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DYNAMICALLY TRIANGULATED SURFACES - SOME ANALYTICAL RESULTS

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1. INTRODUCTION

In this talk (which is in a sense complementary to these of J. Ambjorn and A. Krzywicki) we give a brief review of the analytical results concerning the model of dynamically triangulated surfaces[1-3]. We will discuss the possible types of critical behaviour (depending on the dimension D of the embedding space) and the exact solutions obtained for D=0 and D=-2. The latter are important as a check of the Monte Carlo simulations applyed to study the model in more "physical" dimensions. They give also some general insight of its critical properties.

2. DESCRIPTION OF THE MODEL

The model of dynamically triangulated surfaces has been proposed as a discrete version of the Polyakov string model[4]. The functional measure in the space of parametrized surfaces is replaced by the sum over gaussian embeddings in the D-dimensional space of a class of two-dimensional abstract simplicial lattices (triangulations) with a given topology.

Before going into details, let us remark that the discretization of the parameter space of the surface through triangulations is convenient (especially for Monte Carlo simulations) but not obligatory. The critical behaviour of the random surface should not depend on the details of the discretization, and we can use other collections of two-dimensional lattices provided they are generated by some local algorithm. This means that each lattice can be obtained from some standard one by a sequence of local transformations. This reflects the locality of the functional measure over internal matrices in the continuum theory[4].

Following this remark, we slightly generalize the definition of the model by taking instead of triangulations a set $\mathcal{G}^{(g)}$ of graphs with the topology of a sphere with g handles. The topology of a graph G is defined by its Eules characteristic χ = # vertices − # lines + # faces = 2-2g. The partition functions $F_g(\beta)$ of the surfaces of genus g is defined by

$$\int d^D x \, F_g(\beta) = \sum_{G \in \mathcal{G}^{(g)}} \frac{1}{K(G)} \tag{1}$$

$$\int \prod_{i \in G} \frac{d^D x_i}{(2\pi)^{D/2}} \, e^{-\frac{1}{2}\sum_{\langle ij \rangle}\left[(x_i - x_j)^2 + \beta\right]}$$

where X_i^μ, $\mu = 1, \ldots, D$, are the coordinates of the point i of the graph G, $\langle ij \rangle$ is the line connecting i and j (if there are more than one such line, a sum is implied), and k(G) is the volume of the symmetry group of the graph G.

Locally the internal geometry of a surface is characterized by the coordination numbers $q_i = \{ \# \text{ lines } \langle ij \rangle, \; j \in G \}$ and $q_{n^*} = \{ \# \text{lines} \langle u^* v^* \rangle, v^* \in G^* \}$ of the graph G and its dual graph G*. the standard definition in terms of triangulations corresponds to taking all q_{n^*} equal to 3.

The weight of each graph G depends on the result of the gaussian integration (the entropy of the embedding) and in this sense the discretization of the surfaces is chosen dynamically.

3. RELATION TO THE BOSONIC STRING

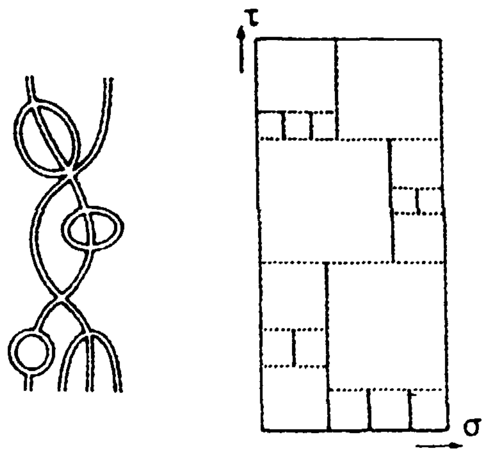
Let us restrict ourselves to the case of surfaces with g=1. Following 't Hooft[6], we introduce light-cone coordinates $X^{\pm} = \frac{1}{\sqrt{2}}(X^{D-1} \pm X^D)$ and go to a mixed momentum coordinate representation. The factor corresponding to a link $\langle ij \rangle$ becomes

$$\delta\left(P_{\langle ij \rangle} - i\left(X_i^- - X_j^-\right)\right) \exp\left\{ i \, P_{\langle ij \rangle}\left(X_i^- - X_j^-\right) - \frac{1}{2}\left(X_i^\perp - X_j^\perp\right)^2 \right\} \tag{2}$$

The integration over X_i^-, $i \in G$, yields a conservation law for the momenta : $\sum_j P_{\langle ij \rangle} = 0$.

Therefore we can assign to each point u^* of the dual graph G^* a coordinate σ_{u^*} such that $P^*_{\langle ij \rangle} = \sigma_{u^*} - \sigma_{v^*}$, where $\langle u^* v^* \rangle$ is the link of G^* dual to $\langle ij \rangle$.

Further, we assign to each point $j \in G$ a coordinate $\tau_j = i X^*_j$. In this way the graph G defines a discretization of the parameter manifold (σ, τ). Each link $\langle ij \rangle$ corresponds to a square fragment of the string with a width $P^*_{\langle ij \rangle}$. A vertex $i \in G$ can be seen as a process of splitting (joining) of several string pieces at time τ_i and is represented by a horizontal segment in the (σ, τ) plane. Finally, a vertex $u^* \in G^*$ corresponds to a cut separating string pieces and is represented by a vertical segment.



(a) (b)

FIGURE 1

A planar graph (a) and the corresponding discretization of the world sheet (b)

The mass spectrum of the string excitations can be extracted from the partition function (1) with the sum going over graphs G with fixed ratio $\Delta\sigma_{total}/\Delta\tau_{total}$ (this quantity depends only on the structure of the graph !). However, we do not know a simple way to impose this condition.

We assume that the (internal) volume of each square is 1. Then the invariant volume in the parameter manifold can be written as $\rho(\sigma,\tau) d\sigma d\tau$ where $\rho(\sigma,\tau)$ is the density of squares at the point (σ,τ).

4. CRITICAL EXPONENTS

The sum in the r.h.s. of (1) is finite for all values of the coupling (or "cosmological constant") β larger than some critical β_c. At $\beta \to \beta_c$ the susceptibility behaves as

$$\partial^2 F_g(\beta)/\partial\beta^2 \sim (\beta-\beta_c)^{-\gamma(g,D)} + \text{const.} \qquad (3)$$

The critical coupling β_c depends on the discretization, but once the discretization is fixed, it does not depend on the genus g.

Another important characteristic of the model if the mean square extent of the surfaces with given area ($\equiv$ # links) A

$$\langle X^2 \rangle_A = \left\langle \frac{\sum_{i,j} (X_i - X_j)^2}{\sum_{i,j} 1} \right\rangle_A \qquad (4)$$

whose scaling behaviour gives the fractal dimension d_F of the random surface

$$\langle X^2 \rangle_A \underset{A \to \infty}{\sim} N^{2\nu} \quad , \quad \nu = 1/d_F \qquad (5)$$

A convenient formula for calculating this quantity is [3]

$$\langle X^2 \rangle_A = -D \frac{\partial}{\partial P^2} \log G_A(P^2) \Big|_{P=0} \qquad (6)$$

where $G(P^2,\beta) = \sum_A G_A(P^2) e^{-\beta A}$ is the Fourier transform of the two-point correlator in the model of random surfaces. The latter is defined by the sum over surfaces with two points fixed.

5. EQUIVALENCE WITH A GAUSSIAN MATRIX MODEL

It is convenient to choose $\mathcal{G}$ in eq.(1) as the set of all Feynaman diagrams corresponding to some scalar field theory. Then the partition function (1) can be identified with the free energy of a matrix field with gaussian propagator[2,3].

Let $\Phi(x)$ be an $N \times N$ matrix field ($\Phi^+ = \Phi$) defined by the action

$$\mathcal{A}[\Phi] = -\frac{1}{2} N \,\mathrm{tr} \int d^D x\, d^D y\, e^{\frac{1}{2}(x-y)^2 + \beta}\, \Phi(x)\Phi(y)$$

$$+ (2\pi)^{-D/2} N \,\mathrm{tr} \int d^D x\, V(\Phi(x)) \;, \qquad (7)$$

with potential

$$V(\Phi) = \lambda_1 \Phi + \frac{1}{2}\lambda_2 \Phi^2 + \frac{1}{3}\lambda_3 \Phi^3 + \dots \qquad (8)$$

Then the $1/N^2$ expansion of the free energy of this model is related to the partition function (1) by

$$F(\beta) = \sum_{g=0}^{\infty} F_g(\beta)\, N^{-2g} + \text{const.} \qquad (9)$$

where $\exp N^2 \int d^2x\, F(\beta) = \int \mathcal{D}\Phi \exp \mathcal{A}\{\Phi\}$ (10)

6. ZERO DIMENSIONS

In the case $D = 0$ the partition function of the model is given by the $1/N^2$ expansion of a single matrix integral. This problem can be solved exactly[7] and one obtains[1,8] a critical singularity of the form (3) with

$$\gamma(g,0) = 2 + \frac{5}{2}(g-1)\ , \quad g = 0,1,2\ . \qquad (11)$$

The result (11) was obtained for the φ^4 potential but it is easy to see that it holds for a generic potential of the form (8).

Eq.(11) is proved only for g=0,1,2 where the explicit formulas for the free energy are known. However it is very likely to be true for all g. Indeed, eq.(3) implies that the number of connected graphs of genus g with 2n lines behaves as $n^{\gamma(g)-3}\exp(2n\beta_c)$ when $n \gg g$. Let assume that this asymptotics holds also for $g/n \lesssim \alpha$ where $\alpha < 1$ does not depend on n. Then the total number of connected diagrams with 2n lines will be about $\exp(2n\,\beta_c)n^{\gamma(\alpha n)}$. On the other hand, this number is $\sim n^n$ which implies that γ grows linearly with n.

Another quantity which can be calculated in zero dimensions is the probability distribution P(q) of the coordination numbers q_i. For the case of triangulations (i.e., graphs dual to Feynman diagrams generated by a Φ^3-potential) and $g = 0$ this was done in [5]. The exact form of P(q) is not universal but in all cases it decays exponentially with q.

7. -2 DIMENSIONS

The partition function can be defined for negative dimensions by analytical continuation from positive D. By the Kirchoff theorem the gaussian integration in (1) can be reduced to pure combinatories[2,3]:

$$F_g(\beta) = \sum_{G\in\mathcal{G}_g} e^{-\beta(\text{number of lines of } G)}\, (T(G))^{-D/2} \qquad (12)$$

where T(G) is the number of spanning trees of the graph G (a spanning tree is a subgraph without cycles which contains all points of G). At $-D/2=1$ one can interchange the sum over trees and the sum over graphs which permits to evaluate without pain the partition function[2,5].

Alternatively, we can assume that the coordinates x_i^μ of the points of the surface are Grassmann variables. This makes sense if $D = -2n$: then each point x is defined by n pair at anticommuting variables : x_i, $\bar{x}_i$, $i = 1, \ldots, n$. The group O(D) of Euclidean rotations is replaced by the simplectic group Sp(n) and the corresponding scalar product is

$$x.y = \sum_{i=1}^{n} \left(x_i y_i + \bar{y}_i x_i\right) \qquad (13)$$

Each field $\Phi(x)$ has a finite Taylor series and can be described by a set of independent N×N matrices. Thus the model of discretized random surfaces in -2n dimensions is equivalent to a zero-dimensional matrix model with n-fold supersymmetry.

In the special case $D = -2$ it is convenient to apply the "dimensional reduction" tric of [9], as it was suggested in [10]. This allows to reduce the model essentially to the gaussian ensemble of hermitean N × N matrices. Using the translational invariance we can write the derivative of the free energy as

$$-\frac{\partial F}{\partial \beta} = \left\langle \frac{\text{tr}}{N} \int dx\, d\bar{x}\; \Phi(0)\Phi(x) e^{x\bar{x}+\beta} \right\rangle \qquad (14)$$

By the standard manipulations[9,8] the r.h.s. can be evaluated as

$$-\frac{\partial F}{\partial \beta} = e^{\beta/2}\ \overline{\varphi_h(\varphi_h - h)} \qquad (15)$$

where the bar means the average with respect to the gaussian random N×N matrix $h=h^+$, and φ_h is the (unique) solution of the algebraical equation

$$e^\beta \varphi = h + V'(\varphi)\ . \qquad (16)$$

thus the $1/N^2$-expansion of the vacuum energy of the -2-dimensional N × N matrix model is generated by the $1/N^2$-expansion of the momenta

$$W_{2n} = \int d^{N^2}h\, e^{-\frac{N}{2}h^2} \Big/ \int d^{N^2}h\, e^{-\frac{N}{2}h^2}\ . \qquad (17)$$

of the gaussian random matrix h. Thus, the $1/N^2$ expansion of the D=-2 matrix model is generated by that of the gaussian ensemble of $N \times N$ matrices.

It follows from (16) that for a generic potential of the form (8) the coefficients φ_n behave as

$$\varphi_n \sim (\text{const})^{N} n^{-3/2} . \qquad (18)$$

On the other hand [8]

$$W_{2n}^{(g)} \propto n^{3g-3/2} 4^n/\sqrt{\pi} \qquad (19)$$

where $W_{2n}^{(g)}$ are the coefficients in the $1/N^2$ expansion of the moments (17). Combining the two asymptotics, we obtain from eq.(15)

$$\gamma(g,-2) = 2 + 3(g-1) \text{ for all } g \qquad (20)$$

In a similar way one can proceed to calculate the correlation functions of the matrix model.

Of special interest are the two-point correlators since they permit to calculate the mean square extent of the surface through eq.(6). The simplest one is

$$G(P_1,P_2) = \langle \text{tr}\,\Phi(P_1)\,\text{tr}\,\Phi(P_2)\rangle \qquad (21)$$
$$=\delta(P_1+P_2)\delta(\bar{P}_1+\bar{P}_2)\left[\overline{\frac{\text{tr}}{n}\varphi'_n+\frac{1}{2}(P_1\bar{P}_1+P_2\bar{P}_2)(\text{tr}\,\varphi_h)^2}\right]$$

where the bar meeans average with respect to h. Expanding this in the parameter $e^{-\beta}$ we can prove that the mean square extent $\langle X^2\rangle_A$ of planar surfaces grows logarithmically with A for a generic potential of the form (8).

The explicit calculations in the case of triangulations were presented in ref.[5].

8. CRITICAL BEHAVIOUR AND UNIVERSALITY

We have seen that at least in the solvable cases ($D=0,-2$) the critical exponent $\gamma(g,D)$ does not depend on the choice of the set $\mathcal{G}$ of discretizing graphs. Such universality allows us to hope that the model can indeed serve as a nonperturbative definition of the bosonic string. We have to compare eq.(11) and (18) with the one-loop perturbative result in the continuum theory[11]

$$\gamma(g,D) = 2 + (1-g)(D-19)/6 \ , \qquad (22)$$

which is qualitatively the same, but with different slope.

The result $\langle X^2\rangle_A \sim \ell n\, A$ obtained for $D = -2$ is in accord with the perturbative calculation[12]. Such logarithmic behaviour is typical for two-dimensional surfaces and is related to the infrared fdivergence of the two-dimensional massless field.

When D goes down to $-\infty$, no change of the critical behaviour is expected. In the quasiclassical limit $D \to -\infty$ the fluctuations of the internal metric are suppressed and the internal geometry freezes to that of a surface with constant curvature. On the other hand, we know that the $(\text{mass})^2$ of the lowest excitations of the bosonic string becomes negative above some critical dimension D_c, which is argued to be 1 [13]. Therefore we can expect that the critical behaviour of the model will become pathological at $D > D_c$.

In order to realize what is going on above the critical dimension, let us consider the limit $D \to \infty$. By the "tree formula" (12) the dominant graphs in this limit are those with minimal number of spanning trees[5]. They correspond to degenerated discretizations of the world sheet (Fig. 1) which consist of a number of thin strips. We can say that in this limit the string dissipates into a system of non interacting point particles.

We can try to improve our model by suppressing the discretizations containing long cuts (they correspond to points $i \in G$ with very high coordination number q_i). This can be done by multiplying the weight of each graph G in the r.h.s. of (1) by a factor

$$\left[\prod_{i \in G} q_i \prod_{u^* \in G^*} q_{u^*}\right]^{\alpha} \qquad (23)$$

where α is some real number. By taking α large we tollerate surfaces with locally flat internal curvature (all coordination numbers are close to their mean value). Therefore in the limit $\alpha \to \infty$ and D large and positive the surfaces which survive will be made out of long thin tubes. One can recognize here the phase of branch polymers which has been shown to be the only possible phase for the surfaces on a hypercubical lattice[13]. From the point of view of the continuum theory this phase can be explained by the existance of a nonperturbative mechanism for tachion condensation: a branched tree of tubes describes a of decay of the closed string through a number of tachions. The resulting phase diagram is shown in fig.2.

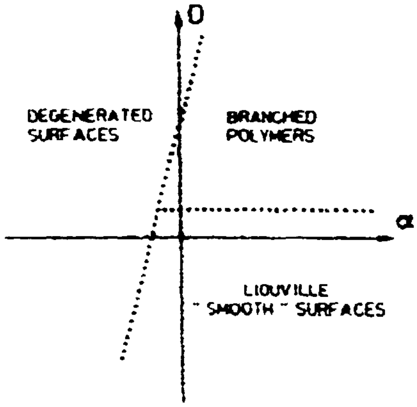


FIGURE 2

Tentative phase diagram for the random surfaces with dynamical discretization

9. CONCLUDING REMARKS

It has been proved in ref.[14] that the string tension does not vanish in the limit $\beta\to\beta_c$. We note that the same is true also for the mass of the lowest excitation in the phase of "smooth" surfaces. In the case D=-2 this follows from the explicite form of the two-point correlator (19). In fact, there are no reasons to expect that the masses will scale to zero with $\beta\to\beta_c$. We discretize the world sheet of the surface and not the embedding space and the scaling will affect only the distances on the world sheet.

An important question is whether the critical behaviour is universal with respect to the parameter α. In the naïve continuum limit the α-dependent factor should correspond to a term in the string action proportional to the square of the internal curvature[5] and therefore irrelevant. A numerical study of the zero-dimensional model with very high statistics[15] showed that $\gamma(0,0)$ does not depend on α up to the point where the transition to degenerated surfaces occurs. Moreover, the authors of [15] succeeded to prove analytically, using the known exact distribution of the coordination numbers, that $d\gamma/d\alpha=0$ at the point $\alpha=0$.

As for the critical dimension D_c, the only thing we know for sure is that it is positive. Unfortunately, the present status of the Monte Carlo data does not allow to locate it precisely. There is even no consensus whether the different phases are separated by sharp transitions or smooth crossover.

Finally, in my opinion, there are at least two possible directions of further investigation :

- try to find a discretization which allows to make connection with two-dimensional conformal field theories. This would permit to define the model above the critical dimension.
- introduce (fermionic ?) degrees of freedom on the world sheet of the surface to obtain a model with larger critical dimension (4 ?).

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