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SOME EXAMPLES OF INSTANTONS IN SIGMA MODELS.
II: K3 MANIFOLDS

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Examples of Ricci-flat Kähler manifolds of type K3 are given including new ones.

Fig. - 19, ref. -

In this paper we describe instantons on some manifolds of type K3. The count of zero modes in two-dimensional sigma-models on such manifolds was produced in our preceding publication [1]. The paper can be divided into two parts. In the first we expose the necessary facts on K3 manifolds, essentially following the monographs [2,3], and in the second we describe instanton confi9 gurations.

1. Basic Facts on K3 Manifolds [2,3]

1. By a K3 manifold one means a two-dimensional simply connected complex manifold M with trivial first Chern class $C_1(M)$ All such manifolds are Kähler. By Yau's theorem [4] they admit a Kähler Ricci-flat metric. This metric is uniquely determined by its cohomology class. If one gives up the condition that M be simply connected, then, besides K3, only complex Tori $T^2_{\mathbb{C}}$ will fulfill the conditions [1)]. Later we will see, how one can construct K3-manifolds from them.

2. It follows from Ricci-flateness that the homology group of a K3 is SU(2). Therefore there exists a nowhere vanishing holomorphic (2,0)-form on it (the invariant of SU(2)). The same thing is to say that the highest exterior power of the cotangent

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1) By $C_1(M)$ we mean the divisor class of an exterior meromorphic 2-form in $H^2(M,\mathbb{Z})$. The class $C_1^{DR}(M)$ in the De Rham cohomology group $H^2_{DR}(M,\mathbb{R})$ is a more coarse invariant. For instance, $C_1^{DR}(M)=0$ for Enriques surfaces , which satisfy the condition $2C_1(M)=0$, but $C_1(M)\neq 0$.

bundle $K=\Lambda^2(T^*M)$, the canonical bundle, is holomorphically trivial: $K\equiv 0$. The Hodge numbers $h^{0,2}$,$h^{2,0}$ are equal to one: $h^{0,2}=h^{2,0}=1$. According to the Noether formula [2], p.504) one can get the Euler number from the Hodge numbers:

$$h^{0,0}-h^{0,1}+h^{0,2}=\frac{\chi(M)+K\cdot K}{12}$$

Because of the simple connecteness of M, we have $h^{0,1}=0$, and taking into account that K = 0, we get $\chi(M)=24$.

On the other hand, we have for the second Betti number $b_2=\chi-2b_0=22$. Hence $h^{1,1}=b_2-2h^{0,2}=20$.

So the Hodge diamond of a K3 surface has the form:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

3. As was already noted, the Yau metric $G_{a\bar{b}}$ is uniquely defined by its cohomology class $G_{a\bar{b}}=i\,b_{a\bar{b}}$, provided a complex structure J^a_b to be fixed. As $h^{1,1}=20$, we obtain the 20-dimensional family of Ricci-flat metrics. Note that such metrics describe compact gravitational instantons[5].

4. Now we come to the discussion of the moduli space of complex structures. A complex structure is defined by the cohomology class of a two-form ω (2,0) which is holomorphic in this structure. The moduli space of a K3 can be locally realized as a submanifold in $\mathbb{C}P^{21}$ whose homogenous coordinates are $(a^i, i=1,\dots,22)$, $a^i=\int_{\gamma_i}\omega$ where $\{\gamma_i\}$ is a basis

of $H_2(M,Z)$. The exterior square $\omega\wedge\omega$ of a holomorphic two-form ω is zero, so this submanifold is a quadric hypersurface. The matrix of the quadratic form defining the quadric coincides with the intersection matrix of the two-cycles in the K3. The latter is the Gramm matrix of an even auto-dual lattice of signature $(3^+, 19^-)$. Such form is unique and equal to $3U+2\Gamma_8$ in an appropriate basis, where $U=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and Γ_8 is the negative Cartan matrix of Lie algebra E_8. Thus, the moduli space of complex structures on K3 manifolds is a compact 20-dimensional complex manifold. All K3 surfaces are diffeomorphic to each other and differ when choosing the complex structure and metric.

5. Now we turn to the problem of algebraicity of K3 which states as follows: given a K3 manifold, when is it possible to realize it as a zero set of homogeneous polynomials in $\mathbb{C}P^n$? Let $a=(a^1,\dots,a^{22})\in\mathbb{C}P^{21}$ be a vector in the moduli space $\left(a^i=\int_{\gamma_i}\omega\right)$ defining a K3 manifold M_a. For M_a to be algebraic it is necessary that there would exist at least one non-zero (holomorphic) algebraic cycle $C=\sum c_j\gamma_j$ on M_a, and for such cycle

$$\int_C \omega=0 \longleftrightarrow \sum a^j c_j=0. \qquad (*)$$

The number $\rho(M_a)$ of linearly independent integer solutions of this equation is called the Picard number of M_a. It varies from 0 to 20. For the Picard number to be fixed one needs imposing additional restrictions $(*)$ on possible deformations of the complex structure. For instance, a generic algebraic K3 (p=1) has the 19-dimensional moduli space.

6. Algebraic cycles form a subgroup in $H_2(M,\mathbb{Z})$ denoted by S or $\operatorname{Pic} M$. It is a sublattice in $H_2(M,\mathbb{Z})$ of signature $(1,n)$, $0 \leqslant n \leqslant 19$:

$$S = S(M) = \operatorname{Pic}(M) \subset H_2(M,\mathbb{Z}).$$

The orthogonal complement T to S in $H_2(M,\mathbb{Z})$ is called the lattice of transcendental cycles. From the formula for the genus $p_a(C)$ of an irreducible curve C on M

$$p_a(C) = \frac{C \cdot C}{2} + 1$$

one can deduce the following correspondence between lattice points and curves on K3:

$(C \cdot C) = -2$ $\qquad$ $p_a(C) = 0$ (rational curves),

$(C \cdot C) = 0$ $\qquad$ $p_a(C) = 1$ (elliptic curves),

and so on.

We see that S is an even **Lorentz** lattice. Non-singular rational curves correspond to 2-roots of S, i.e. lattice points with $(e,e) = -2$, to nonsingular tori lattice points on the isotropic cone, and to curves of higher genus the lattice points in the interior of the cone. It immediately follows from such considerations that there are no smooth curves of genus 0 or 1 on a generic algebraic K3 (p=1, n=0).

Remind also that a curve C of genus 0 on a K3 manifold has no deformations, or it is rigid, and a genus 1 curve has a one-parameter family of deformations. We shall see later that it is parametrized by $\mathbb{C}P^1$.

7. Let Π_2 be a positive 2-root system of lattice S defined by $\Pi_2=\{e:(e,e)=-2,\ |e|\neq\emptyset\}$, where $|e|$ denotes the set of effective (or positive) divisors representing the divisor class $e\in S=\operatorname{Pic}M$. Let $O^+(S)$ denote a subgroup of the transformation group of the lattice which sends the upper half of the isotropic cone to itself and preserves the root system Π_2:

$$O^+(S)=\left\{\alpha \;\middle|\; \begin{array}{l} \text{i) } (\alpha x,\alpha x)>0,\ (x,\alpha x)>0,\ \text{if } (x,x)>0 \\ \text{ii) } \alpha(\Pi_2)=\Pi_2 \end{array}\right\}$$

The automorphism group Aut M of M is isomorphic to a central extension of $O^+(S)$ by some finite group acting on transcendental cycles from T whose structure is insignificant for us [6]. Thus, Aut (M) acts on the curves of genus p by permutations. Finite automorphism groups were completely classified in [9, 7] while infinite ones were studied in [8, 9]. If $\rho(M)=20$, then $O^+(S)$ is an infinite group.

8. Let h be a positive class in PicM (i.e. $|h|\neq\emptyset$) and $(h,e)\geqslant 0$ for all $e\in\Pi_2$. Then one has the map φ_h from M to $\mathbb{C}P^N$ where $N=\frac{1}{2}(h,h)+1$.

This map is either a birational isomorphism onto a surface of degree (h,h), or a double covering of a surface of degree $\frac{1}{2}(h,h)$ [10]. The map φ_h sends each point $x\in M$ to the hyperplane of effective divisors $D\in|h|$ passing through x, and hyperplanes in the projective space $|h|\simeq\mathbb{C}P^N$ also form the projective space $|h|^\vee\simeq\mathbb{C}P^N$. For more details see [11]. We will use only the fact of existence of φ_h.

9. It is well known that for any $\rho\geqslant 2$ there is an al-

gebraic manifold M_a such that it has a curve of arithmetic genus p on it, but no curves of smaller arithmetic genera. On such a K3-manifold each smooth curve has genus $\geqslant$ p.

10. Let p_m denote the smallest possible value of the arithmetic genus of irreducible curves of genus $\geqslant 2$ on M_a. Clearly, $p_m \geqslant 2$.

The following statements hold:

(i) If $p_m = 2$, then M_a can be realized as a two-sheet cover of $\mathbb{C}P^2$ branched in a curve of degree 6 in $\mathbb{C}P^2$ (see Fig.1).

(ii) If $p_m > 2$ and the curve C of genus p_m on M is non-hyperelliptic, then there is a birational map from M into $\mathbb{C}P^{p_m}$ such that the degree of its image is equal to $2(p_m - 1)$.

(iii) If $p_m > 2$ and C is hyperelliptic, then M_a can be mapped birationally into $\mathbb{C}P^{4p_m-3}$ with degree $8(p_m-1)$. Besides, it can be mapped by a 2-to-1 map onto a rational surface of degree $p_m - 1$ in $\mathbb{C}P^{p_m}$.

For a generic M_a, the classes of all curves on M_a are multiples of the class of C, $\rho(M_a) = 1$. In this case C is a hyperplane section of M_a in $\mathbb{C}P^{p_m}$ (or the inverse image of a line in $\mathbb{C}P^2$, if $p_m = 2$, which we will also call a hyperplane section). But for special M_a the Picard number may jump and even attain the maximal value $\rho(M_a) = 20$. In this case M_a acquires new algebraic curves, other than hyperplane sections. For examples see Figs.2,3.

2. Examples of K3 Manifolds and Instantons on them

One can in an obvious way identify algebraic curves of genus p on K3 manifolds as holomorphic instantons of genus p. The degree of a curve in this identification is nothing else than the topological charge of the instanton. Some general properties of curves on K3 were described in [1]. Here we are conserned with curves on some special K3 manifolds.

We will make use of birational transformations of K3 manifolds under consideration. A birational transformation is a rational (i.e. not everywhere defined) map $f: M \to N$ which has the inverse $g: N \to M$ such that $f \circ g$ and $g \circ f$ are identical maps on N and M, correspondingly. Two manifolds which can be birationally transformed to each other are said to be birationally equivalent. The birational transformations can contract some curves to points or, conversely, blow up points to curves. They also can charge intersection numbers of curves and their degrees.

1. Consider a K3 manifold M embedded in $\mathbb{C}P^n$ by the linear system of the hyperplane section C. As the vector bundle $[C]$ is positive, by Kodaira vanishing theorem $h^1(M,C)=h^2(M,C)=0$. According to Riemann-Roch and the equality $K\cdot K=0$, we have

$$n+1=h^0(M,C)=\frac{C\cdot C}{2}+\chi(\mathcal{O}_M),$$

so,

$$C\cdot C=\deg M=2n-2.$$

Let n=3. Then M is a quartic in $\mathbb{C}P^3$ with the equation

$$x^4+y^4+z^4+w^4+a_{ijkl}x^iy^jz^kw^l=0$$

The space of symmetric complex tensors a_{ijkl} has dimension 34. But the tensors are defined up to transformations from $SL(4,\mathbb{C})$. So we have a family of K3 depending on 34-15=19 parameters.

One can show that each quartic posesses a holomorphic (0,2) form. This is a consequence of the fact that the canonical class of $\mathbb{C}P^3$ is K = -4H and the adjunction formula

$$K_M=\left(K_{\mathbb{C}P^3}+M\right)\Big|_M=\left(-4H+4H\right)\Big|_M=0.$$

Similarly, for n=4 we have a sextic surface in $\mathbb{C}P^4$, which is the complete intersection of a quadric and cubic[2]. Some particular examples of these surfaces will be discussed later. The Picard number may change between its minimal value 1 for a generic sextic and 20.

Let M be a generic quartic in $\mathbb{C}P^3$, i.e. $\rho(M)=1$. Consider all its hyperplane sections. We get a three-parameter family of curves of genus 3. But these curves can degenerate to curves of smaller geometric genus. If the hyperplane H in $\mathbb{C}P^3$ is tangent to M in one point, the intersection $H\cap M$ is a curve of genus 2. Thus, we get a two-parameter family of curves of genus 2. If H is tangent to M in two points, we get curves of genus 1 paramwtrized by some curve C. Finally, if M is tritangent plane, the hyperplane section degenerates to a curve of geometric genus 0. The number of such hyperplanes is finite. All curves appearing in such a way are singular. On a generic M there are no smooth curves of genus 0,1,2. Note also that in this case any curve E on M is the intesection of M and some surface of degree d in $\mathbb{C}P^3$ and

$$p_a(E) = \frac{E\cdot E}{2} - 1 = 2d^2 + 1.$$

The dimension of the family of such curves is $2d^2 + 1$, so it seems likely that there always exist degenerations to curves of genus 0 with nodes.

If M is the Fermat surface

$$x^4 + y^4 + z^4 + w^4 = 0$$

then $\rho(M) = 20$. In this case, the transcendental cycles form the lattice $T = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$, and algebraic ones - the lattice $S = (-2E_8) \oplus (-T) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The automorphism group on this manifold is infinite (see Sec.1.6). We are going to indicate all lines on it. The equations of a generic line in $\mathbb{C}P^3$ can be written in the form

$$z_1 = u, \quad z_2 = v, \quad z_3 = au + bv, \quad z_4 = cu + dv.$$

The condition that the line lies in M gives the equation

$$u^4 + v^4 + (au + bv)^4 + (cu + dv)^4 = 0$$

from which we get:

$$a^4+c^4+1=0,\quad b^4+d^4+1=0,$$
$$a^3b+c^3d=0,\quad ab^3+cd^3=0,$$
$$a^2b^2+c^2d^2=0$$

Now, if a) $c\neq 0$ and $b\neq 0$, then

$$\left(\frac{a}{c}\right)^3=-\frac{d}{b},\quad \frac{a}{c}=-\left(\frac{d}{b}\right)^3,\quad \left(\frac{a}{c}\right)^2=\left(\frac{d}{b}\right)^2$$

and therefore

$$a=0,\quad d=0,\quad b^4=-1,\quad c^4=-1$$

But if b) $a\neq 0$, $d\neq 0$ then

$$b=c=0,\quad a^4=-1,\quad d^4=-1.$$

Thus, we get 48 lines:

A. $z_1=u,\ z_2=v,\ z_3=\varepsilon_i u,\ z_4=\varepsilon_k v,\ \varepsilon_i^4=-1$

B. $z_1=u,\ z_2=\varepsilon_i u,\ z_3=v,\ z_4=\varepsilon_k v,$

C. $z_1=u,\ z_2=v,\ z_3=\varepsilon_k v,\ z_4=\varepsilon_i u.$

Now we turn to the description of plane curves of degree 2, or conics, in M. Consider the curves defined by the equations

$$z_1=u,\quad z_2=v,\quad z_3=au+bv,\quad z_4^2=cu^2+duv+e$$

Then we get the identity in u and v:

$$u^4 + v^4 + (au + bv)^4 + (cu^2 + duv + ev^2)^2 = 0,$$

from which it follows

$$a^4 + c^4 + 1 = 0, \quad b^4 + e^4 + 1 = 0,$$

$$4a^3b + 2cd = 0, \quad 4ab^3 + 2de = 0,$$

$$6a^2b^2 + d^2 + 2ce = 0$$

Note that this system has obvious solutions when $cu^2 + duv + ev^2$ is a perfect square. We omit them and find out that

$$6a^2b^2 + d^2 + 8\frac{a^4b^4}{d^2} = 0,$$

or

$$(d^2 + 3a^2b^2)^2 = a^4b^4$$

and hence $d^2 = -3a^2b^2 \pm a^2b^2$. The solution with minus yields pairs of the old curves and $d^2 = -2a^2b^2$ yields

$$\alpha) \quad a = b = \pm 1, \quad c = e = d = \pm i\sqrt{2}$$

$$\beta) \quad a = -b = \pm 1, \quad c = e = -d = \pm i\sqrt{2}$$

Thus, we obtain a set of plane conics lying in M. There are 128 such conics. They are also curves of genus 0.

The pencils of planes passing through lines in M yield 48 one-parameter families of curves of degree 3 and genus 1.

2. The other class of K3-surfaces consists of double coverings $\pi : M \to \mathbb{C}P^2$ branched along a curve C of sixth degree:

$$z_0^2 + a_{i_1 \dots i_6} z_{i_1} \cdot z_{i_2} \dots z_{i_6} = 0 \quad (i_1, \dots i_6 = 1, 2, 3)$$

The canonical class of such M is $K_M = \pi^* K_{CP^2} + \pi^{-1}(C) = \pi^*(-3M) + \pi^{-1}(C)$. This is the divisor of a meromorphic function on M, say, z_0 / z_1^3, so the canonical class of M is trivial. For the Euler number, we have

$$\chi(M) = 2(\chi(P^2) - \chi(C)) + \chi(C) = 24$$

Hence, M is really a K3 surface.

Variations of C give rise to variations of the complex structure on M. The equation of a curve of degree 6 in three variables depends on $\binom{8}{6} - 1 = 27$ parameters. Taking into account the action of $SL_3(\mathbb{C})$ depending on 8 parameters, we get 19-parameter family of K3 manifolds, as above. Here the simplest curve is a hyperplane section, a hyperelliptic curve with six branch points of genus 2. The set of such curves in M is parametrized by $\mathbb{C}\mathbb{P}^2$.

Let D be a line which is tangent to C in one point (Fig.5). Then the arithmetic genus of $\pi^{-1}(D)$ is 2, as before, but the geometric genus is 1. The set of such (singular) curves is parametrized by the curve C with some points identified.

If, finally, D is a bitangent to C (see Fig.6), then the geometric genus of $\pi^{-1}(D)$ is zero. By Plückar formula, there are $\frac{1}{2} d(d-2)(d^2-9)$ bitangents to a curve of degree d. For d=6 this number equals 324.

3. Now we show how one can construct K3 manifolds from complex tori $T_2^{\mathbb{C}}$

a) $T_2^{\mathbb{C}} = T_1^{\mathbb{C}} \times T_1^{\mathbb{C}} \qquad T_1^{\mathbb{C}} = \{1, i\}$

(the generators of the torus in $\mathbb{C}$ are indicated). The automorphism $\sigma: z_i \to -z_i$ acts on this torus. It preserves the form $dz_1 \wedge dz_2$. 16 points $\{ z_j^{(\nu)} = (0, \frac{1}{2}, \frac{i}{2}, \frac{i+1}{2}) \}$ are fixed under σ. The quotient $T_2^{\mathbb{C}}/\sigma = X_2'$ is called a Kummer surface. It has sixteen singular points. Each of them can be resolved by pasting in a copy of $\mathbb{CP}^1$. The resolved surface X_2 is a K3 surface [2]. The pasted $\mathbb{CP}^1$'s represent 16 linearly independent algebraic cycles in Fig.2. Each of them generates an additional direction in deformations of the complex structure, as was described in Sec.1.5. The Euler number equals to

$$\chi(X_2) = \frac{\chi(T_2^{\mathbb{C}}) - 16}{\mathrm{ord}\ \sigma} + \chi(\mathbb{CP}^1)\cdot 16 = \frac{0-16}{2} + 2\cdot 16 = $$
$$= 24$$

The matrix of trascendental cycles is $T = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ and
$S = (-2E_8) \oplus \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
For the Picard number we have $\rho(X_2) = 20$.

We present now some examples of instantons of genus 0. There are 16 instantons $E_{\mu\nu}$ $(\mu, \nu = 1, \ldots, 4)$ coming from the

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[2] This is true for any two-dimensional torus, not only for a decomposable $T_2^{\mathbb{C}} = T_1^{\mathbb{C}} \times T_1^{\mathbb{C}}$. The decomposability of the torus will be used below to define the curves F_ν, G_ν.

resolution of singularities (see Fig.7). Besides, there are rational curves $F_\mu = \{ Z^{(\mu)}, T_1^{C}/\sigma \}$ and $G_\mu = \{ T_1^{C}/\sigma , Z^{\mu} \}$. They have the following intersection indoces: $G_\mu F_\nu = 0$, $E_{\mu\nu}^2 = G_\mu^2 = F_\mu^2 = -2$, $F_\mu E_{\gamma\nu} = \delta_{\mu\gamma}$, $G_\mu E_{\gamma\nu} = \delta_{\mu\nu}$ (see Fig.9). One can represent the scheme of rational curves by the Coxeter graph (Fig.10), 24 vertices of which correspond to the curves $F_\mu, G_\nu, E_{\mu\nu}$.

b) Let $T_1^{C} = \{1, \omega = e^{2\pi i/3}\}$ and σ: $(Z_1 \to \omega Z_1, Z_2 \to \omega^2 Z_2)$. Then σ has 9 fixed points $Z_j^{(\nu)} = \left(0, \frac{i}{\sqrt{3}}, \frac{1}{2}\left(1 + \frac{i}{\sqrt{3}}\right)\right)$. This transformation also preserves the canonical form $dZ_1 \wedge dZ_2$. As before, after resolving singular points of the orbifold $X_3' = T_2^{C}/\sigma$ we obtain a smooth manifold X_3. The resolution is made by pasting into each singular point a pair of rational curves $(E_{\mu\nu}, E'_{\mu\nu})$. with the intersection matrix $-A_2$. The Euler number is

$$\chi(X_3) = \frac{0-9}{3} + \chi\left(\cup\{E_{\mu\nu}, E'_{\mu\nu}\}\right) = -3 + 9\cdot 3 = 24$$

This manifold has the Picard number $\rho(X_3) = 20$. The intersection matrix of transcendental cycles has the form $T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and that of algebraic ones $-2E_8 \oplus (-T) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Similar to the above, we have 24 instantons of genus 0 (see Fig.8):

$$E_{\mu\nu}^2 = E'^2_{\mu\nu} = G_\mu^2 = F_\nu^2 = -2, \; E_{\mu\nu} E'_{\lambda\eta} = \delta_{\mu\lambda} \delta_{\nu\eta} ,$$

and so on. One can describe them alternatively using the Coxeter graph 9. The indicated cycles correspond to 2-roots of the lattice S (see Sec.1.6). It turns out that these roots generate all the lattice S. As follows from the above scheme, the Coxeter graph has the form as shown in Fig.12.

The group $Aut\, X_3$ (see [9]) is infinite and contains the group of symmetries of the Coxeter graph.

Let h be the sum of all the 2-roots from the Coxeter graph. Then, for any root from the graph,

$$(h, e) = 0 \quad \text{or} \quad 1$$

The corresponding vertices are denoted 0 or ⊖ . Furthermore, (h, h) = 6 and h corresponds to a curve of genus 4. According to Sec.1.8 we have the birational isomorphism

$$\varphi_h : \quad X_3 \to \mathbb{C}P^4$$

As was noted above, K3 surfaces in $\mathbb{C}P^4$ are complete intersections of a quadric and a cubic. In the case under consideration the equations of the image of φ_h can be written as

$$Y^1 : \begin{array}{l} y^2 + x_1^2 + x_2^2 + x_3^2 - 2(x_1x_3 + x_3x_2 + x_1x_2) = 0 \\ z^3 - x_1x_2x_3 = 0 \end{array}$$

The rational curves corresponding to vertices 0 go over to singular points on Y^1, and those corresponding to vertices ⊖ go over to conics.

There is another map φ_h from X_3 to $\mathbb{C}P^4$. Take for h the sum of roots with weights indicated in Fig.13. Then (h,e) = = 0,2 or 3, corresponding to vertices of type 0, ◎ or ● .

As before, $(h,h) = 6$.

The image of X_3 is the surface

$$Y: \begin{cases} y^2 + x_2x_3 + x_1x_3 + x_1x_2 = 0 \\ z^3 - x_1x_2x_3 = 0 \end{cases}$$

The rational curves O go over to three singular points on Y, ◎ to three conics, and ● to three cubics. One can easily see in the graph that the singular points have the E_6 resolution scheme and that each conic passes through two singular points.

c)Consider the torus $T_2^{\mathbb{C}} = T_1^{\mathbb{C}} \times T_1^{\mathbb{C}}$, $T_1^{\mathbb{C}} = \{1, i\}$ and $\sigma: (z_1 \to iz_1,\ z_2 \to -iz_2)$, $\sigma^4 = 1$. As the form $dz_1 \wedge dz_2$ is preserved by the action of σ , the quotient $X_4' = T_2^{\mathbb{C}}/\sigma$ can be resolved to give a K3 surface X_4 . The transformation σ has 4 fixed points, and σ^2 has 12 more (see a)). The Picard number $\rho(X_4) = 20$ and $T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ The lattice S of algebraic cycles is $-2E_8 \oplus \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The resolution of singularities yields the following set of rational curves/8/. Let $F(\sigma)$ and $F(\sigma^2)$ be sets of fixed points of σ and σ^2 in $T_2^{\mathbb{C}}$, correspondingly. Then

$$\operatorname{ord} F(\sigma) = 4, \qquad \operatorname{ord} F(\sigma^2) = 16$$

$$F(\sigma) = \{(v_i, v_j), i,j = 1,2\}, \ v_1 = 0,\ v_2 = \frac{\sqrt{2}}{2} e^{\frac{i\pi}{4}}$$

Denote the canonical map $T_2^{\mathbb{C}} \to X_4'$ by π . To describe the singular set of of X_4' we note that the set $\pi(F(\sigma)) = \{e_{ij}\}, i,j=1,2$ consists of 4 points of type A_3 and the set $F(\sigma^2)$ is divided into 7 orbits under the action of σ : four orbits e_{ij} from one point and six orbits from two points (see Fig.7):

$$\pi F(\sigma^2) = \{ e_{ij}, z_1, .., z_6 \}.$$

The singular points z_j are of the type A_1. Besides, the 4.3+6 = 18 rational curves pasted into singular points by the resolution, there are 6 more rational curves

$$F_j = \{V_j, T_1^c/\sigma\}, \quad G_j = \{T_1^c/\sigma, V_j\} \quad j=1,2$$

$$F_3 = \{(3,4)\times T_1^c/\sigma\}, \quad G_3 = \{T_1^c/\sigma, (3,4)\}$$

where (3,4) denotes the orbit of σ in T_1^c. Thus, there are 24 components $\mathbb{P}^1$. Their intersection scheme is depicted in Fig.14. The X_4 contains also elliptic curves forming a pencil with $\mathbb{P}^1$ as a base and three degenerate fibers, as shown in Fig.15.

More complete information on rational curves can be obtained from the structure of the lattice S /9/. Consider the Coxeter graph woth 25 vertices (see Fig.16). The vertices correspond to 25 smooth rational curves, and the deges to their intersection points. There are 20 more rational curves, each meeting four curves from the Coxeter graph. Applying transformations from the infinite group $\mathrm{Aut}\,X_4$ to these 45 curves one gets an infinite sequence of smooth rational curves on X_4 (for description of $\mathrm{Aut}\,X_4$ see /9/).

There are two different projective models Y, Y' of X_4 in $\mathbb{C}P^3$. The first comes from the class $h \in \mathrm{Pic}\,X_4$

whose intersection indeces with curves of the type $\bigcirc$, $\ominus$, $\circledcirc$ are 0,1,2, respectively, and (h,h) = 4. The equation for it is

$$Y: \quad x_0^4 + x_1 x_2 x_3 (x_1 + x_2 + x_3) = 0$$

The surface Y has 6 singular points of the type A_3 . The curves of the type $\ominus$ correspond to 5 lines (curves of degree 1 in $\mathbb{C}P^3$) on Y. Three curves of the type $\circledcirc$ correspond to conics on Y.

The other quartic birationally equivalent to X_4 has the equation

$$Y': \quad x_0^4 = x_2^2 x_3^2 + x_3^2 x_1^2 + x_1^2 x_2^2 - 2 x_1 x_2 x_3 (x_1 + x_2 + x_3)$$

The cycle h' defining the map to Y' has the intersection indices with the vertices O, $\circledcirc$, $\bullet$, 0,2,4, respectively (see Fig.16). The coordinates of h' with respect to vertices of the graph are indicated as their weights.

The exceptional curves that go over to three singular points of the type E_6 correspond to the vertices of the type O. 6 conics correspond to the vertices of the type $\circledcirc$, and one rational curve of degree 4 corresponds to the vertice $\bullet$. Each conic passes through two singular points.

d) Let $T_2^{\mathbb{C}} = T_1^{\mathbb{C}} \oplus T_1^{\mathbb{C}}$, where $T_1^{\mathbb{C}}$ is the torus from b), $T_1^{\mathbb{C}} = \{1, \omega = e^{2\pi i/3}\}$, but σ is now the automorphism of order 6:

$$\sigma(z_1, z_2) = (a z_1, \varepsilon^{-1} z_2), \quad \varepsilon = \exp\left(\frac{\pi i}{3}\right)$$

We have the following numbers of fixed points:

$$\operatorname{ord} F(\sigma) = 1, \quad \operatorname{ord} F(\sigma^2) = 9, \quad \operatorname{ord} F(\sigma^3) = 16$$

The first equality is verified immediately, the others were obtained in examples b) and c). One can use the formulae of /12/ to calculate the Euler number of the resolved surface K. Let $\pi : T_2^{\mathbb{C}} \to K'$ denote the canonical map. Then $\pi F(\sigma) = P$ is a point of type A_5. The set $F(\sigma^2)$ is divided into 5 orbits: P, Q_1, Q_2, Q_3, Q_4 (see Fig.8). As $(\sigma^2)^3 = 1$, the points Q_i are of type A_2. The set $F(\sigma^3)$ is divided into 6 orbits: P, $R_1, \dots, R_5$. These are of type A_1. The resolution scheme is shown in Fig.17. Here again we have 5 + 2.4 + 5 = 18 exceptional components. There are also 6 more rational curves F_1, F_2, F_3, G_1, G_2, G_3, where $F_1 = (1, T_1^{\mathbb{C}}/\sigma)$, $F_2 = (\{2,3\}, T_1^{\mathbb{C}}/\sigma)$, $F_3 = (\{4,5,6\}, T_1^{\mathbb{C}}/\sigma)$, and G_j are obtained from F_j by permutation of factors. These 24 curves meet each other in accordance with the scheme in Fig.18. One can easily find also instantons of genus 1. They are parametrized by $\mathbb{P}^1$ and from an elliptic pencil with three degenerate fibers (see Fig.19).

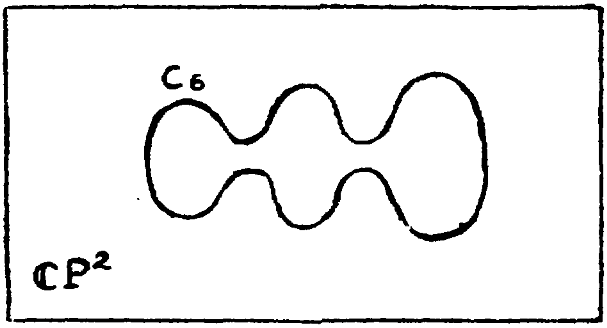

Fig. 1. A K3 manifold doubly covering $\mathbb{C}P^2$ with a curve C_6 of degree G as a branch locus.

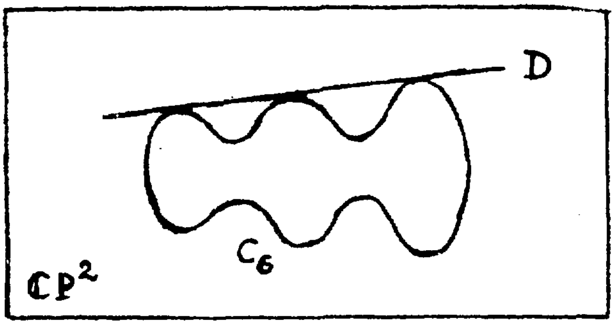

Fig. 2. A special choice of the curve C_6 with a tritangent line.

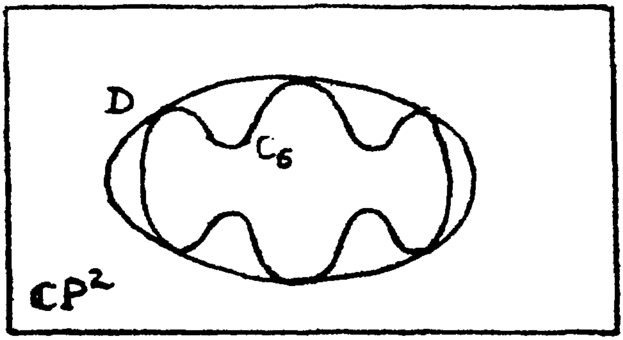

Fig. 3. A special choice of the curve C_6 such that there exists a conic tangent to C_6 at 6 points.

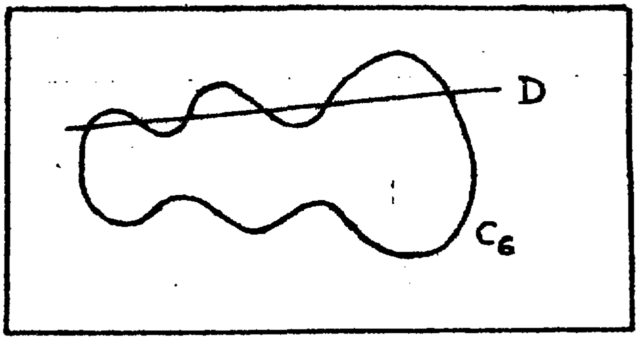

Fig.4. The hyperplane section D is a hyperelliptic curve with 6 branch points.

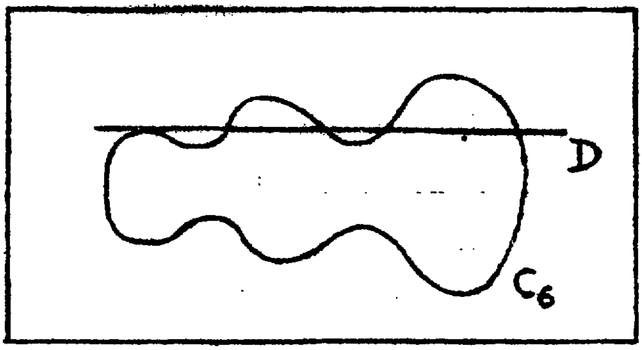

Fig.5. D is tangent to C_6 at one point.

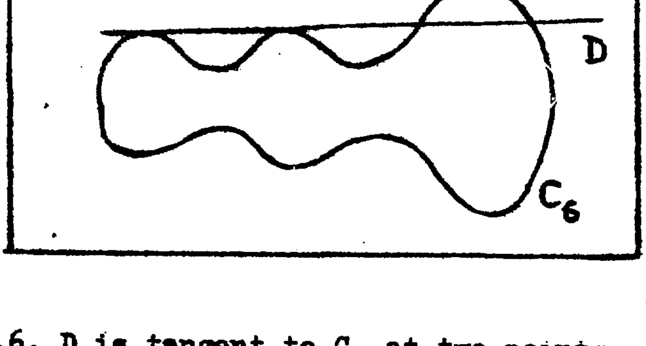

Fig.6. D is tangent to C_6 at two points.

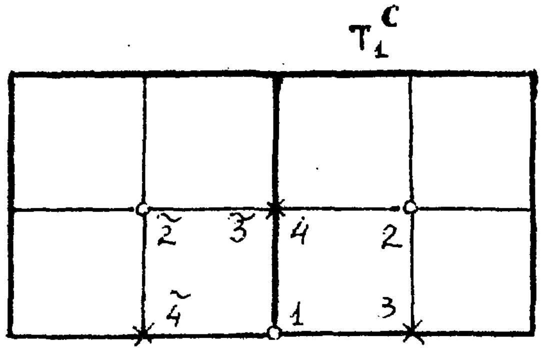

Fig.7. On the one-torus $T_1^{\mathbb{C}}$ the points 1 and 2 stay fixed under the transformation $\sigma: z \to iz$, but the points 4 and 3 are permuted. On the direct product $T_2^{\mathbb{C}} = T_1^{\mathbb{C}} \times T_1^{\mathbb{C}}$ we have 4 orbits from one point $E_{ij} = \{(1,1), (2,2), (1,2), (2,1)\}$ and 6 orbits from two points: $R_1 = \{(1,3),(1,4)\}$, $R_2 = \{(2,3),(2,4)\}$, $R_3 = \{(3,1),(4,1)\}$, $R_4 = \{(3,2),(4,2)\}$ $R_5 = \{(3,3),(4,4)\}$, $R_6 = \{(4,3),(3,4)\}$

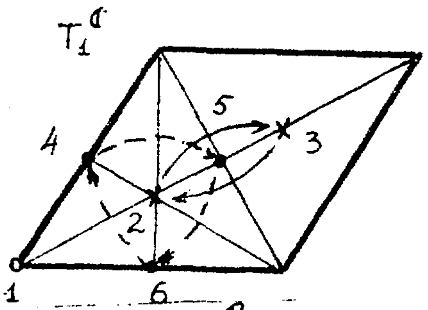

Fig.8. On the torus $T_1^{\mathbb{C}}$ the fixed point set has the following structure: $F(\sigma) = (1)$, $F(\sigma^2) = (1,2,3)$, $F(\sigma^3) = (1,4,5,6)$. The points 2 and 3 are permuted by σ , and (4,5,6) rotate cyclically. In the direct product $T_2^{\mathbb{C}} = T_1^{\mathbb{C}} \times T_1^{\mathbb{C}}$ the fixed point set is divided into orbits: $P = \{(1,1)\}$, $Q_1 = \{(1,2),(1,3)\}$, $Q_2 = \{(2,1),(3,1)\}$, $Q_3 = \{(2,2),(3,3)\}$ $Q_4 = \{(3,2),(2,3)\}$, $R_1 = \{(1,4),(1,5),(1,6)\}$, $R_2 = \{(4,1),(5,1),(6,1)\}$, $R_3 = \{(4,4),(5,5),(6,6)\}$, $R_4 =$

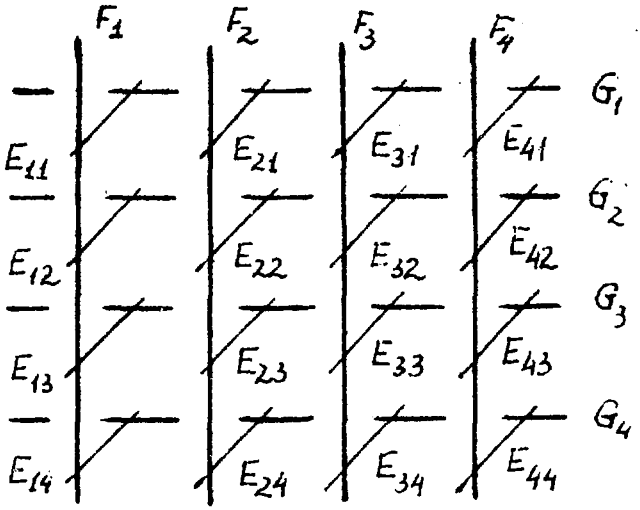

Fig.9. The intersection scheme of rational curves on X_2 .

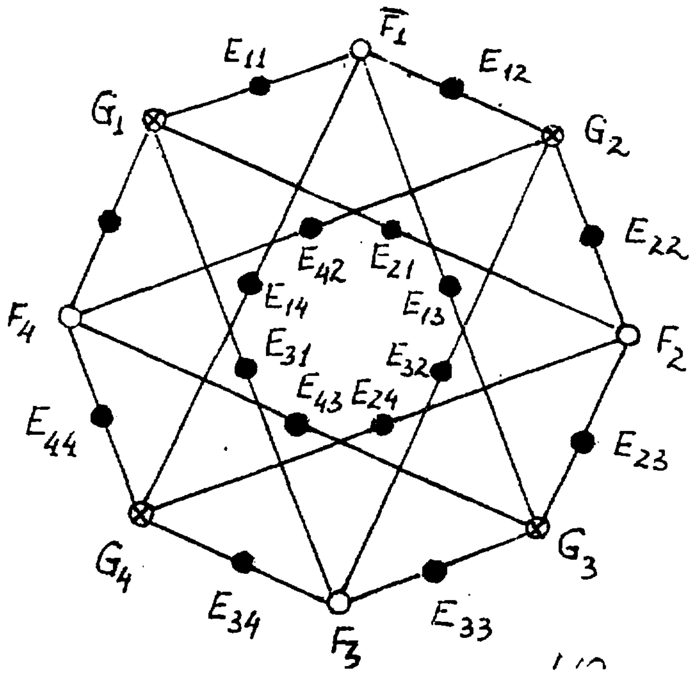

Fig. 10. The Coxeter graph for X_2 .

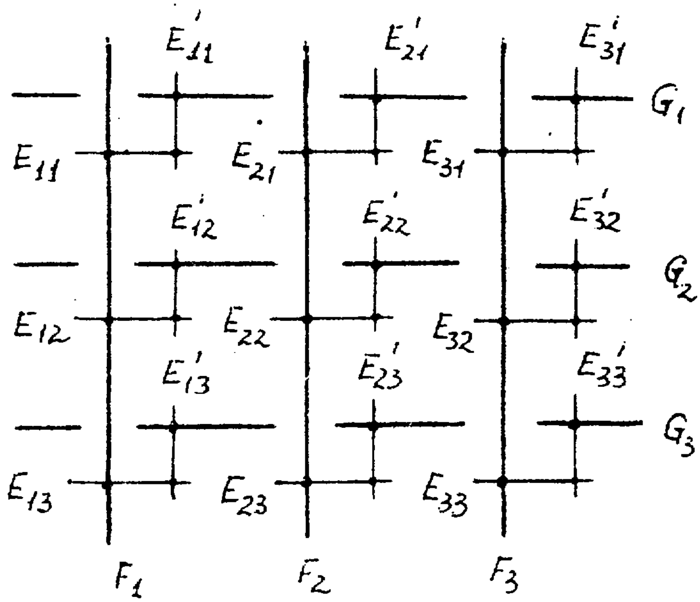

Fig.11. The intersection scheme of rational curves on X_3 .

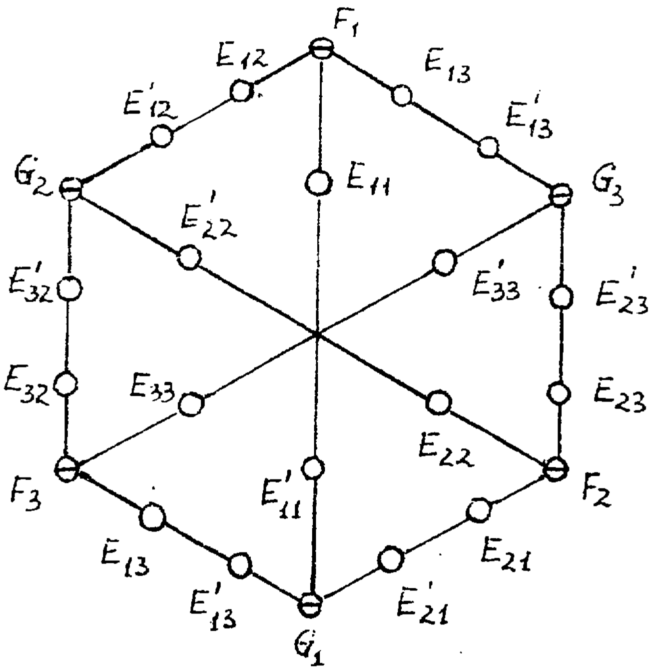

Fig.12. The Coxeter graph for X_3 .

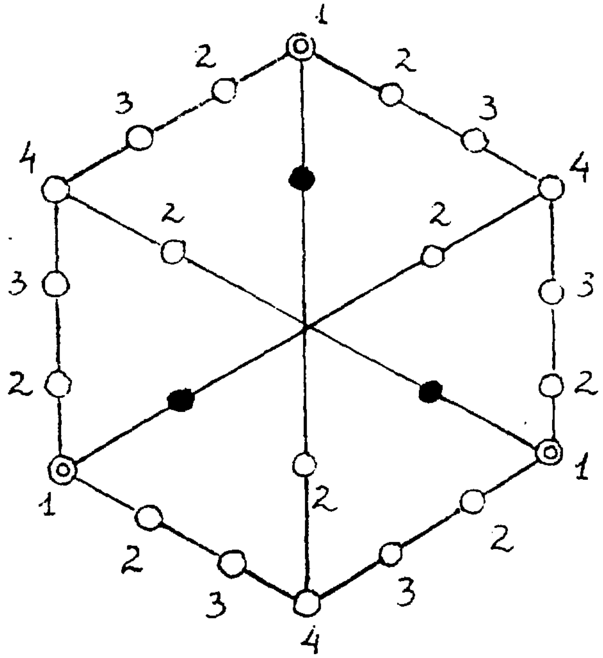

Fig.13. The coordinates of the vector h at the Coxeter graph.

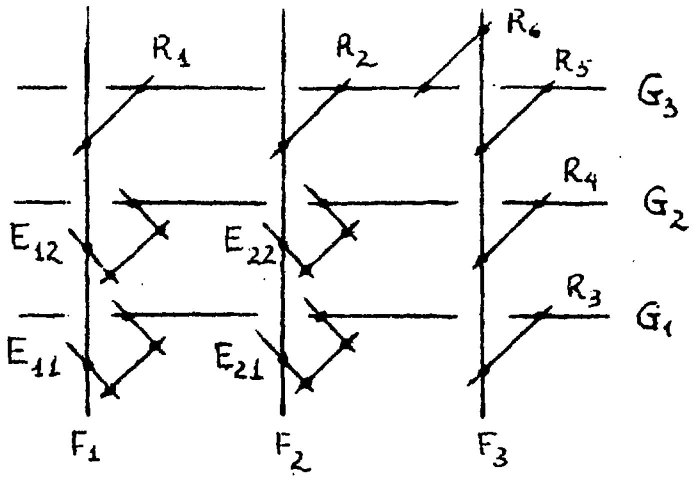

Fig.14. The intersection scheme of rational curves on X_4 .

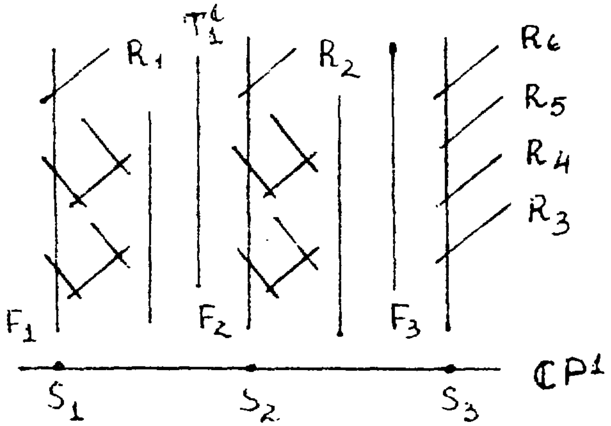

Fig.15. The structure of an elliptic pencil with three degenerate fibers over points S_1, S_2, S_3 of the base $\mathbb{C}P^1$

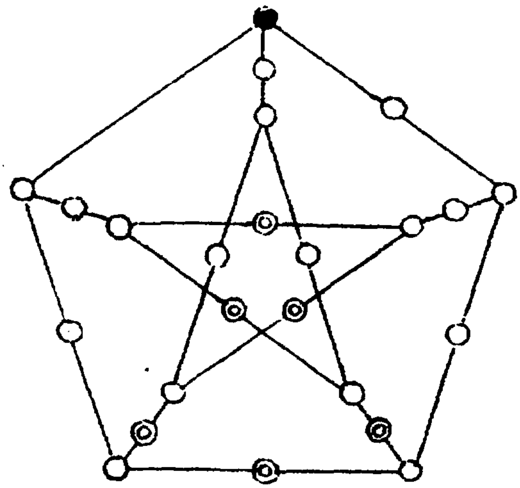

Fig.16. The Coxeter scheme of X_4 .

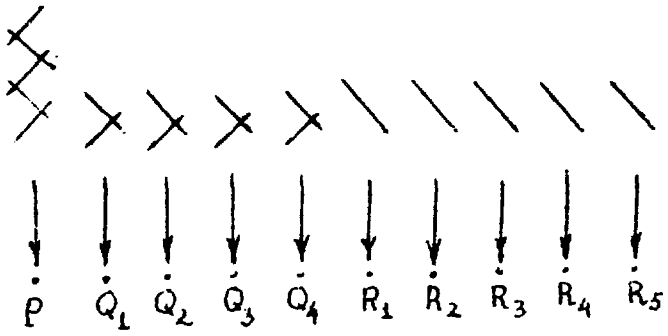

Fig.17. The desingularization scheme for X_6 .

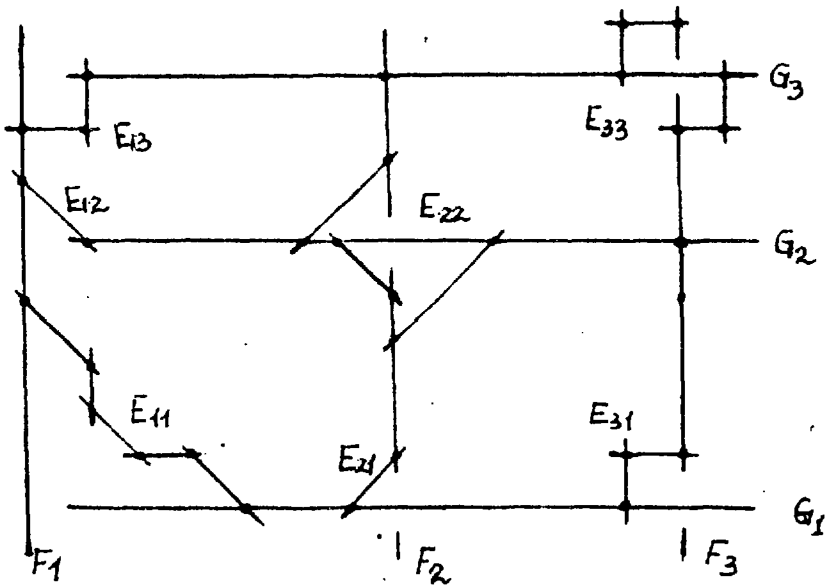

Fig.18. The intersection scheme of rational curves on X_6 .

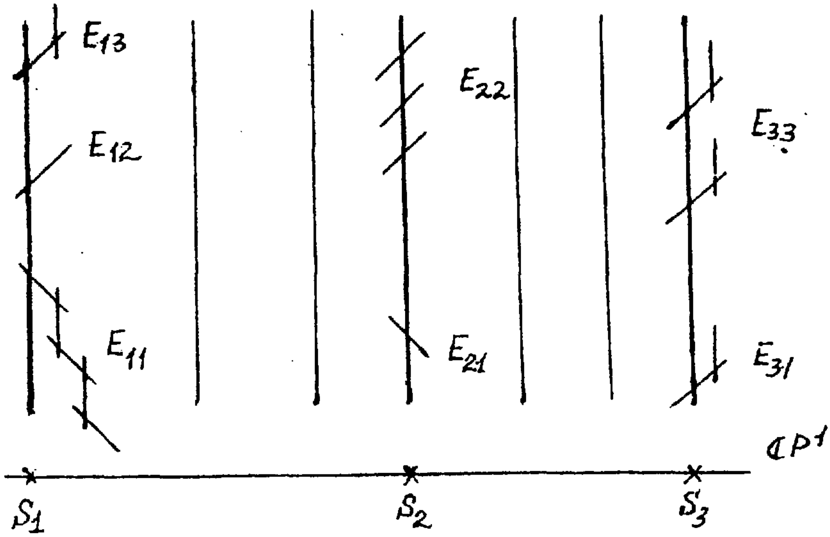

Fig. 19. X_6 as an elliptic pencil with three degenerate fibers over points S_1, S_2, S_3 of the base $\mathbb{C}P^1$

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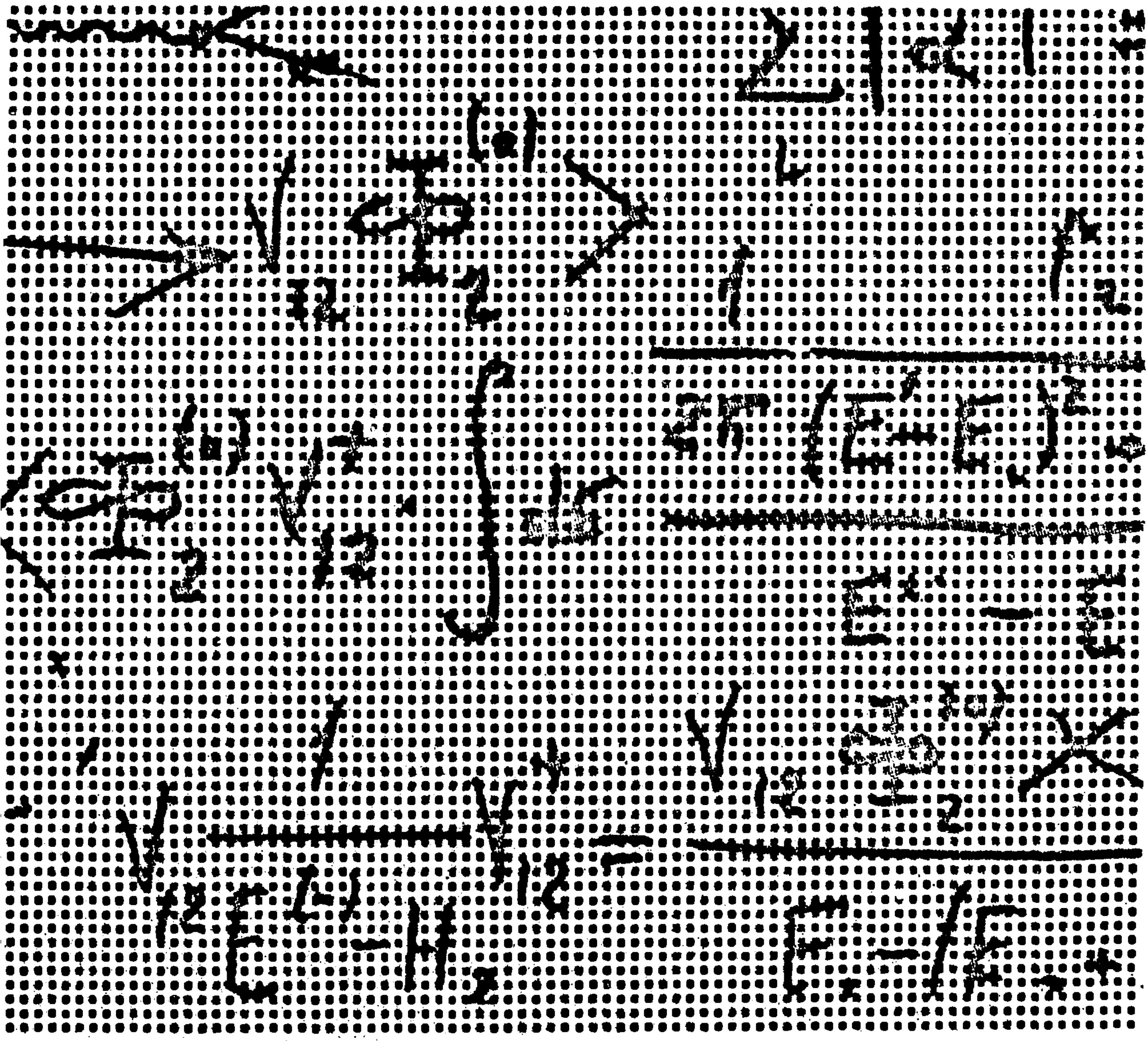

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