USSR STATE COMMITTEE FOR UTILIZATION OF ATOMIC ENERGY

INSTITUTE FOR HIGH ENERGY PHYSICS

И Ф В Э 88-39
ОТФ

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# NEW CONTINUAL ANALOGS OF TWO-DIMENSIONAL TODA LATTICES RELATED WITH NON-LINEAR INTEGRO-DIFFERENTIAL EQUATIONS

Serpukhov 1988

УДК 539.I.0I

М-24


Abstract

Saveliev M.V. New continual Analogs of Two-Dimensional Toda Lattices Related with Non-Linear Integro-Differential Equations: IHEP Preprint 88-39. - Serpukhov, 1988. - p. 4, refs. 5.

Continual "extensions" of two-dimensional Toda lattices are proposed. They are described by integro-differential equations, generally speaking, with singular kernels, depending on new (third) variable. The problem of their integrability on the corresponding class of the initial discrete system solutions is discussed. The latter takes place, in particular, for the kernel coinciding with the causal function.

Аннотация

Савельев М.В. Новые континуальные аналоги двумерных цепочек Тода, связанные с нелинейными интегро-дифференциальными уравнениями: Препринт ИФВЭ 88-39. - Серпухов, 1088. - 4 с., библиогр.: 5.

Предложены континуальные "расширения" двумерных цепочек Тода, описываемые интегро-дифференциальными уравнениями с, вообще говоря, сингулярными ядрами, зависящими от новой (третьей) переменной. Обсуждается вопрос об их интегрируемости на соответствующем классе решений исходной дискретной системы. Последнее имеет место, в частности для ядра, являющегося причинной функцией.

1. Besides the known (see, for example [1,2]) continual extensions of one-dimensional discrete systems of the Toda lattices type which do not lead out off the framework of differential equations and local integrals of motion it is possible to introduce a class of principally different integro-differential analogs of such systems. As far as I know they were not proposed earlier, although they probably are of interest for various physical applications, in particular, in particle physics, plasma and nonlinear optics. In distinction with the construction proposed in[3] the equations contain integration over a new variable which is a continuous limit of a lattice knot number rather than over the spatial argument of the corresponding wave or evolution equation.

Consider two-dimensional generalized Toda lattice with fixed end-points, $\partial^2 x_i/\partial z_+ \partial z_- = \exp(kx)_i$, $1 \leqslant i \leqslant r$, which is exactly integrable[4] for the matrices k coinciding with the Cartan matrices of the simple Lie algebras of rank r. The proposed continual analog of this system has the form

$$\partial^2 x(z_+, z_-;t)/\partial z_+ \partial z_- = \exp \int_L dt' K(t, t') x(z_+, z_-; t') \equiv \exp \hat{K} x, \qquad (1)$$

where $\hat{K}$ is some integral operator, generally speaking, with singular kernel. The latter should be understood in the sense of the principal value. New variable t is just a continuous limit of a lattice index, $x_i(z_+, z_-) \to x(z_+, z_-; t)$. The range of variation of t and the contour in (1) are not fixed yet. In the same way it is possible to write a continual analog of the periodic problem

$$\partial^2 x/\partial z_+ \partial z_- = \exp \hat{K} x - \exp(-\hat{K} x), \qquad (2)$$

and their multicomponent generalizations.

2. Let us appeal now to the integrability problem for non-linear integro-differential equations (1) and (2) whose

solution is highly nontrivial and is defined as in the discrete case by the properties of the operators $\hat{K}$. At first confine ourselves by consideration of equation (1). It is easy to get convinced that this equation can be represented in zero curvature form, $[\partial/\partial z_+ + A_+, \partial/\partial z_- + A_-]_- = 0$, with the functions

$$A_{\pm} = \int dt [ \pm \frac{1}{2} h(t) \partial x(t)/\partial z_{\pm} + X_{\pm}(t) \exp(\frac{1}{2}\hat{K}x(t))], \qquad (3)$$

taking values in the local part $\hat{G}$ of some Lie algebra G. (Here and in what follows the dependence of the function x on the arguments $z_{\pm}$ is omitted). The generators satisfy the commutation relations

$$[h(t), h(t')] = 0, \quad [h(t), X_{\pm}(t')] = \pm K(t', t) X_{\pm}(t'),$$

$$[X_+(t), X_-(t')] = \delta(t-t')h(t). \qquad (4)$$

They generate the whole infinite-dimensional Lie algebra G by repeated commutations $[X_+(t_n)[X_+(t_{n-1})\ldots[X_+(t_2), X_+(t_1)]\ldots]]$ . (Note, that in the simplest case with $\delta$-type kernel the algebra G coincides with the current algebra without Schwinger term which, however, can be involved in the play).

Application of the group-algebraic methods for investigation of the integrability problem for equation (1) requires the knowledge of the properties of these algebras. In particular, one needs the conditions on the kernels K with which the algebras have a finite (functional) growth. However as far as I know, such algebras were not considered previously and it is not known how to work with them. Thus, leaving the solution of the integration problem for equation (1) in general statement till "better days" confine ourselves by some heuristic considerations which lead to two interesting integrable examples.

3. Consider a requirement of presence of the characteristic (nonlocal on t) integrals on a class of the solutions to equation (1). It follows already for the second order integrals, that the operator $\hat{K}$ should be either symmetrizable, i.e., there is such a function $v(t)$ that $v(t)K(t,t') = v(t')K(t',t)$, or eigen for the function $\exp\hat{K}x$, i.e., $\hat{K}\cdot\exp\cdot\hat{K}x = c\cdot\exp\hat{K}x$, $c = const \neq 0$ (in further we put $c=2$ without waste of generality). In the first case the higher order integrals arise under the fulfilment of definite polynomial relations for the kernel. In the second case it

is obvious that $\hat{K}$ can be taken in the form $\hat{I}+i\hat{H}\equiv\hat{K}_{\pm}$, where $\hat{H}$ is the Hilbert operator, $\hat{H}^2=-\hat{I}$ or $\hat{K}^2=2\hat{K}$, i.e. $K_{\pm}(t,t')\equiv\equiv 2\delta_{\pm}(t-t')\equiv\pm i/\pi(t-t'\pm i\varepsilon)^{-1}$ is the causal function for zero mass. Such a choice of the kernel gives the simplest integrable example of equations (1) and (2).*)

Indeed, choosing $\hat{K}=\hat{K}_+$ (or $=\hat{K}_-$) and acting by it from the left on equations (1) and (2) we obtain for the functions $y\equiv\hat{K}x$ the Liouville and sh-Gordon equations, respectively. As a result the problem of finding the solutions to equations (1) and (2) for the operator under consideration reduces to the solution of the degenerated case of the singular characteristic integral equation with the Cauchy kernel,

$$\pm x(t)+\frac{1}{\pi i}\int_L dt'\frac{x(t')}{t'-t}=\pm y(t), \qquad (5)$$

where the dependence of functions x and y on variables $z_+$ and $z_-$ is omitted for brevity. Take as inhomogenity (y(t)) the well-known general solution of the Liouville equation and N-soliton solution of the sh-Gordon equation. Here arbitrary functions of $z_{\pm}$ of the first case and the parameters of the second one depend on the variable t. Equation (5) can be solved by the standard methods of the singular integral equations theory/5/ applicable to the considered case, if we impose the Hölder condition on y(t). In this the inhomogeneous boundary value problem is solvable if $y^-(t)\equiv\hat{K}_-y(t)=0$ or $y^+(t)\equiv\hat{K}_+y(t)=0$, respectively, for the upper or lower sign in equation (5). The general solution of (5) is $x(t)=y^+(t)-\Phi_-(t)$ and, $x(t)=-y^-(t)+\Phi_+(t)$, respectively. Here $\Phi_+(z)(\Phi_-(z))$ is an arbitrary function analytic inside (outside) the contour, whose boundary value $\Phi_+(t)$ $(\Phi_-(t))$ satisfies the condition $\hat{K}_-\Phi_+(t)=0$ $(\hat{K}_+\Phi_-(t)=0)$. Therefore the integration procedure for equation (5) puts the corresponding additional requirements on the analytic t-dependence of the general solutions of the Liouville equation and soliton solutions of the sh-Gordon equations.

*) It is interesting to note that if in the equation for the induced Langmuir plasmon scattering by ions (see, i.e. review by V.E.Zakharov, S.L.Musher, and A.M.Rubenchik in Phys. Repts. 129, 1985, 287) with zero linear damping, $\partial n_k/\partial t=n_k\int dk' T_{kk'}n_{k'}$, which is tightly related in a certain sense with (1), we formally take as a matrix element $T_{kk'}$ the Cauchy kernel, the equation reduces to the Emden-Fowler equation. This is evident due to the Poincaré-Bertrand formula/5/.

The author would like to thank O.I.Bogoyavlensky, A.A.Kirillov, I.M.Krichever, A.N.Leznov and A.V.Razumov for fruitful discussions.

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Received January 11, 1988.

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Новые континуальные аналоги двумерных цепочек Тода, связанные с нелинейными интегро-дифференциальными уравнениями.

Редактор А.А.Антипова. Технический редактор Л.П.Тимкина.

Корректор Е.Н.Горина.

Подписано к печати 10.02.88. Т-04548. Формат 60х90/16. Офсетная печать. Печ.л. 0,25. Уч.-изд.л. 0,41. Тираж 250. Заказ 140. Индекс 3622. Цена 6 коп.

Институт физики высоких энергий, 142284, Серпухов Московской обл.

6 коп. Индекс 3622

П Р Е П Р И Н Т 88-39, И Ф В Э, 1988