz $R^4 \times S^3 \to R^2 \times S^2$ of *noncompact* fiber $R^2 \times S^1$.

We shall see in § 3 that there are two types of Lie algebras under constraints that we can associate to the latter three types of Hurwitz transformations.

2.2. The case $2m = 4$

This case is especially simple to present since it can be obtained from the case $2m = 8$ by omitting everything involving c_3, (u_4, u_5, u_6, u_7), (x_4, x_5), (ω_6, ω_7) and (X_6, X_7). This yields the $R^4 \to R^3$ map defined by

$$
\begin{aligned}
x_0 &= u_0{}^2 - c_1 u_1{}^2 + c_2 u_2{}^2 - c_1 c_2 u_3{}^2 \\
x_2 &= 2\,(- u_0 u_2 + c_1 u_1 u_3) \qquad (12) \\
x_3 &= 2\,(-u_0 u_3 + u_1 u_2)
\end{aligned}
$$

and subjected to the constraint

$$
\omega_1 = 2\,(-u_1 du_0 + u_0 du_1 + c_2 u_3 du_2 - c_2 u_2 du_3) = 0. \qquad (13)
$$

In this case there is only one vector field, namely

$$
X_1 = c_1 u_1 \frac{\partial}{\partial u_0} + u_0 \frac{\partial}{\partial u_1} + c_1 u_3 \frac{\partial}{\partial u_2} + u_2 \frac{\partial}{\partial u_3} \qquad (14)
$$

which belongs to the Lie algebra $sp(8,R)$ with the property that $X_1\ \psi\ (x_0, x_2, x_3) = 0$ for ψ in $C^1(R^3)$. The operator X_1 generates the subalgebra $so(2)$ for $(c_1, c_2) = (-1, \pm 1)$ and $so(1,1)$ for $(c_1, c_2) = (1, \pm 1)$.

We thus get a transformation which coincides with the right Hurwitz transformation $\mathcal{H}_R^{(1)}$ associated to the anti-involution j_1 of $A(c_1, c_2)$, see Lambert and Kibler (1987b). The special situation where $c_1 = c_2 = -1$ leads to the transformation worked out by Kustaanheimo and Stiefel (1965). The transformation recently introduced by Iwai (1985) is obtained by taking $c_1 = - c_2 = -1$.

Here again, we have three types of Hurwitz transformations which will give two types of Lie algebras under constraints. We extract from the work of Lambert and Kibler (1987b) the following classification that may be readily understood as a restriction of the corresponding one for $2m = 8$.

Type (c') : For $(c_1, c_2) = (-1, -1)$, the map $R^4 \to R^3$ corresponds to the famous Hopf fibration on spheres $S^3 \to S^2$ of *compact* fiber S^1.

Type (c") : For $(c_1, c_2) = (-1, 1)$, the map $R^4 \to R^+ \times R^2 \subset R^3$ corresponds to a fibration on hyperboloids, viz $R^2 \times S^1 \to R^2$ of *compact* fiber S^1.

Type (nc) : For $(c_1, c_2) = (1, -1)$ or $(1, 1)$, the map $R^4 \to R^3$ corresponds to another fibration on hyperboloids, viz $R^2 \times S^1 \to R \times S^1$ of *noncompact* fiber R.

2.3. The case 2m = 2

This limiting case presents some specific features, with respect to the cases 2m = 4 and 8, as can be seen in terms of Laplacian and d'Alembertian operators. Nevertheless, those points of relevance for what follows may be deduced from the case 2m = 4 by simply suppressing the expressions with c_2, (u_2, u_3) and (x_2, x_3). We are thus left with the $R^2 \to R$ map

$$x_0 = u_0^2 - c_1 u_1^2 \tag{15}$$

accompanied by the constraint

$$\omega_1 = 2(-u_1 du_0 + u_0 du_1) = 0. \tag{16}$$

The corresponding vector field

$$X_1 = c_1 u_1 \frac{\partial}{\partial u_0} + u_0 \frac{\partial}{\partial u_1} \tag{17}$$

is defined in the Lie algebra $sp(4,R) \sim so(3,2)$ and satisfies $X_1 \psi(x_0) = 0$ for ψ in $C^1(R)$. The operator X_1 generates the subalgebra $so(2)$ for $c_1 = -1$ and $so(1,1)$ for $c_1 = 1$. Equations (15)-(17) correspond to the right Hurwitz transformation $\mathcal{K}_R^{(1)}$ associated to the anti-involution $j_1 \equiv j_0$ of $A(c_1)$, cf Lambert and Kibler (1987b).

It is obvious in this case that there are only two distinct Hurwitz transformations, which will produce two types of Lie algebras under constraints in § 3. Indeed, we have the following classification.

Type (c) : For $c_1 = -1$, the map $R^2 \longrightarrow R^+ \subset R$ corresponds to the fibration $S^1 \longrightarrow \{1\}$ of *compact* fiber S^1.

Type (nc) : For $c_1 = 1$, the map $R^2 \longrightarrow R$ corresponds to the fibration $R \longrightarrow \{1\}$ of *noncompact* fiber R.

3. Lie algebras under constraints

The study of nonbijective canonical transformations has lead us to a mathematical problem that is of independent interest and has a wider realm of applications. It can be formulated as follows.

Consider a finite dimensional Lie algebra L and one of its proper subalgebras L_0. Let L have a faithful finite dimensional representation on some linear space M. Consider a nonbijective mapping from M to some lower dimensional space $\tilde{M}$ such that on $\tilde{M}$ the subalgebra L_0 is represented trivially by

$$D : L_0 \to D(L_0) = 0. \tag{18}$$

The questions that we pose are:

1. Is there a uniquely defined largest subalgebra $\tilde{L}$ of L such that $L_0 \subset \tilde{L} \subseteq L$ and having a nonfaithful linear representation $D : \tilde{L} \to D(\tilde{L})$ on $\tilde{M}$ with L_0 as its kernel, i.e. satisfying equation (18).

2. If $\tilde{L}$ exists, how one does find it and which is the largest subalgebra L_1 of $\tilde{L}$ that is represented faithfully in the representation $D(\tilde{L})$?

We start with some general Lie algebraic considerations and answer the above questions under some restrictions on L_0 and L. We then specialise to the cases of interest in the context of the Hurwitz transformations of §2, where we have $L = sp(4m, R)$ with $2m = 2$, 4 and 8, $L_0 = \{X_1\}$ for $2m = 2$ or 4 and $L_0 = \{X_1, X_6, X_7\}$ for $2m = 8$.

As far terminology is concerned, we call L_0 a "constraint Lie algebra" (the constraint being (18)) and L_1 a "Lie algebra under constraints" (the constraints being brought by (18)).

3.1. General discussion

Let us first introduce some concepts that we shall need below. Here, L stands for an arbitrary Lie algebra, the Lie brackets $[,]$ of which identify with commutators in a given linear representation.

Definition 1. The normaliser of a Lie algebra L_0 in a Lie algebra L, with $L_0 \subset L$, is defined as

$$nor_L L_0 = \{Z \in L \mid [Z, L_0] \subseteq L_0\}. \tag{19}$$

Thus, $nor_L L_0$ is the largest subalgebra of L in which L_0 is an ideal. Given L and L_0, $nor_L L_0$ is uniquely determined.

Definition 2. The centraliser of a Lie algebra L_0 in a Lie algebra L, with $L_0 \subset L$, is defined as

$$cent_L L_0 = \{Z \in L \mid [Z, L_0] = 0\}. \tag{20}$$

Clearly, the subalgebra $cent_L L_0$ of L is uniquely determined once L and L_0 are given.

Directly from the definitions we see that we have

$$L_0 \subseteq nor_L L_0 \qquad nor_L L_0 \subseteq L \qquad cent_L L_0 \subseteq nor_L L_0. \tag{21}$$

Let us now turn to the problem at hand. The algebra L_0 is the kernel of the Lie algebra homomorphism $D : \tilde{L} \to D(\tilde{L})$. Then, the Lie brackets

$$[D(\tilde{L}), D(L_0)] = 0 \tag{22}$$

are compatible with those of $\tilde{L}$ only if we have

$$[\tilde{L}, L_0] \subseteq L_0. \tag{23}$$

Hence, L_0 must be an ideal in $\tilde{L}$ and consequently we must have

$$\tilde{L} \subseteq nor_L L_0. \tag{24}$$

Let us now introduce a basis $\{X_i;\ 1 \leq i \leq n\}$ for the Lie algebra L_0 (of dimension n) and complement it to a basis $\{X_i, Y_\alpha;\ 1 \leq i \leq n,\ 1 \leq \alpha \leq \nu\}$ for the Lie algebra $nor_L L_0$ (of dimension $n + \nu$). The Lie brackets for $nor_L L_0$ in this basis are

$$[X_i, X_j] = a_{ij}^k X_k \tag{25a}$$

$$[X_i, Y_\alpha] = b_{i\alpha}^j X_j \tag{25b}$$

$$[Y_\alpha, Y_\beta] = c_{\alpha\beta}^\gamma Y_\gamma + d_{\alpha\beta}^i X_i. \tag{25c}$$

If the basis $\{Y_\alpha;\ 1 \leq \alpha \leq \nu\}$ of the factor "algebra" $F = nor_L L_0 / L_0$ can be so chosen that $d_{\alpha\beta}^i = 0$ $(1 \leq \alpha, \beta \leq \nu,\ 1 \leq i \leq n)$, then the factor algebra F is itself a Lie algebra. Moreover, in this case we have

$$L_1 = F = \{Y_\alpha;\ 1 \leq \alpha \leq \nu\} \tag{26}$$

i.e. the factor algebra F, that can be characterised as the external normaliser of L_0 in L, is itself the Lie algebra L_1 that is represented faithfully in $D(\tilde{L})$ with $\tilde{L} = nor_L L_0$.

Relations (25b) provide an outer derivation of the Lie algebra L_0 unless all structure constants $b_{i\alpha}^j$ vanish. To proceed further we restrict ourselves to constraint algebras L_0 that do not have any outer derivation. According to a theorem proven by Zassenhaus (1952, see also Jacobson 1979), this will be the case if L_0 is a finite dimensional semi-simple Lie algebra over a field of characteristic zero. On the other hand, in the case where L_0 is abelian, a given element X_i of L_0 will either commute with all basis elements Y_α or will be represented by a nilpotent matrix in the adjoint representation of $\tilde{L}$. We thus arrive at the following results.

Lemma 1. Let the constraint Lie algebra L_0 be a semi-simple Lie algebra over a field of characteristic zero. Then, the structure constants in (25b) satisfy

$$b^j_{i\alpha} = 0 \qquad 1 \leq i,j \leq n \qquad 1 \leq \alpha \leq \nu \tag{27}$$

and we have

$$nor_L L_0 = L_0 \ (+) \ cent_L L_0 \tag{28}$$

where $(+)$ denotes the direct sum of vector spaces.

Proof. Equation (27) follows from the fact that a semi-simple Lie algebra has no outer derivation. The result (28) is a consequence of (27) and the fact that a semi-simple Lie algebra does not have a center, hence the condition $[X, L_0] = 0$ implies that X does not belong to L_0. *Q.E.D.*

Lemma 2. Let L_0 be a subalgebra of a Cartan subalgebra of a finite dimensional semi-simple Lie algebra L over a field of characteristic zero. Then, the structure constants in (25b) satisfy

$$b^j_{i\alpha} = 0 \qquad 1 \leq i,j \leq n \qquad 1 \leq \alpha \leq \nu \tag{29}$$

and we have

$$nor_L L_0 = cent_L L_0 \,. \tag{30}$$

Proof. A Cartan subalgebra of a semi-simple Lie algebra L consists entirely of elements that are represented by simultaneously diagonalisable matrices in the adjoint representation of L, at least after a field extension. A set of such matrices does not contain any nilpotent matrix. The algebra L_0 has no outer derivation so that $b^j_{i\alpha} = 0$ in (25b). Since L_0 is abelian, we have $a^k_{ij} = 0$ in (25a) and the result (30) follows. *Q.E.D.*

We now turn to our main results on Lie algebras under constraints.

Theorem 1. Let L be a finite dimensional Lie algebra over a field of characteristic zero and let L_0 be a semi-simple proper subalgebra of L. The largest subalgebra $\tilde{L}$ of L that has a linear representation $D(\tilde{L})$ with L_0 as its kernel is the normaliser

$$\tilde{L} = nor_L L_0 = L_0 \oplus cent_L L_0 . \tag{31a}$$

The largest subalgebra L_1 of $\tilde{L}$ that can be represented faithfully in $D(\tilde{L})$ is the centraliser

$$L_1 = cent_L L_0 = nor_L L_0 / L_0 . \tag{31b}$$

Proof. From lemma 1 we already know that $nor_L L_0$ is the direct sum of the two disjoint vector spaces L_0 and $cent_L L_0$ and that we have $b^j_{i\alpha} = 0$ in (25b). Since $cent_L L_0$ is a Lie algebra and X_i $(1 \leq i \leq n)$ does not belong to $cent_L L_0$, we must have $d^i_{\alpha\beta} = 0$ in (25c) and we obtain (30). Thus, $\tilde{L} = nor_L L_0$ is a direct sum of Lie algebras and setting $D(L_0) = 0$ is consistent with representing $L_1 = cent_L L_0$ faithfully. *Q.E.D.*

Theorem 2. Let L be a classical Lie algebra over the field R having an even dimensional self-representation, i.e. a real form of A_{2N-3}, C_N or D_N $(N = 2, 3, \ldots)$ in Cartan's notations. Let L_0 be a one-dimensional subalgebra of a Cartan subalgebra of L, namely one of the "diagonal subalgebras" $[o(2) \oplus o(2) \oplus \ldots \oplus o(2)]_d$ or $[o(1,1) \oplus o(1,1) \oplus \ldots \oplus o(1,1)]_d$. Then, the largest subalgebra $\tilde{L}$ of L that has a nonfaithful representation $D(\tilde{L})$ with L_0 as its kernel is uniquely determined to be

$$\tilde{L} = nor_L L_0 = cent_L L_0. \tag{32a}$$

The largest subalgebra L_1 of $\tilde{L}$ that can be represented faithfully in $D(\tilde{L})$ is the factor algebra

$$L_1 = cent_L L_0 / L_0 \tag{32b}$$

which in this case is itself a Lie algebra.

Proof. From lemma 2 we already have $nor_L L_0 = cent_L L_0$. We must show that under the conditions of the theorem we have

$$cent_L L_0 = L_0 \oplus L_1. \tag{33}$$

By hypothesis we have $n = 1$ and therefore the Lie brackets (25a, b, c) reduce to

$$[X_1, Y_\alpha] = 0 \qquad [Y_\alpha, Y_\beta] = c^\gamma_{\alpha\beta} Y_\gamma + d^1_{\alpha\beta} X_1. \tag{34}$$

Equation (34) describes a central extension of the Lie algebra $\{Y_\alpha;\ 1 \leq \alpha \leq \nu\}$ and we must show that this extension is trivial, i.e. $d^1_{\alpha\beta} = 0$ $(1 \leq \alpha, \beta \leq \nu)$.

Consider first the noncompact case $L_0 = [o(1,1) \oplus o(1,1) \oplus \ldots \oplus o(1,1)]_d$. We can choose a realisation of the defining faithful linear representation of L in which L_0 is represented by the matrices

$$X = a\ diag[I_N, -I_N] \qquad a \in R. \tag{35a}$$

A simple calculation shows that in this representation we have

$$nor_L L_0 = cent_L L_0 = \{diag[A, B] \qquad A, B \in R^{N\times N}\} \tag{35b}$$

with possibly further restrictions on the matrices A and B depending on which particular classical Lie algebra L we are considering. In any case, independently of the choice of L, the derived algebra $[cent_L L_0, cent_L L_0]$ of $cent_L L_0$ is represented by matrices of the form $diag[A, B]$ with $tr\ A = tr\ B = 0$. Hence, $L_0 \not\subset [cent_L L_0, cent_L L_0]$ and we obtain $d^1_{\alpha\beta} = 0$ in (34) so that (33) holds.

Now consider the compact case $L_0 = [o(2) \oplus o(2) \oplus \ldots \oplus o(2)]_d$. In an appropriate realisation of the defining representation of L we have L_0 represented by the matrices

$$X = b\ diag[J, J, \ldots, J] \qquad b \in R \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{36a}$$

where the matrix J occurs N times along the diagonal of X. We obtain

$$nor_L L_0 = cent_L L_0 = \{ \begin{bmatrix} X_{11} & \cdots & X_{1N} \\ \vdots & \cdots & \vdots \\ X_{N1} & \cdots & X_{NN} \end{bmatrix} \; X_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{bmatrix} \; a_{ij},\ b_{ij} \in R \quad 1 \leq i, j \leq N\} \tag{36b}$$

for the normaliser (and centraliser) of L_0 in $\Delta = R^{2N\times 2N}$. Since (36b) provides a real representation of $gl(N, C)$, we have $nor_\Delta L_0 = gl(N, C)$. The normaliser of L_0 in L will be a subalgebra of $gl(N, C)$, obtained by imposing the appropriate involution condition, reducing Δ to L. In any case, L_0 is not contained in the derived algebra $sl(N, C)$ of $gl(N, C)$ and still less in that of any subalgebra of $gl(N, C)$. We again conclude that $d^1_{\alpha\beta} = 0$ in (34) so that we obtain (33). *Q.E.D.*

Finding the maximal subalgebra L_1 of L that is represented faithfully when L_0 is represented trivially is thus a simple task of linear algebra and boils down, in the cases of relevance in §3.2, to constructing the set of elements commuting elementwise with the elements of L_0. The Lie algebra L_1 coincides with what Kibler and Négadi (1983a, b, 1984a) refer to as a *Lie algebra under constraints*. In their terminology, L_1 is isomorphic to the algebra L subjected to the constraints

$$X_i = 0 \qquad 1 \leq i \leq n \tag{37}$$

and may thus be thought of as the Lie algebra surviving when the constraints (37) are introduced inside L.

3.2. Application to Hurwitz transformations

Returning to the nonbijective quadratic transformations described in §2, we identify L as $sp(4m, R)$ with $2m = 2, 4$ or 8. The basic problem is for $L = sp(16, R)$ and $L_0 = so(3) \sim su(2)$ or $so(2,1) \sim su(1,1)$ and corresponds to $2m = 8$. The two remaining problems concern $L = sp(8, R)$ for $2m = 4$ and $L = sp(4, R)$ for $2m = 2$ and both correspond to $L_0 = so(2)$ or $so(1,1)$. The problems for $2m = 4$ and 2 can be solved at the same time as the problem for $2m = 8$ by adapting the restriction process of §2.

We realize the algebra $sp(4m, R)$ by matrices X of $R^{4m \times 4m}$ satisfying

$$XK + K\tilde{X} = 0 \qquad \text{with} \qquad K = \begin{bmatrix} 0 & I_{2m} \\ -I_{2m} & 0 \end{bmatrix} \tag{38}$$

so that we have

$$X = \begin{bmatrix} A & B \\ C & -\tilde{A} \end{bmatrix} \quad A \in R^{2m \times 2m} \quad B = \tilde{B} \in R^{2m \times 2m} \quad C = \tilde{C} \in R^{2m \times 2m} \tag{39}$$

[see Moshinsky and Winternitz (1980) for details]. The matrix X depends on $2m(4m + 1)$ parameters as it must be. The Lie algebra $sp(4m, R)$ is, on one hand, realised by the matrices (39) and, on the other, by the bilinear forms

$$\alpha_{ij} = \partial_i u_j + u_j \partial_i \qquad \beta_{ij} = \partial_i \partial_j \qquad \gamma_{ij} = u_i u_j. \tag{40}$$

The representatives of the operators α_{ij}, β_{ij} and γ_{ij} in terms of matrices X may be obtained according to a simple prescription (Moshinsky and Winternitz 1980).

For $2m = 2$ and 4, L_0 ($= so(2)$ or $so(1,1)$) is spanned by X_1 of (17) and (14), respectively. For $2m = 8$, L_0 ($= so(3)$ or $so(2,1)$) is spanned by the three operators X_1, X_6 and X_7 of

(10). It is easy to represent the constraint operators X_1, X_6 and X_7 for $sp(16,R)$ in terms of matrices X of equation (39) with $2m=8$ by applying the above-mentioned prescription. The representative matrix of X_1 so-obtained may serve to generate the matrices that represent the constraint operators X_1 for $sp(8,R)$ and $sp(4,R)$ by means of a subduction process which parallels the restriction process described in §2 for the coordinate transformations.

Then, it is a simple matter of calculation to find the centraliser of $\{X_1, X_6, X_7\}$ in $sp(16,R)$. It is sufficient to search for the general matrix X which commutes with the representative matrices of the operators $X_1(c_1)$ and $X_6(c_2,c_3)$ corresponding to the case $2m=8$. [The representative matrix of the operator $X_7(c_1,c_2,c_3)$ does not need to be considered since it imposes no further restriction.] This has been done in a brute force way by using the algebraic and symbolic programming sytem REDUCE. As a net result, the general representative matrix $X(c_1,c_2,c_3)$ of the centraliser of $\{X_1(c_1), X_6(c_2,c_3), X_7(c_1,c_2,c_3)\}$ in $sp(16,R)$ is given by equation (39) with

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18}\\ c_1a_{12} & a_{11} & c_1a_{14} & a_{13} & c_1a_{16} & a_{15} & c_1a_{18} & a_{17}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38}\\ c_1a_{32} & a_{31} & c_1a_{34} & a_{33} & c_1a_{36} & a_{35} & c_1a_{38} & a_{37}\\ c_3a_{37} & -c_3a_{38} & -c_2c_3a_{35} & c_2c_3a_{36} & a_{33} & -a_{34} & -c_2a_{31} & c_2a_{32}\\ -c_1c_3a_{38} & c_3a_{37} & c_1c_2c_3a_{36} & -c_2c_3a_{35} & -c_1a_{34} & a_{33} & c_1c_2a_{32} & -c_2a_{31}\\ -c_2c_3a_{17} & c_2c_3a_{18} & c_3a_{15} & -c_3a_{16} & -c_2a_{13} & c_2a_{14} & a_{11} & -a_{12}\\ c_1c_2c_3a_{18} & -c_2c_3a_{17} & -c_1c_3a_{16} & c_3a_{15} & c_1c_2a_{14} & -c_2a_{13} & -c_1a_{12} & a_{11} \end{bmatrix} \tag{41a}$$

$$B=\begin{bmatrix} b_{11} & 0 & b_{13} & b_{14} & b_{15} & b_{16} & 0 & 0\\ 0 & -c_1b_{11} & -b_{14} & -c_1b_{13} & -b_{16} & -c_1b_{15} & 0 & 0\\ b_{13} & -b_{14} & b_{33} & 0 & 0 & 0 & c_2b_{15} & c_2b_{16}\\ b_{14} & -c_1b_{13} & 0 & -c_1b_{33} & 0 & 0 & -c_2b_{16} & -c_1c_2b_{15}\\ b_{15} & -b_{16} & 0 & 0 & c_2c_3b_{33} & 0 & -c_3b_{13} & -c_3b_{14}\\ b_{16} & -c_1b_{15} & 0 & 0 & 0 & -c_1c_2c_3b_{33} & c_3b_{14} & c_1c_3b_{13}\\ 0 & 0 & c_2b_{15} & -c_2b_{16} & -c_3b_{13} & c_3b_{14} & c_2c_3b_{11} & 0\\ 0 & 0 & c_2b_{16} & -c_1c_2b_{15} & -c_3b_{14} & c_1c_3b_{13} & 0 & -c_1c_2c_3b_{11} \end{bmatrix} \tag{41b}$$

$$C = \text{the same as } B \text{ with } b_{ij} \to c_{ij}. \tag{41c}$$

From the matrix $X(c_1, c_2, c_3)$ so-obtained, we can perform the calculation of the rank and dimension of the Lie algebra under constraints L_1, as well as the dimension of the maximal compact subalgebra of L_1 in each of the cases $L = sp(4m, R)$ with $2m = 8$, 4 and 2. This makes it possible to identify L_1 in the following way. In the case $2m = 2$ or 4, we find that $cent_L\{X_1(c_1)\}$ is a Lie algebra of dimension $4m^2$ and rank $2m$ with a maximal compact subalgebra of dimension $2m^2$ for $c_1 = -1$ and $m(2m-1)$ for $c_1 = 1$. Therefore, in the cases $2m = 2$ and 4, we have $cent_L\{X_1(c_1)\} = u(m,m)$ or $gl(2m, R)$ for $c_1 = -1$ or 1, respectively. Consequently, $L_1 = cent_L\{X_1(c_1)\}/\{X_1(c_1)\}$ is identified as $su(m,m)$ for $c_1 = -1$ and $sl(2m, R)$ for $c_1 = 1$. In the case $2m = 8$, we find that $cent_L\{X_1(c_1), X_6(c_2,c_3), X_7(c_1,c_2,c_3)\}$ is a Lie algebra of dimension 28, of rank 4 and of character (i.e., the number of noncompact generators minus the number of compact generators) -4 for $(c_1,c_2,c_3) = (-1, \pm1, \pm1)$ and $+4$ for $(c_1,c_2,c_3) \neq (-1,\pm1,\pm1)$. Consequently, $L_1 = cent_L\{X_1(c_1), X_6(c_2,c_3), X_7(c_1,c_2,c_3)\}$ is identified as $so^*(8)$ for $(c_1,c_2,c_3) = (-1,\pm1,\pm1)$ and $so(4,4)$ for $(c_1,c_2,c_3) \neq (-1,\pm1,\pm1)$. The results for $2m = 8$, 4 and 2 can be summed up and further documented as follows.

<u>The case $2m = 8$, $L = sp(16, R)$:</u>

(c) $L_0 = so(3)$ and $L_1 = so^*(8) \sim so(6,2)$ for $(c_1,c_2,c_3) = (-1,-1,-1)$ or $(-1,1,1)$.

(nc) $L_0 = so(2,1)$ and $L_1 = so(4,4)$ for $(c_1,c_2,c_3) \neq (-1,\pm1,\pm1)$.

<u>The case $2m = 4$, $L = sp(8, R)$:</u>

(c) $L_0 = so(2)$ and $L_1 = su(2,2) \sim so(4,2)$ for $(c_1,c_2) = (-1,-1)$ or $(-1,1)$.

(nc) $L_0 = so(1,1)$ and $L_1 = sl(4,R) \sim so(3,3)$ for $(c_1,c_2) = (1,-1)$ or $(1,1)$.

The case $2m = 2$, $L = sp(4, R)$:

(c) $L_0 = so(2)$ and $L_1 = su(1,1) \sim so(2,1)$ for $c_1 = -1$.

(nc) $L_0 = so(1,1)$ and $L_1 = sl(2,R) \sim so(2,1)$ for $c_1 = 1$.

It is to be mentioned that the result (c) for $2m = 4$ agrees with the one derived by Kibler and Négadi (1983a, b, 1984a) in the frame of a study of the hydrogen-oscillator connection based on a bosonisation of the Pauli equations for the hydrogen atom.

We note the important result that, in each of the cases (n) and (nc), there is a correspondence between the types of Lie algebras under constraints and the types of fibers described in §2. More precisely, the cases (c) correspond to *compact* fibers and the cases (nc) to *noncompact* fibers.

4. Symplectic Lie algebras under constraints

The results of §3.2 can be generalised to arbitrary symplectic Lie algebras L and various constraint Lie algebras L_0. Indeed, the results obtained in §3.2 may be derived in an alternative and more rational manner that points to further generalisations.

We shall first deal with two cases where L_0 is a one-dimensional constraint algebra and shall thus apply theorem 2. We shall then turn to two cases where L_0 is a simple Lie algebra and shall thus apply theorem 1.

4.1. The case $L_0 = [o(1,1) \oplus o(1,1) \oplus \ldots \oplus o(1,1)]_d$ and $L = sp(2N, R)$

We realize the noncompact diagonal algebra L_0 by matrices of the type (39) with $2m \to N$ and

$$A_0 = a\ I_N \qquad a \in R \qquad B_0 = C_0 = 0. \tag{42}$$

We immediately obtain

$$nor_L L_0 = cent_L L_0 = \{diag[A, -\tilde{A}]\} = gl(N, R) = L_0 \oplus sl(N, R). \tag{43}$$

Thus, the Lie algebra under constraints is

$$L_1 = sl(N, R) \subset \tilde{L} = gl(N, R). \tag{44a}$$

In particular, we have

$$L_1 = sl(2, R) \sim so(2,1) \text{ for } N = 2 \qquad L_1 = sl(4, R) \sim so(3,3) \text{ for } N = 4 \tag{44b}$$

in agreement with the results of §3.2.

4.2. The case $L_0 = [o(2) \oplus o(2) \oplus .. \oplus o(2)]_d$ and $L = sp(4N, R)$

We realize the compact diagonal algebra L_0 as in (39) with $m \to N$ and

$$A_0 = a\ diag[J, J, \ldots, J] \qquad a \in R \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B_0 = C_0 = 0 \qquad (45)$$

where the matrix J occurs N times along the diagonal of A_0. A simple calculation yields the centraliser of L_0 in L in the form (39) where the matrix A is an $N \times N$ matrix of elements

$$\begin{bmatrix} a_{ij}^1 & a_{ij}^2 \\ -a_{ij}^2 & a_{ij}^1 \end{bmatrix} \qquad a_{ij}^k \in R \qquad 1 \le i, j \le N \qquad 1 \le k \le 2 \qquad (46a)$$

and where the matrices B and C are given by

$$B = A \text{ with } a_{ij}^k \to b_{ij}^k \qquad C = A \text{ with } a_{ij}^k \to c_{ij}^k \qquad b_{ii}^2 = c_{ii}^2 = 0 \qquad 1 \le i \le N. \quad (46b)$$

We thus obtain

$$nor_L L_0 = cent_L L_0 = u(N, N) = L_s \oplus su(N, N) \qquad (47)$$

and the Lie algebra under constraints is

$$L_1 = su(N, N) \subset \tilde{L} = u(N, N). \qquad (48a)$$

In particular, we have

$$L_1 = su(1,1) \sim so(2,1) \text{ for } N = 1 \qquad L_1 = su(2,2) \sim so(4,2) \text{ for } N = 2 \qquad (48b)$$

as in §3.2.

4.3. The case $L_0 = [sl(2,R) \oplus sl(2,R) \oplus \ldots \oplus sl(2,R)]_d$ *and* $L = sp(8N,R)$

We realize the noncompact diagonal gebra L_0, of dimension 3, by the matrices (39) with $m \to 2N$ and

$$A_0 = diag[J, J, \ldots, J] \qquad J = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \qquad a,b,c \in R \qquad B_0 = C_0 = 0 \tag{49}$$

where the matrix J occurs $2N$ times along the diagonal of A_0. A straightforward calculation yields the centraliser of L_0 in $L = sp(8N,R)$ in the form (39) where the matrices A, B and C are given by

$$A = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & a_{1,2N} & 0 \\ 0 & a_{11} & 0 & a_{12} & \cdots & 0 & a_{1,2N} \\ a_{21} & 0 & a_{22} & 0 & \cdots & a_{2,2N} & 0 \\ 0 & a_{21} & 0 & a_{22} & \cdots & 0 & a_{2,2N} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{2N,1} & 0 & a_{2N,2} & 0 & \cdots & a_{2N,2N} & 0 \\ 0 & a_{2N,1} & 0 & a_{2N,2} & \cdots & 0 & a_{2N,2N} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & b_{12} & \cdots & 0 & b_{1,2N} \\ 0 & 0 & -b_{12} & 0 & \cdots & -b_{1,2N} & 0 \\ 0 & -b_{12} & 0 & 0 & \cdots & 0 & b_{2,2N} \\ b_{12} & 0 & 0 & 0 & \cdots & -b_{2,2N} & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & -b_{1,2N} & 0 & -b_{2,2N} & \cdots & 0 & 0 \\ b_{1,2N} & 0 & b_{2,2N} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$C = B \text{ with } b_{ij} \to c_{ij} \qquad a_{ij}, b_{ij}, c_{ij} \in R \tag{50}$$

The latter realisation shows that $cent_L L_0$ is a simple Lie algebra of dimension $2N(4N-1)$ and rank $2N$ with a maximal compact subalgebra of dimension $2N(2N-1)$. We can thus identify the centraliser and the Lie algebra under constraints as

$$L_1 = cent_L L_0 = so(2N, 2N). \tag{51}$$

For $N = 2$, we recover the result $L_1 = so(4,4)$ of §3.2 corresponding to $L_0 = so(2,1)$ and $L = sp(16, R)$.

4.4. The case $L_0 = [o(3) \oplus o(3) \oplus \ldots \oplus o(3)]_d$ and $L = sp(8N, R)$

We realize the compact diagonal algebra L_0, of dimension 3, by the matrices (35) with $m \to 2N$ and

$$A_0 = diag[J, J, \ldots, J] \qquad J = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{bmatrix} \qquad a, b, c \in R \qquad B_0 = C_0 = 0 \tag{52}$$

where the matrix J occurs N times along the diagonal of A_0. The centralizer of L_0 in $L = sp(8N, R)$ can be easily calculated. It is realised by matrices of the form (39) where A, B and C are $N \times N$ matrices in which each entry is a 4×4 (real) matrix of the type

$$X_{ij} = \begin{bmatrix} x_{ij}^0 & x_{ij}^1 & x_{ij}^2 & x_{ij}^3 \\ -x_{ij}^1 & x_{ij}^0 & x_{ij}^3 & -x_{ij}^2 \\ -x_{ij}^2 & -x_{ij}^3 & x_{ij}^0 & x_{ij}^1 \\ -x_{ij}^3 & x_{ij}^2 & -x_{ij}^1 & x_{ij}^0 \end{bmatrix} \qquad x_{ij}^k \in R \qquad 1 \le i, j \le N \qquad 0 \le k \le 3. \tag{53}$$

For $X \equiv B$ and C, we have $B_{ij} = \hat{B}_{ji}$ and $C_{ij} = \tilde{C}_{ji}$ $(1 \le i, j \le N)$. We find that $cent_L L_0$ is a simple Lie algebra of dimension $2N(4N-1)$ and rank $2N$ with a maximal compact subalgebra of dimension $4N^2$. We can thus identify $cent_L L_0$ as $so^*(4N)$ so that we end up with

$$L_1 = cent_L L_0 = so^*(4N). \tag{54}$$

For $N = 1$, we have $so^*(4) \sim so(3) \oplus so(2,1)$. For $N = 2$, we recover the result $L_1 = so^*(8) \sim so(6,2)$ of §3.2 corresponding to $L_0 = so(3)$ and $L = sp(16, R)$.

5. Concluding remarks

The main mathematical result of this paper can be summarised in the following manner. Consider a finite dimensional Lie algebra L and a proper subalgebra L_0 of L. Then, the largest Lie algebra $\tilde{L}$, satisfying $L_0 \subset \tilde{L} \subsetneq L$ and having a nonfaithful linear representation in which L_0 is represented trivially, is the normaliser $nor_L L_0$ of L_0 in L. If the normaliser allows a decomposition

$$nor_L L_0 = L_0 \oplus L_1 \tag{55}$$

into the direct sum of L_0 and a Lie algebra L_1, then L_1 can be represented faithfully in a Lie algebra homomorphism $D : nor_L L_0 \to D(nor_L L_0)$ with L_0 as its kernel.

The condition that the decomposition (55) should hold is a restriction on the algebras L_0 and L. We have shown that equation (55) always holds for the constraint algebras L_0 and the algebras L occurring in the $R^{2m} \to R^{2m-n}$ nonbijective canonical transformations with $(2m, 2m-n) = (2,1)$, $(4,3)$ and $(8,5)$.

If the decomposition (55) does not hold, then it may be necessary to enlarge the kernel of the homomorphism for $nor_L L_0/L_0$ to be a Lie algebra. To see this, consider the example where $L = sp(4, R)$ (realised as in (39) with $m = 1$) and L_0 is the one-dimensional nilpotent Lie algebra

$$L_0 = \{ \begin{bmatrix} 0 & 0 & b_{11} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad b_{11} \in R\}. \tag{56}$$

We find that

$$nor_L L_0 = \{ \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ 0 & a_{22} & b_{12} & b_{22} \\ 0 & 0 & -a_{11} & 0 \\ 0 & c_{22} & -a_{12} & -a_{22} \end{bmatrix} \qquad a_{ij}, b_{ij}, c_{ij} \in R\}. \tag{57}$$

In this case, $nor_L L_0$ is a Lie algebra isomorphic to the "optical Lie algebra" $opt(2,1)$ (see Patera, Sharp, Winternitz and Zassenhaus 1977, Burdet, Patera, Perrin and Winternitz 1978). Denoting A_{ij} the element of $nor_L L_0$ obtained by setting $a_{ij} = 1$ and all other entries equal to zero in equation (57), and similarly for B_{ij} and C_{ij}, we have

$$[A_{12}, B_{12}] = 2B_{11} \in L_0 \tag{58}$$

and hence $nor_L L_0/L_0$ is not a Lie algebra. To obtain a consistent homomorphism, we must enlarge the kernel to include

$$L_0' = \{A_{12}, B_{12}, B_{11}\}. \tag{59}$$

Then, we have

$$nor_L L_0' = nor_L L_0$$

$$L_1 \sim nor_L L_0'/L_0' \sim \{A_{11} \oplus (A_{22}, B_{22}, C_{22})\} \sim o(1,1) \oplus sl(2, R) \tag{60}$$

and L_1 is the algebra represented faithfully.

The motivation, stressed in this article, for introducing Lie algebras under constraints comes from the study of nonbijective canonical transformations. In this respect, the mathematical results obtained here are of interest in the determination of invariance and noninvariance algebras of dynamical systems (cf. Kibler and Négadi 1983a, b, 1984a, Lambert and Kibler 1987b). They may also be useful in related fields as in atomic and nuclear shell theory (cf. Quesne 1986) and in nuclear models of the type interacting vector boson model (cf. Georgieva, Ivanov, Raychev and Roussev 1986).

A different application that suggests itself concerns symmetry reduction for partial differential equations. Thus, let L be the Lie algebra of the Lie group G of local point symmetries of a system of partial differential equations (see Olver 1986) and let L_0 be a subalgebra of L corresponding to a subgroup G_0 of G. The construction of solutions invariant under the subgroup G_0 involves a nonbijective transformation from the space of independent and dependent variables $\{x_1, ..., x_n, u_1, ..., u_N\}$ to the space of G_0 invariants $\{\xi_1, ..., \xi_k, w_1, ..., w_N\}$ $(k < n)$. The transformation involves precisely the conditions

$$X\Phi(x_1, ..., x_n, u_1, ..., u_N) = 0 \qquad X \in L_0. \tag{61}$$

Hence, the Lie algebra under constraints L_1 is in this case the Lie algebra of a group G_1 leaving invariant the space $\tilde{M}$ of invariants of L_0. Either G_1 or a subgroup of G_1 will then be the symmetry group of the reduced equations.

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