1. Wave Equations for Pulse Propagation

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Abstract

Theoretical discussions of the propagation of pulses of laser radiation through atomic or molecular vapor rely on a number of traditional approximations for idealizing the radiation and the molecules, and for quantifying their mutual interaction by various equations of propagation (for the radiation) and excitation (for the molecules). In treating short-pulse phenomena it is essential to consider coherent excitation phenomena of the sort that is manifest in Rabi oscillations of atomic or molecular populations. Such processes are not adequately treated by rate equations for excitation nor by rate equations for radiation. As part of a more comprehensive treatment of the coupled equations that describe propagation of short pulses, this memo presents background discussion of the equations that describe the field.

This memo discusses the origin, in Maxwell's equations, of the wave equation used in the description of pulse propagation. It notes the separation into lamellar and solenoidal (or longitudinal and transverse) and positive and negative frequency parts. It mentions the possibility of separating the polarization field into linear and nonlinear parts, in order to define a susceptibility or index of refraction and, from these, a phase and group velocity.

The memo discusses various ways of characterizing the polarization characteristics of plane waves, that is, of parameterizing a transverse unit vector, such as the Jones vector, the Stokes vector, and the Poincare sphere. It discusses the connection between macroscopically defined quantities, such as the intensity or, more generally, the Stokes parameters, and microscopic field amplitudes.

The material presented here is a portion of a more extensive treatment of propagation to be presented separately. The equations presented here have been described in various books and articles. They are collected here as a summary and review of theory needed when treating pulse propagation.

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§II.2 Radiation Intensity

Traditionally one characterizes radiation fields with several macroscopically defined quantities. Chief of these quantities is the energy of the electromagnetic field. This energy is distributed throughout the region occupied by radiation, and it is therefore natural to introduce the notion of energy density as a limiting value for the increment of energy contained within a small incremental volume. By definition the *mean spectral energy density* $\bar{u}(\omega,\mathbf{r},t)$ is the energy per unit volume per unit (angular) frequency at point $\mathbf{r}$ at time t. That is, the increment

$$dE(\mathbf{r},t) = \bar{u}(\omega,\mathbf{r},t)\, d\omega\, dV \tag{1.4-1}$$

is the energy in the frequency increment $d\omega$ about ω within volume element dV around point r. The adjective *mean* here denotes a temporal average over a time sufficiently long to remove possible fluctuations of thermal origin. The adjective *spectral* signifies a function of frequency.

We must also parameterize the flow of radiation energy or radiation flux. This we can do by means of the energy *flux vector* $\mathbf{F}(\omega,\mathbf{r},t)$, defined such that the projection $\mathbf{n}.\mathbf{F}(\omega,\mathbf{r},t)$ of the flux vector $\mathbf{F}(\omega,\mathbf{r},t)$ upon a unit vector $\mathbf{n}$ is the mean energy per unit time per unit frequency interval flowing through unit area of a surface normal to $\mathbf{n}$ at point $\mathbf{r}$ and time t. Thus the increment

$$dE(\mathbf{n},\mathbf{r},t) = \mathbf{n} \cdot \mathbf{F}(\omega,\mathbf{r},t)\; d\omega\, dA\, dt \tag{1.4-2}$$

is the energy within frequency increment $d\omega$ about ω flowing through the oriented surface element $\mathbf{n}dA$ at point r in time dt. Radiation detectors, such as photoelectric devices, respond to radiant energy. Suitable detectors reveal time variations in the energy, and so measure power. Because such detectors have fixed surface area, they provide measurements of the power per unit area, or radiation intensity.

Both the energy density and the flux vector, as well as other quantities defined below, may vary with position and time. For simplicity I shall omit explicity reference to the space-time value $\mathbf{r},t$ in the following, and shall write $dE \equiv dE(\mathbf{r},t)$ and $\mathbf{F}(\omega) \equiv \mathbf{F}(\omega,\mathbf{r},t)$, etc.

In treatments of radiation flow in which the radiation comprises a collimated beam, as emanates from a laser, it is customary to define spectral *intensity* $I(\omega) \equiv I(\omega,\mathbf{r},t)$ as the magnitude of the electromagnetic energy flux vector.

$$I(\omega) = |\, \mathbf{F}(\omega)\, |. \tag{1.4-3}$$

The spectral intensity so defined, having units of power per unit area per unit frequency, can be converted into spectral *photon flux* $\mathcal{F}(\omega)$ (the number of photons per unit area per unit time per unit frequency) by dividing $I(\omega)$ by the energy carried by one photon, $\hbar\omega$.

$$\mathcal{F}(\omega) = \frac{I(\omega)}{\hbar\omega}\,. \tag{1.4-4}$$

It follows from these definitions that when the energy can be assigned a mono-directional flow velocity c, then the intensity is c times the energy density:

$$I(\omega) = \hbar\omega\, \mathscr{F}(\omega) = c\, \bar{u}(\omega) \qquad (\textit{mono-directional}). \tag{1.4-5}$$

Treatments of processes in thermodynamic equilibrium deal with an isotropic radiation field, in which energy flows with equal likelihood into any element of solid angle. One can then define a mean intensity and mean photon flux by apportioning these quantities uniformly over 4π solid angle. This procedure leads to the definition

$$\bar{I}(\omega) = \hbar\omega\, \bar{\mathscr{F}}(\omega) = \frac{c}{4\pi}\, \bar{u}(\omega) \qquad (\textit{isotropic}). \tag{1.4-6}$$

This definition of intensity is widely used in theoretical treatments of laser induced excitation. However, the disciplines of photometry, optics, and radiative transfer introduce various alternative definitions for the term "intensity." The most common alternative definition deals with the distribution of energy flow into differential solid angle $d\hat{k}$. This definition exhibits the incremental energy of Eqn. (1.4-2) as the integral

$$dE = \int d\hat{k}\; \mathbf{k} \cdot \mathbf{n}\; I(\omega,\mathbf{k})\; d\omega\; dA\; dt. \tag{1.4-7}$$

The differential coefficient $I(\omega,\mathbf{k})$, with units of power per unit area per unit solid angle per unit frequency, is then called the *specific spectral intensity*. Because I shall be concerned with collimated plane waves I shall adopt the definition (1.4-3) for intensities. This nomenclature follows the common practice in laser theory

Integrated Intensity

Spectral intensity $I(\omega)$ refers to power per unit frequency per unit area. For nearly-monochromatic radiation it proves more convenient to introduce the frequency-integrated intensity I,

$$I = \int d\omega\; I(\omega) \tag{1.4-8}$$

bearing units of power per unit area. This quantity is of particular utility when we deal with collimated beams that are nearly monochromatic. The intensity I measures the total power in such a beam, regardless of direction or frequency.

Discussions of incoherent excitation, epitomized by Einstein's rate equations employ intensity or energy density as the principal characteristics of the radiation field. These are the parameters used in the theory of radiative transfer and in the astrophysics of stellar interiors. They represent macroscopic aspects of radiation. To treat coherent excitation we must examine more microscopic properties of radiation.

§12.2 Wave Equations

Discussions of steady radiation flow through matter often begin with postulated rate equations describing the gains and losses to rays of radiation as it passes through matter. These equations use intensity (or energy density) as a measure of radiation, and atomic populations as a measure of excitation. When treating transient effects it is preferable to begin with a more microscopic description, in order to admit coherent effects.

A complete description of coherent excitation must include both atom and field variables. For the atom the relevant equations derive from the Schrödinger equation--either as a multistate equation for probability amplitudes or as equations for density matrix elements. For the fields the underlying equations are those of Maxwell, suitably interpreted in terms of operators to incorporate quantum mechanics. When decomposed into normal modes, these equations too lead to a Schrödinger equation (or Heisenberg equation). Thus a starting point for the treatment of atomic influence upon radiation fields is the set of Maxwell equations in the presence of matter.

Maxwell's Equations with Charges and Currents

In free space Maxwell's equations provide equations of motion for an electric field $\mathbf{E} \equiv \mathbf{E}(\mathbf{r},t)$ and a magnetic field $\mathbf{B} \equiv \mathbf{B}(\mathbf{r},t)$ that together constitute the electromagnetic field. Maxwell's equations relate space and time derivatives of these two vector fields at any space-time point $(\mathbf{r},t)$. The presence of matter requires additional fields. Most simply, we may describe the electromagnetic properties of matter by a (scalar) charge density $\rho \equiv \rho(\mathbf{r},t)$ and a (vector) current density $\mathbf{j} \equiv \mathbf{j}(\mathbf{r},t)$. When matter is present the Maxwell equations comprise the equations of motion

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = \frac{4\pi}{c}\mathbf{j} \tag{12.2-1}$$

together with the constraints

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0. \tag{12.2-2}$$

The charges and currents originate in elementary charges. For our purposes these are electrons and nuclei. It is often useful to distinguish those electrons and nuclei that are bound together as neutral atoms or molecules, from those "free" charges that are not so bound. In the preceeding equations ρ denotes the charge density of all charges (bound and free), and $\mathbf{j}$ denotes the current density of all moving charges (bound and free).

A consistent description of electromagnetic waves must include the effect of the waves upon charges and currents. For un-ionized matter considered here, those effects consist of excitation. That is, the fields modify the electron distribution within an atom, or the structure of a molecule. The modified charge and current distributions in turn affect the fields, which affect the charges, and so on *ad infinitim*. All of these effects derive from the Lorentz force density

$$\mathbf{F} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B} \tag{12.2-3}$$

or from equivalent expressions of interactions.

In the visible and infrared portions of the spectrum excitations of atoms and molecules are primarily driven by the electric field (interacting with dipole moments) rather than by the magnetic field (interacting with magnetic moments). Thus our interest is in the electric field. We require equations descriptive of electric-field propagation, together with equations describing the effect of the electric field upon matter.

Poynting Vector and Energy Density

From the two dynamical Maxwell equations, together with the vector identity

$$\nabla.(\mathbf{A} \times \mathbf{B}) = \mathbf{B}.(\nabla \times \mathbf{A}) - \mathbf{A}.(\nabla \times \mathbf{B}), \tag{12.2-4}$$

we obtain the following relationship:

$$\nabla. \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) = - \frac{\partial}{\partial t} \frac{c}{8\pi} (\mathbf{E}.\mathbf{E} + \mathbf{B}.\mathbf{B}) - \mathbf{E}.\mathbf{j}. \tag{12.2-5}$$

This has the form of a continuity equation, relating the divergence of a flux vector to the time rate of change of a density, less an additional term:

$$\nabla.\mathbf{S} = - \frac{\partial}{\partial t} u - \mathbf{E}.\mathbf{j}. \tag{12.2-6}$$

The flux vector is the Poynting vector $\mathbf{S} \equiv \mathbf{S}(\mathbf{r},t)$ (power per unit area) and the density is the electromagnetic field energy density $u \equiv u(\mathbf{r},t)$ (power per unit volume),

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}, \qquad u = \frac{1}{8\pi} (\mathbf{E}.\mathbf{E} + \mathbf{B}.\mathbf{B}). \tag{12.2-7}$$

The term $\mathbf{E}.\mathbf{j}$ represents the density of power lost from the fields $\mathbf{E}$ and $\mathbf{B}$ as a result of radiation-matter interaction. This is energy gained by the charges, either as kinetic energy of free charges or as excitation energies of bound charges.

The Vector-Potential Wave Equation

Rather than deal with Maxwell's coupled first order equations, it is often useful to introduce uncoupled second order equations. The simplest example is obtained by introducing a vector potential $\mathbf{A} \equiv \mathbf{A}(\mathbf{r},t)$ and a scalar potential $\Phi \equiv \Phi(\mathbf{r},t)$ such that

$$\mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{A} - \nabla\Phi, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{12.2-8}$$

In place of Maxwell's equations we then obtain the wave equation

$$\nabla \times \nabla \times \mathbf{A} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = \frac{4\pi}{c} \mathbf{j} \tag{12.2-9}$$

and Poisson's equation

$$\nabla^2\Phi = 4\pi\rho, \tag{12.2-10}$$

These equations are particularly useful when applied to collections of structureless point particles, say electrons and protons. However, alternative wave equations prove more convenient when we consider charges bound into deformable aggregates.

The Polarization and Magnetization Fields

We are interested in propagation through a region in which the charges and currents are primarily those of electrons and nuclei bound together as neutral atoms (or molecules). To treat this situation we express the total charge density ρ and current density $\mathbf{j}$ as the contributions ρ' and $\mathbf{j}'$ of free charges plus charges and currents bound within atoms. These latter we write as contributions of a polarization field $\mathbf{P} \equiv \mathbf{P}(\mathbf{r},t)$ and a magnetization field $\mathbf{M} \equiv \mathbf{M}(\mathbf{r},t)$, defined such that

$$\mathbf{j} = \mathbf{j}' + \frac{\partial}{\partial t}\mathbf{P} + c\nabla\mathbf{x}\mathbf{M}, \qquad \rho = \rho' + -\nabla.\mathbf{P}. \tag{12.2-11}$$

The introduction of these vector fields maintains the continuity equation for electrical charge,

$$\nabla.\mathbf{j} = -\frac{\partial}{\partial t}\rho. \tag{12.2-12}$$

The polarization field $\mathbf{P}$ expresses the distribution of atomic electric-dipole moments. The magnetization field $\mathbf{M}$ expresses the distribution of Amperian currents, or magnetic-dipole moments. If the matter comprises a single species of atom, distributed with number density $\mathcal{N}$, then in the dipole approximation these fields are expectation values of the electric and magnetic dipole moments:

$$\mathbf{P} = \mathcal{N} \langle\mathbf{d}\rangle, \qquad \mathbf{M} = \mathcal{N} \langle\mathbf{m}\rangle. \tag{12.2-13}$$

More generally, the expectation value must be interpreted as a sum over various species and various Doppler shifts and orientations. It is generally the case that magnetic moments are smaller than electric moments by roughly the fine structure constant (i.e. by the ratio of electron speeds in the atom to the speed of light.) I shall assume that the magnetic dipole moments are negligible compared with the electric dipole moments. This assumption allows us to neglect the magnetization field $\mathbf{M}$.

Although ρ' and $\mathbf{j}'$ are called "free" charge and current density, it should be evident that these are actually those contributions to the charges and currents that are not represented by the expectation values of Eqn. (12.2-13). We therefore have the opportunity to partition the atomic basis states into two portions, and to define the polarization and magnetization fields as those fields that arise from selected states. The influence of the remaining atomic states appears in the current $\mathbf{j}'$. The separation is, to this extent, somewhat arbitrary.

Wave Equations for E and B

With the introduction of the polarization and magnetization fields Maxwell's equations of motion become

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0 \qquad (12.2\text{-}14)$$

$$\nabla \times (\mathbf{B} - 4\pi\mathbf{M}) - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E} + 4\pi\mathbf{P}) = \frac{4\pi}{c}\mathbf{j}' \qquad (12.2\text{-}15)$$

and the Maxwell constraint equations become

$$\nabla.\mathbf{B} = 0, \qquad \nabla.(\mathbf{E} + 4\pi\mathbf{P}) = 4\pi\rho'. \qquad (12.2\text{-}16)$$

The two equations of motion combine to produce the single wave equation

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P} - \frac{4\pi}{c}\frac{\partial}{\partial t}\nabla\times\mathbf{M} - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\mathbf{j}'. \qquad (12.2\text{-}17)$$

This wave equation determines the electric field **E** at all space-time points, given an initial condition over a spatial surface and given the space-time dependence of the polarization field **P** and magnetization field **M**. These latter fields must be determined from a theory of the response of the medium to the field, i.e. from a constitutive connection between **P** and atomic properties, together with a theory for the atomic response to the field (the Schrödinger equation).

Given a solution to this equation, we can determine the companion magnetic field by the equation

$$\frac{\partial}{\partial t}\mathbf{B} = -c\nabla \times \mathbf{E}. \qquad (12.2\text{-}18)$$

Alternatively, we can determine the magnetic field by means of the wave equation

$$\nabla \times \nabla \times \mathbf{B} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B} = 4\pi\nabla \times \nabla \times \mathbf{M} + \frac{4\pi}{c}\frac{\partial}{\partial t}\nabla\times\mathbf{P} + \frac{4\pi}{c}\nabla\times\mathbf{j}' \qquad (12.2\text{-}19)$$

and then determine the electric field from the equation

$$\frac{\partial}{\partial t}\mathbf{E} = c\nabla \times (\mathbf{B} - 4\pi\mathbf{M}) - \frac{\partial}{\partial t}4\pi\mathbf{P} - 4\pi\mathbf{j}'. \qquad (12.2\text{-}20)$$

With either approach the time and spatial variations of the polarization field **P** and magnetization field **M** act as sources for the electric and magnetic fields **E** and **B**. We shall neglect the magnetization field **M** and consider only the effects of the polarization field **P**.

The Auxiliary Fields D and H

One can also treat electromagnetic propagation by means of two auxiliary fields **D** and **H**, defined as

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \qquad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}. \qquad (12.2\text{-}21)$$

These fields include that portion of the charges and currents that have been expressed

as polarization and magnetization. The Maxwell equations then read

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0, \qquad \nabla \cdot \mathbf{B} = 0 \tag{12.2-22}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D} = \frac{4\pi}{c}\mathbf{j}', \qquad \nabla \cdot \mathbf{D} = 4\pi\rho' \tag{12.2-23}$$

The wave equation for the electric displacement vector **D** is

$$\nabla \times \nabla \times \mathbf{D} + \frac{1}{c^2}\frac{\partial}{\partial t^2}\mathbf{D} = 4\pi \nabla \times \nabla \times \mathbf{P} + \frac{4\pi}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{M} + \frac{4\pi}{c^2}\frac{\partial}{\partial t}\mathbf{j}'. \tag{12.2-24}$$

The source term for **D** involves spatial derivatives of **P**, whereas the source term for **E** involves time derivatives. The equation for the **H** field is

$$\nabla \times \nabla \times \mathbf{H} + \frac{1}{c}\frac{\partial^2}{\partial t^2}\mathbf{H} = -\frac{4\pi}{c}\frac{\partial^2}{\partial t^2}\mathbf{M} + \frac{4\pi}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{P} + \frac{4\pi}{c}\nabla \times \mathbf{j}'. \tag{12.2-25}$$

Given solutions to these equations, one must evaluate the fields $\mathbf{E} = \mathbf{D} - 4\pi\mathbf{P}$ and $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ in order to determine the forces acting upon (bound or free) charges and currents. From these Maxwell equations we derive the field continuity equation

$$\nabla \cdot \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{E} + \mathbf{H}\cdot\mathbf{H}) - \left[\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{M} + \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{P} + \mathbf{E}\cdot\mathbf{j}'\right]. \tag{12.2-26}$$

This equation replaces Eqn. (12.2-10) as an expression for energy conservation. It relates loss of energy in the fields **E** and **H** to power transmitted to bound charges as magnetization **M** and polarization **P**, as well as power exchanged with free charges. By regarding the final bracket as energy lost from the fields (and gained by matter) we can define a field energy density u' and Poynting vector **S'**,

$$u' = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2), \qquad \mathbf{S}' = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H}). \tag{12.2-27}$$

These formulas differ from our earlier expression by the appearance of **H** in place of **B**. This difference accompanies a redefinition of the rate of energy loss by the fields.

Alternative Expressions for Energy Flow

Weak steady electromagnetic fields induce electric and magnetic dipole moments in matter. These induced moments are often responsible for all of the polarization and magnetization fields. The magnitude of these induced moments is, for weak steady radiation, directly proportional to the applied field. Under those circumstances it is possible to regard the energy of polarized or magnetized matter as an additional field energy. Van Vleck (1932) provides the following example of how such induced fields may be treated. Let us assume there exists a scalar relationship between **H** and **B** and between **D** and **E** of the form

$$\mathbf{B} = \mu\mathbf{H}, \qquad \mathbf{D} = \epsilon\mathbf{E}. \tag{12.2-27}$$

The proportionality constants are the magnetic permeability μ and the electric permittivity ϵ. The introduction of these relationships is equivalent to the assumption that the polarization and magnetization fields have the relationships

$$4\pi\mathbf{M} = (\mu-1)\,\mathbf{H} = \frac{\mu-1}{\mu}\,\mathbf{B}, \qquad 4\pi\mathbf{P} = (\epsilon-1)\mathbf{E}. \tag{12.2-28}$$

Then we can define the fields

$$\mathbf{E}' = \sqrt{\epsilon}\,\mathbf{E}, \qquad \mathbf{H}' = \sqrt{\mu}\,\mathbf{H}, \qquad \rho'' = \frac{\rho'}{\sqrt{\epsilon}}, \qquad \mathbf{j}'' = \frac{\mathbf{j}'}{\sqrt{\epsilon}}. \tag{12.2-29}$$

Using these fields we can write Maxwell's equations as

$$\nabla \times \mathbf{E}' + \frac{1}{c'}\frac{\partial}{\partial t}\mathbf{H}' = \mathbf{H}'\,\frac{1}{c'}\frac{\partial}{\partial t}\left(ln\sqrt{\mu}\right) - \mathbf{E}' \times \nabla\left(ln\sqrt{\epsilon}\right) \tag{12.2-30}$$

$$\nabla \times \mathbf{H}' - \frac{1}{c'}\frac{\partial}{\partial t}\mathbf{E}' = \frac{4\pi}{c'}\,\mathbf{j}'' + \mathbf{E}'\,\frac{1}{c'}\frac{\partial}{\partial t}\left(ln\sqrt{\epsilon}\right) - \mathbf{H}' \times \nabla\left(ln\sqrt{\mu}\right) \tag{12.2-31}$$

$$\nabla.\mathbf{H}' = -\mathbf{H}'.\nabla\left(ln\sqrt{\mu}\right), \qquad \nabla.\mathbf{E}' = 4\pi\rho'' - \mathbf{E}'.\nabla\left(ln\sqrt{\epsilon}\right). \tag{12.2-32}$$

The velocity c' that occurs in these equations is the phase velocity

$$c' = \frac{c}{\sqrt{\mu\epsilon}}. \tag{12.2-33}$$

The continuity equation for the fields $\mathbf{E}'$ and $\mathbf{H}'$ reads

$$\nabla.\,\frac{c'}{4\pi}(\mathbf{E}' \times \mathbf{H}') = -\frac{\partial}{\partial t}\,\frac{1}{8\pi}\,[\mathbf{E}'.\mathbf{E}' + \mathbf{H}'.\mathbf{H}'] - \mathbf{E}'.\mathbf{j}''$$

$$- \frac{1}{4\pi}\left[\mathbf{E}'.\mathbf{E}'\,\frac{\partial}{\partial t}\left(ln\sqrt{\epsilon}\right) + \mathbf{H}'.\mathbf{H}'\,\frac{\partial}{\partial t}\left(ln\sqrt{\mu}\right)\right]. \tag{12.2-34}$$

This expression leads to the introduction of a Poynting vector S' and energy density u'' through the definitions

$$\mathbf{S}' = \frac{c'}{4\pi}\,(\mathbf{E}' \times \mathbf{H}') = \frac{c}{4\pi}\,(\mathbf{E} \times \mathbf{H}) \tag{12.2-35}$$

$$u'' = \frac{1}{8\pi}\,[\,\mathbf{E}'.\mathbf{E}' + \mathbf{H}'.\mathbf{H}'] = \frac{1}{8\pi}\,[\epsilon\mathbf{E}.\mathbf{E} + \mu\mathbf{H}.\mathbf{H}] = \frac{1}{8\pi}\,[\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B}]. \qquad (12.2\text{-}36)$$

This approach counts as part of the field energy that portion of the magnetization and polarization that is expressed by the scalars ϵ and μ. Apart from the possible time dependence of $\sqrt{\mu\epsilon}$, the remaining energy is the interaction

$$\mathbf{E}'.\mathbf{j}'' = \mathbf{E}.\mathbf{j}'. \qquad (12.2\text{-}37)$$

When one can neglect the spatial variations of $\sqrt{\mu\epsilon}$ one obtains the constraint equations

$$\nabla.\mathbf{H} = 0, \qquad \nabla.\mathbf{E} = \frac{4\pi}{\epsilon}\,\rho'. \qquad (12.2\text{-}38)$$

That is, the electric charges embodied in the "free" charge density ρ' are modified by the dielectric constant ϵ.

Transverse Fields

We can decompose the fields **E** and **P** into lamellar (longitudinal) parts $\mathbf{E}^L$, $\mathbf{P}^L$ and solenoidal (transverse) parts, $\mathbf{E}^T$, $\mathbf{P}^T$ defined by the property [*]

$$\nabla \times \mathbf{E}^L = 0 \qquad \nabla.\mathbf{E}^T = 0. \qquad (12.2\text{-}39)$$

By so doing we obtain (in the absence of free currents and magnetization) the solenoidal-field equation

$$\left[\nabla^2 - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\right]\mathbf{E}^T = \frac{4\pi}{c^2}\,\frac{\partial^2}{\partial t^2}\,\mathbf{P}^T \qquad (12.2\text{-}40)$$

and the lamellar-field equation

$$\frac{\partial^2}{\partial t^2}\,[\mathbf{E}^L + 4\pi\mathbf{P}^L] = 0. \qquad (12.2\text{-}41)$$

It is the solenoidal part of the electric field that constitutes the propagating radiation, and it is this field that is to be determined. We shall omit the label T upon this field; the field **E** is to be understood as transverse. The source term for this field is the transverse part of the polarization field.

Positive and Negative Frequency Parts

[*] *footnote*: For plane waves the solenoidal (divergenceless) fields are transverse to the propagation direction whereas the lamellar fields are directed longitudinally, along the propagation axis. Because we usually deal with plane waves it is customary to refer to radiation fields as transverse.

Although classical electromagnetic fields are real-valued functions, their description is greatly simplified by introducing complex valued fields whose real parts are the physical fields. This we do by introducing complex-valued positive and negative frequency components of each field, as in the electric-field expression

$$\mathbf{E} = \mathbf{E}^{(+)} + \mathbf{E}^{(-)} = 2\,\mathscr{R}e\,\mathbf{E}^{(+)}. \tag{12.2-42}$$

The Linear and Nonlinear Polarization Fields; Susceptibility

For weak steady-state monochromatic radiation (as contrasted with transient pulsed radiation), the atomic dipole moment expectation value <d> is an induced moment directly proportional to the electric field E. Hence the macroscopic polarization field P is also linearly related to the electric field. This proportionality occurs if the atomic excitation is weak and if there is sufficient time for the atom to relax into equilibrium with the excitation and its other surroundings. The proportionality is explained quantitatively with the Lorentz oscillator model of an atom or molecule, and it applies so long as the field is weak and monochromatic. The proportionality constant (the polarizibility) depends upon the frequency of the radiation and upon atomic properties but is constant in time. For intense fields and for transient phenomena the linear approximation for P is not adequate. The remaining portion of the polarization field is termed the nonlinear portion. One therefore writes the polarization field as two parts:

$$\mathbf{P} = {}^{\text{lin}}\mathbf{P} + {}^{\text{non}}\mathbf{P}. \tag{12.2-43}$$

The proportionality between $^{\text{lin}}\mathbf{P}$ and $\mathbf{E}$ is generally a frequency-dependent tensor relationship. To quantify this proportionality one introduces the following quantities:

$\alpha \equiv$ polarizibility

$\chi = \mathscr{N}\alpha \equiv$ linear electric susceptibility

$\epsilon = 1 + 4\pi\chi \equiv$ electric permittivity (dielectric constant)

$\eta = \sqrt{\mu\epsilon} \equiv$ index of refraction

$\mu \equiv$ magnetic permeability

Here $\mathscr{N}$ is the atomic number density. These are defined through the expressions

$$4\pi\,{}^{\text{lin}}\mathbf{P}^{(+)} = 4\pi\,\mathscr{N}\,\alpha\,\mathbf{E}^{(+)} = 4\pi\chi\,\mathbf{E}^{(+)} = (\epsilon - 1)\,\mathbf{E}^{(+)} \tag{12.2-44}$$

relating positive frequency components of the fields $^{\text{lin}}\mathbf{P}$ and $\mathbf{E}$. In general both χ and ϵ as well as η will be complex-valued time-dependent (or frequency-dependent) tensors. With these quantities one writes the polarization field as

$$4\pi\mathbf{P}^{(+)} = (\epsilon - 1)\,\mathbf{E}^{(+)} + 4\pi\,{}^{\text{non}}\mathbf{P}^{(+)}. \tag{12.2-45}$$

This presentation suggests that the nonlinear portion of the polarization field is that

part that does not vary linearly with the electric field. However, there is no reason why one cannot incorporate a portion of the linear response into χ and leave the remainder in $^{non}\mathbf{P}$. Alternatively, one could place some nonlinearity into χ. It is only neccessary to have a well-defined prescription for separating out a portion of the linear response. The separation may be made, for example, on the basis of fast and slow response time of the atom, as discussed by Ackerhalt, Drummond and Eberly (1986). A similar decomposition may be made for the magnetization field,

$$4\pi\mathbf{M}^{(+)} = (\mu - 1)\,\mathbf{H}^{(+)} + 4\pi\,{}^{non}\mathbf{M}^{(+)}. \tag{12.2-46}$$

In nonmagnetic material $\mu = 1$. With these definitions Maxwell's equations become

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mu\mathbf{H} = -\frac{4\pi}{c}\frac{\partial}{\partial t}\,{}^{non}\mathbf{M} \tag{12.2-47a}$$

$$\nabla \times \mathbf{H} + \frac{1}{c}\frac{\partial}{\partial t}\epsilon\mathbf{E} = \frac{4\pi}{c}\frac{\partial}{\partial t}\,{}^{non}\mathbf{P} + \frac{4\pi}{c}\,\mathbf{j}' \tag{12.2-47b}$$

and the field continuity equation becomes

$$\nabla\cdot\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}\frac{1}{8\pi}(\epsilon\mathbf{E}^2 + \mu\mathbf{H}^2)$$

$$-\left[\mathbf{E}^2\frac{\partial}{\partial t}\frac{\epsilon}{8\pi} + \mathbf{H}^2\frac{\partial}{\partial t}\frac{\mu}{8\pi} + \mathbf{E}\cdot\frac{\partial}{\partial t}\,{}^{non}\mathbf{P} + \mathbf{H}\cdot\frac{\partial}{\partial t}\,{}^{non}\mathbf{M} + \mathbf{E}\cdot\mathbf{j}'\right] \tag{12.2-48}$$

We can now identify the field energy u' and Poynting vector S' as

$$u' = \frac{1}{8\pi}(\epsilon\mathbf{E}^2 + \mu\mathbf{H}^2), \qquad \mathbf{S}' = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}). \tag{12.2-49}$$

Changes in the field energy are balanced by changes in the nonlinear polarization and magnetization, in the atomic structure responsible for the coefficients ϵ and μ, and in the free currents.

The Index of Refraction in Non-magnetic Matter

By splitting the solenoidal polarization field into these two parts, and neglecting magnetization ($\mu = 1$), we can write the wave equation for the positive frequency components of the electric field as

$$\nabla^2\mathbf{E}^{(+)} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\eta^2\mathbf{E}^{(+)} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\,{}^{non}\mathbf{P}^{(+)}. \tag{12.2-50}$$

The simplest case occurs when $\eta = \sqrt{\epsilon}$ is a time-independent scalar. Then the linear portion of the $\mathbf{P}^{(+)}$ field acts to modify the speed of wave propagation from c to the phase velocity $c/\sqrt{\epsilon} \equiv c/\eta$, while the nonlinear portion of $\mathbf{P}^{(+)}$ acts as a source term for propagation at this altered speed.

One can generalize the notion of an index of refraction to include all effects of the medium, both linear and nonlinear. One simply takes the ratio of the instantaneous $4\pi P^{(+)}$ and the instantaneous $E^{(+)}$ as the definition of a time-varying $4\pi\chi = \epsilon - 1 = \eta^2 - 1$.

Alternatively, one can incorporate all properties of the medium into the total polarization field, and set the index of refraction to unity. If one obtains reliable equations of motion for the atomic dipole moment, then the slowed speed of propagation will emerge as a natural consequence, without further assumption. That is, the introduction of an index of refraction may be viewed as a convenient but not neccessary artifice.

When there are several atomic species present it may be useful to treat one species in the linear approximation and to treat another species without making the linear approximation. Then the index of refraction incorporates the influence of the linearized species, while the polarization-field source term incorporates the influence of the other species.

The Coupled Equations

The complete description of radiation flowing through matter consists of the following equations. One has, for the field, some variant of the Maxwell equations, say

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}. \qquad (12.2\text{-}51)$$

The polarization field is expressible as a statistical average of dipole moments,

$$\mathbf{P} = \mathcal{N} \langle \mathbf{d} \rangle. \qquad (12.2\text{-}52)$$

When only a single species of atom occurs we can write this average as

$$\mathbf{P} = \mathcal{N}\, \mathcal{T}race(\rho \mathbf{d}). \qquad (12.2\text{-}53)$$

More generally, the polarization field must be constructed from a summation of species and velocities. The atoms are described by probability amplitudes that obey the Schrödinger equation, or by a density matrix ρ that obeys an equation of motion

$$\frac{\partial}{\partial t}\rho = -i[H^0 - \mathbf{d}.\mathbf{E}\ ,\ \rho\ ] - \Gamma\rho. \qquad (12.2\text{-}54)$$

These sets of partial differential equations are known as the coupled Maxwell-Bloch equations. The following sections describe simplified examples of these equations.

§1.4 Microscopic Plane Waves: Polarization

Radiation may be regarded, on a microscopic scale, as a superposition of traveling electromagnetic waves, that is, of quasi-periodic electric and magnetic fields $\mathbf{E} \equiv \mathbf{E}(\mathbf{r},t)$ and $\mathbf{B} \equiv \mathbf{B}(\mathbf{r},t)$. In free space (i.e. in regions devoid of matter) these fields are solenoidal (divergenceless) fields,

$$\nabla.\mathbf{E} = 0, \qquad \nabla.\mathbf{B} = 0 \tag{1.4-9}$$

and each field satisfies the *wave equation* [*]

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial}{\partial t^2}\right] \mathbf{A} \equiv \left[\nabla^2 \quad \frac{1}{c^2}\frac{\partial}{\partial t^2}\right] \mathbf{A} = 0. \tag{1.4-10}$$

In addition, the field $\nabla\times\mathbf{B}$ is proportional to $\partial\mathbf{E}/\partial t$ and the field $\nabla\times\mathbf{E}$ is proportional to $-\partial\mathbf{B}/\partial t$. The precise expression of this proportionality depends upon the choice of units for $\mathbf{E}$ and $\mathbf{B}$ (I shall use Gaussian cgs units, wherin the constant is $1/c$, with c the speed of light). This equation has solutions $\mathbf{A}(\xi)$ that are arbitrary differentiable functions of the argument $\xi_+ = k(x - ct)$ or $\xi_- = k(x + ct)$ or, more generally, of the argument

$$\xi_\pm = k.r \mp kct \equiv k_x x + k_y y + k_z z \mp \sqrt{k_x^2 + k_y^2 + k_z^2}\, ct. \tag{1.4-1}$$

The functional dependence of the field upon the argument ξ remains fixed, but the pattern moves (as a wave) with velocity c. The field is the same everywhere over a plane defined by $\mathbf{k}.\mathbf{r} = constant$, and so the fields are known as plane waves. The argument ξ_+ produces waves that advance toward increasing r with increasing time. The spatial scale of the pattern is set by the magnitude $k = |\mathbf{k}|$ of the *wave vector* k, while the direction of wave advance is specified by the components k_x, k_y, and k_z of this vector. The vectors $\mathbf{E}$ and $\mathbf{B}$ are constrained, by the divergence condition of Eqn. (1.4-9) to be perpendicular to the vector $\mathbf{k}$ (i.e. transverse to the propagation direction),

$$\mathbf{k}.\mathbf{E} = \mathbf{k}.\mathbf{B} = 0. \tag{1.4-12}$$

That is, plane wave vector fields that are solenoidal are transverse waves. The wavelength λ and angular frequency ω of a field of this type are

$$\lambda = \frac{2\pi}{k}, \qquad \omega = \frac{k}{c}. \tag{1.4-13}$$

Because the wave equation is homogeneous, the sum of two solutions will again be a

[*] *footnote*: The presence of matter makes several changes in the equation, most notably the presence of an inhomogeneous source term on the right hand side and an altered phase velocity in place of c.

solution. We can regard general fields as superpositions of different plane waves having different propagation vectors. Other types of solutions, such as those in which the field takes constant value over a spherical surface rather than a plane surface, may also be used in constructing such superpositions.

Example

A simple example of such an electromagnetic field is the pair of vector fields

$$\mathbf{E}(\mathbf{r},t) = \mathbf{e}(x) \mid \mathcal{E} \mid \cos(\xi) \tag{1.4-14a}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{e}(y) \mid \mathcal{B} \mid \cos(\xi), \tag{1.4-14b}$$

The space and time variation of these fields occurs through the argument

$$\xi = \omega\left(\frac{z}{c} - t\right) + \phi \equiv kz - \omega t + \phi, \tag{1.4-15}$$

while the vector nature of the fields appears in two mutually perpendicular unit vectors,

$$\mathbf{e}(x) \cdot \mathbf{e}(x) = \mathbf{e}(y) \cdot \mathbf{e}(y) = 1, \quad \mathbf{e}(x) \cdot \mathbf{e}(y) = 0. \tag{1.4-16a}$$

These properties of the unit vectors may be conveniently expressed by the single equation

$$\mathbf{e}(i) \cdot \mathbf{e}(j) = \delta(i,j) \equiv \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \tag{1.4-16b}$$

where $\delta(i,j)$, often denoted δ_{ij} , is the *Kronecker delta*, defined as unity when $i = j$ and zero otherwise.

These expressions describe a correlated pair of vector fields that, for fixed position, vary periodically in time and, at any instant of time, vary periodically in space. In this particular example the variation affects only the magnitude of the fields; the directional properties (i.e. the vectors contributing to **E** and **B**) are constant over planes perpendicular to the z axis. The vector fields so defined constitute examples of purely periodic, plane-wave electromagnetic fields. They have angular frequency ω and they propagate along the z axis with phase velocity c. The constant ϕ fixes the *phase* of the wave, the quantities $\mid \mathcal{E} \mid$ and $\mid \mathcal{B} \mid$ establish the magnitudes of the fields, and the unit vectors e(x) and e(y) establish the field directions.

Units

I shall adopt the Gaussian cgs units for electromagnetic fields, a system used in traditional texts on quantum electrodynamics and atomic physics. In this system the unit of electric charge is the statcoulomb, the unit of electric field is the statvolt / cm (1 statvolt = 299.79 volt) and the unit of magnetic field is the gauss (1 gauss = 10^{-4} tesla). These units are such that

$$1 \text{ dyne} = 1 \frac{(\text{statcoul})^2}{\text{cm}} = 1 (\text{statvolt})^2 \text{ cm}$$

$$= 1 (\text{abamp})^2 = 1 \text{ gauss abamp cm} \qquad (1.4\text{-}17)$$

This choice of units makes the field amplitudes $\mathcal{E}$ and $\mathcal{B}$ of equal magnitude in free space:

$$| \mathcal{E} | = | \mathcal{B} | \quad \text{in free space.} \qquad (1.4\text{-}18)$$

This equality produces the dimensionality relationships

$$1 \left[\frac{\text{statvolt}}{\text{cm}}\right]^2 = 1 (\text{gauss})^2 = 1 \frac{\text{erg}}{\text{cm}^3}. \qquad (1.4\text{-}19)$$

Polarization

In free space the vectors **E** and **B** of a radiation field are always perpendicular to each other and to the direction of propagation (the z axis of Eqn. (1.4-15)). Thus we require only a single vector to specify the directional characteristics of the field vectors **E** and **B** at a space time point, given the propagation direction. Following standard practice, I shall take the electric field **E** as this basic vector, and shall refer to this directon as the *polarization* direction of the radiation. In free space the polarization direction is always perpendicular to the propagation direction, so that these two directions can be taken as two of the three reference axes of a Cartesian coordinate system.

The plane electromagnetic wave of Eqns. (1.4-14) is *linearly* polarized (in the x direction): throughout space, and for all time, the electric vector points along the x axis, while its magnitude varies sinusoidally with position and time. In a more general plane wave both magnitude and direction may vary sinusoidally with time. The resultant polarization is termed elliptical. Special cases (see Fig. (1.4-14)) are *right circular* polarization

$$\mathbf{E}(\mathbf{r},t) = \frac{\mathcal{E}}{\sqrt{2}} [\mathbf{e}(x) \cos(\xi) + \mathbf{e}(y) \sin(\xi)], \qquad (1.4\text{-}20a)$$

and *left circular* polarization,

$$\mathbf{E}(\mathbf{r},t) = \frac{\mathcal{E}}{\sqrt{2}} [\mathbf{e}(x) \cos(\xi) - \mathbf{e}(y) \sin(\xi)]. \qquad (1.4\text{-}20b)$$

(Recall that $\xi \equiv \frac{\omega z}{c} - \omega t$.) This nomenclature follows the convention of Born and Wolf (1959, p28): for *right* circular polarization the electric and magnetic vectors appear to rotate *clockwise* in the x-y plane, for fixed z, as viewed with the light beam *approaching* the observer. In space the tips of the E and B vectors trace a helix, when the light is circularly polarized. The helix twists in a positive sense for

left-circular polarization: left circular polarization is termed *positive helicity*. The numerical factors $\sqrt{2}$ have been introduced in Eqns. (1.4-20) to maintain a common normalization with Eqn. (1.4-14), so that in both cases the amplitude $\mathcal{E}$ expresses the value of the time-averaged product of 2E.E.

Positive and Negative Frequency Components

Although the electric field $\mathbf{E}(\mathbf{r},t)$ has real-valued components, it proves useful to introduce complex-valued functions to describe the spatial and temporal variation. For example, all of the waves above can be written in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\,[\mathcal{E}\,\boldsymbol{\epsilon}\,e^{i\xi} + (\mathcal{E}\,\boldsymbol{\epsilon}\,e^{i\xi})^*] = \mathcal{R}e\,[\,\mathcal{E}\,\boldsymbol{\epsilon}\,\exp(ikz - i\omega t)], \qquad (1.4\text{-}21)$$

where $k \equiv \omega/c$ and where $\boldsymbol{\epsilon}$ is an appropriate unit vector, possibly complex. By unit vector, here and elsewhere, I mean a vector having unit magnitude:

$$|\,\boldsymbol{\epsilon}\,|^2 = \boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon} = 1, \qquad (1.4\text{-}22)$$

as do the three orthogonal Cartesian unit vectors $\mathbf{e}(x)$, $\mathbf{e}(y)$, and $\mathbf{e}(z)$. These expressions provide the opportunity to adjust the overall phase of the vector $\boldsymbol{\epsilon}$; this phase must be fixed by convention. I shall follow the choice

$$\begin{aligned}
\boldsymbol{\epsilon} &= \mathbf{e}(x) \text{ or } \mathbf{e}(y) && \textit{for linear polarization}\\
&= \frac{1}{\sqrt{2}}\,[\mathbf{e}(x) - i\mathbf{e}(y)] \equiv \mathbf{e}(-1) && \textit{for right circular polarization (negative helicity)}\\
&= -\frac{1}{\sqrt{2}}\,[\mathbf{e}(x) + i\mathbf{e}(y)] \equiv \mathbf{e}(+1) && \textit{for left circular polarization (positive helicity)}
\end{aligned} \qquad (1.4\text{-}23)$$

The unit vectors $\mathbf{e}(\pm 1)$ are complex when the Cartesian vectors $\mathbf{e}(x)$ and $\mathbf{e}(y)$ are real. (Conversely, a representation in which the vectors $\mathbf{e}(\pm)$ are real makes $\mathbf{e}(x)$ real and $\mathbf{e}(y)$ imaginary.) They obey they orthogonality property

$$\mathbf{e}(q)^*\cdot\mathbf{e}(q') = \delta(q,q'), \quad \text{for } q,q' = \pm 1. \qquad (1.4\text{-}24)$$

The introduction of complex exponentials allows us to write any real-valued field $\mathbf{E}(\mathbf{r},t)$ as the sum of two complex-valued vector fields,

$$\mathbf{E}(t) = \mathbf{E}^{(+)}(t) + \mathbf{E}^{(-)}(t). \qquad (1.4\text{-}25)$$

These two vectors are complex conjugates:

$$\mathbf{E}^{(+)}(t) = [\mathbf{E}^{(-)}(t)]^*. \qquad (1.4\text{-}26)$$

They form, respectively, the *positive* and *negative frequency components* of the electric vector. The positive frequency part is, by definition, the portion of the field that contains the time varying exponential $\exp(-i\omega t)$. Quite generally we can express this

complex vector as a combination of any pair of independent unit vectors in the plane perpendicular to the propagation direction. For the z-directed waves above we can write, at a fixed position,

$$E^{(+)}(t) = \tfrac{1}{2}\,[e(x)\,|\,\mathcal{E}_x\,|\,\exp(-i\delta_x) + e(y)\,|\,\mathcal{E}_y\,|\,\exp(-i\delta_y)]\,\exp(-i\omega t)$$

$$= \tfrac{1}{2}\,[e(x)\,\mathcal{E}_x + e(y)\,\mathcal{E}_y\,]\,\exp(-i\omega t) \qquad (1.4\text{-}27)$$

Here the real-valued phases δ_x and δ_y combine with the real-valued amplitudes $|\,\mathcal{E}_x|$ and $|\,\mathcal{E}_y\,|$ to produce complex-valued amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$.

The electric and magnetic fields that together constitute the electromagnetic field, are each examples of three-dimensional vector fields. Such a field comprises, at each point in space, three numbers that represent field components along three independent directions. The three independent coordinates can be chosen from any convenient three-dimensional coordinate system. When the radiation is a collimated beam, as occurs with laser radiation, then it is natural to take one of the reference axes to coincide with the propagation axis. Neither the electric nor the magnetic field have components in this direction. We are left with the problem of defining complex vectors in a two-dimensional plane. Let us examine some of the means of doing this.

Representations of Polarization: The Jones Vector

For any fixed direction of propagation, and point in space, we require two mutually orthogonal unit vectors to describe either the electric or magnetic field vector. In general, any vector **A** in a two-dimensional space can be expressed as a superposition of two independent unit vectors, say e(1) and e(2). The simplest form for this expansion, appropriate when the vectors are all reall, is

$$\mathbf{A} = A_1\,e(1) + A_2\,e(2). \qquad (1.4\text{-}28a)$$

In the space of the x-y plane, the pair of vectors e(x) and e(y) or the pair e(+1) and e(-1), or any other two independent combinations of these pairs, can serve as the unit vectors. Such an independent pair of orthogonal unit vectors is termed a set of *basis vectors*. Any vector in this two-dimensional space can be specified by two numbers (generally complex), the components A_n along two basis vectors. When we deal with complex vectors it is often convenient to write the expansion as

$$\mathbf{A} = A_1\,e(1)^* + A_2\,e(2)^* \qquad (1.4\text{-}28b)$$

so that the components are $A_n = e(n)^*.\mathbf{A}$. We can treat these numbers as elements of a two-component column vector (the *Jones vector*) and write Eqn. (1.4-28) as

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad \text{where} \quad e(1) = e(1)^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad (1.4\text{-}29)$$

In this representation the unit vectors are real, so there is no disctinction between Eqn. (1.4-28a) and (1.4-28b). The values of vector components A_j depend upon our choice of basis vectors -- they depend upon whether j stands for the Cartesian labels x,y or the helicity labels +1,-1. Basis vectors in two different coordinate systems are

related by means of linear equations,

$$e(q) = \sum_i U_{qi}\, e(i), \qquad e(i) = \sum_q (U^{-1})_{iq}\, e(q) \tag{1.4-30}$$

For example, the basis vectors of the two reference systems (x,y) and (+1,-1) are related by the matrix equations

$$\begin{bmatrix} e(+1) \\ e(-1) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -i \\ 1 & -i \end{bmatrix} \begin{bmatrix} e(x) \\ e(y) \end{bmatrix} \tag{1.4-31a}$$

$$\begin{bmatrix} e(x) \\ e(y) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ i & i \end{bmatrix} \begin{bmatrix} e(+1) \\ e(-1) \end{bmatrix}. \tag{1.4-31b}$$

Note that the elements of the inverse matrix U^{-1} are complex conjugates of the transpose U^T of the matrix U, i.e. the matrix U^{-1} is the Hermitian adjoint of U (that is, $U^{-1} = U^\dagger \equiv (U^*)^T$.). The matrix U is therefore *unitary*. It follows that the vector components $A_n = \mathbf{e}(n).\mathbf{A}$ appearing in the equation

$$\mathbf{A} = \sum_q e(q)^*\, A_q = \sum_i e(i)^*\, A_i \tag{1.4-32}$$

share the transformation properties of the unit vectors, Eqn. (1.4-29),

$$A_q = \sum_{i=x,y} U_{qi} A_i, \qquad A_i = \sum_{q=\pm 1} (U^T)_{iq}^*\, A_q. \tag{1.4-33}$$

When we taken Eqn. (1.4-28a) rather than Eqn. (1.4-28b) as the equation of expansion for a complex vector,

$$\mathbf{A} = \sum_q e(q)\, A^q = \sum_i e(i)\, A^i, \tag{$\frac{1}{4}$-34}$$

the vector components $A^q = e(q)^*.A$ obey the transformation rule [*]

$$A^q = \sum_{j=x,y} U_{qi}{}^* A^i \tag{1.4-35}$$

Often the vectors of interest have complex-valued components in our chosen coordinate system. It follows that, whereas the length of a real vector **R** is obtained from the square root of the scalar product **R.R**, the scalar product **A.A** will, in general, be a complex number. In place of **R.R** as a measure of length we require the scalar product of **A** and $\mathbf{A}^*$. This scalar product yields the sum of the absolute squares of components:

$$\mathbf{A}^*.\mathbf{A} \equiv |\mathbf{A}|^2 = |A_1|^2 + |A_2|^2 = \sum_j A_j\, A_j^*. \tag{1.4-36}$$

When we represent **A** as a column vector we can construct this sum as the matrix product of **A** and the complex-conjugated row vector $(\mathbf{A}^*)^T$. This row vector is just the Hermitian adjoint of the column vector **A**:

$$\mathbf{A}^*.\mathbf{A} = \mathbf{A}^\dagger\mathbf{A} = [A_1^*, A_2^*]\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{1.4-37}$$

Table 1.4-1 exhibits examples of common unit vectors in this notation, using a basis of Cartesian vectors.

Dirac Notation

It often proves useful to employ the Dirac notation $|A\rangle$ to denote a column vector **A**, whose components along basis vectors e(j) are the numbers A_j. Dirac notation allows us to simplify the typography for unit vectors by writing $e(+1) = |+1\rangle$ or $e(x) = |x\rangle$ or generally $e(j) = |j\rangle$. In this notation Eqn. (1.4-28) reads

$$|A\rangle = A_1\,|1\rangle + A_2\,|2\rangle. \tag{1.4-38}$$

We use the notation $\langle A|$ to denote the Hermitian adjoint of the column vector $|A\rangle$, that is, $\langle A|$ is the row vector with elements A_j^*. With this notation the scalar product of two vectors is denoted $\langle A|B\rangle$:

[*] *footnote:* The distinction here is between covariant components A_q and contravariant components A^q.

$$\langle A|B\rangle = \sum_j A_j^* B_j. \qquad (1.4\text{-}39)$$

The component [*] A_n appearing in Eqn. (1.4-36) is, in Dirac notation, $A_n = \langle n|A\rangle$. The orthogonality property of unit vectors becomes, in Dirac notation,

$$\langle j|j'\rangle = \delta(j,j') \qquad (1.4\text{-}40)$$

where $\delta(j,j')$ is the Kronecker delta and where j and j' may be elements of the index pairs (x,y), (+1,-1), or any other appropriate orthogonal labels. In Dirac notation the transformation coefficients for basis vectors, Eqn. (1.4-32), become

$$U_{qi} = (i|q), \quad (U^{-1})_{iq} = (q|i) = (i|q)^* \qquad (1.4\text{-}41)$$

where j=x,y denotes a Cartesian index and q=±1 denotes a helicity index.

Representations of Polarization: The Poincaré Sphere

We have noted the possibility of choosing as basis vectors the vectors |x> and |y>, representing two orthogonal linear polarizations, or the alternative pair |+1> and |-1>, representing two independent circular polarizations (helicities). How can we introduce more general pairs of basis vectors, representing orthogonal states of elliptical polarization intermediate between linear and circular polarization?

One possibility is to introduce unit vectors parameterized by two angles θ and ϕ, with ranges $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$, through the definitions

$$|\theta,\phi\rangle = \exp\left(\frac{i\phi}{2}\right)\cos\left(\frac{\theta}{2}\right)|-1\rangle - \exp\left(-\frac{i\phi}{2}\right)\sin\left(\frac{\theta}{2}\right)|+1\rangle \qquad (1.4\text{-}43)$$

$$|\pm 1\rangle = \pm\frac{1}{\sqrt{2}}\left[\,|x\rangle \pm i|y\rangle\right] \qquad (1.4\text{-}44)$$

The two angles θ and ϕ define a direction in an *abstract space* of three dimensions. Let us denote the Cartesian axes in this abstract space as 1,2,3 (corresponding to the triad of axes x,y,z), with the 3 axis aligned vertically and the 1,2 plane horizontal. Let us consider a sphere of unit radius in this abstract space. Then the angles θ and ϕ define a point on this sphere. Figure 1.4 -1 (after O'Neil, 1963 or Glauber, 1986) depicts this *Poincaré sphere*. The north pole of this sphere, $\theta = 0$, corresponds to negative helicity (right-circular polarization), and the south pole, $\theta = \pi$, corresponds to positive helicity (left-circular polarization). The equatorial plane, $\theta = \pi/2$, corresponds to various types of linear polarization. Specifically, the positive 1 axis, $\phi = 0$, repre-

[*] *footnote*: The $\langle n|A\rangle$ are contravariant components; it is the covariant components $\langle A|n\rangle = \langle n|A\rangle^*$ that share the transformation properties $|q\rangle = \Sigma\, U_{qj}\,|j\rangle$ of the basis vectors.

sents polarization along the x axis (horizontal linear polarization), whereas the negative I axis, $\phi = \pi$, represents polarization along the y axis (vertical linear polarization). Table 1.4-1 exhibits these cases. With ϕ fixed at $\phi = 0$, the vector $|\theta,0\rangle$ can describe horizontal or vertical linear polarization, as well as right or left circular polarization, and a range of elliptic polarizations. However, any single-parameter family of vectors does not include all possible polarizations. The set with $\phi = 0$, for example, omits linear polarization in directions other than vertical and horizontal. The most general polarization requires two angles for its specification.

We can recognize that any two diametrically opposed points on the sphere represent orthogonal polarizations. Any two such antipodal points provide angles with which to define a pair of basis vectors for elliptically polarized light. The transformation from one pair of vectors to another orthogonal pair may be regarded as the product of two matrix multiplications:

$$\begin{bmatrix} \epsilon'_{+1} \\ \epsilon'_{-1} \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{bmatrix} \exp\left(\frac{i\phi}{2}\right) & 0 \\ 0 & \exp\left(-\frac{i\phi}{2}\right) \end{bmatrix} \begin{bmatrix} \epsilon_{+1} \\ \epsilon_{-1} \end{bmatrix} \equiv D(\theta,\phi) \begin{bmatrix} \epsilon_{+1} \\ \epsilon_{-1} \end{bmatrix}. \qquad (1.4\text{-}45)$$

The resulting 2x2 unitary matrix $D(\theta,\phi)$ is a two-dimensional representation of the three-dimensional rotation group. (Section §17.5 discusses the properties of this unitary matrix as a special case of more general (2j+1)-dimensional matrix representations of rotations. The two-dimensional case is a representation of $j = \frac{1}{2}$)

Time Averages

Conventional light sources, such as sunlight or fluorescent lamps, produce superpositions of waves traveling into a range of directions and with a range of frequencies. When collimated into directed beams, the phases and polarization directions of such waves vary randomly with time. Suitable optical devices can impose polarization upon any such beam (cf. O'Neil, 1963). That is, the electric vector of a suitably prepared beam will maintain a well-defined phase relationship during the course of time. Let us consider the traditional means of parameterizing such polarization.

We begin with the notion of time averaging. Let the symbol $\{ \ldots \}$ denote a time average, over a time interval 2T sufficiently long to include optical cycles as well as thermal fluctuations:

$$\{ F(t) \} = \frac{1}{2T} \int_{-T}^{+T} dr \; F(t+\tau). \qquad (1.4\text{-}46)$$

With this definition the time average of a sinusoidally varying quantity, such as a component of the electric field, is zero:

$$\{ E(t) \} = \{ E^{(+)}(t) \} = \{ E^{(-)}(t) \} \qquad (1.4\text{-}47)$$

Periodic functions are an example of *stationary* processes, processes for which the time average is independent of the time t about which the average centers. To obtain nonzero averages of electromagnetic quantities we must consider bilinear combinations of field components. The decomposition into positive and negative frequency components proves particularly useful for such averages.

Connection: Fields and Intensity

The macroscopic radiation characteristics $\bar{u}(\omega)$ and I are related to time averages of bilinear products of the microscopic electric and magnetic fields $\mathbf{E} = \mathbf{E}(\mathbf{r},t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{r},t)$. As we shall discuss in Section §9.1 , an electromagnetic field in regions away from any matter (i.e., a free radiation field) has mean energy density given by the expression (in Gaussian cgs units)

$$\int d\omega\ \bar{u}(\omega) \;=\; \frac{1}{8\pi}\,\{\,\mathbf{E}^2 \;+\; \mathbf{B}^2\,\}. \tag{1.4-48}$$

The instantaneous flux of electromagnetic energy is given by the *Poynting vector*

$$\mathbf{S} = \frac{c}{4\pi}\,\mathbf{E} \times \mathbf{B} \tag{1.4-49}$$

This vector is c times the momentum density of the electromagnetic field. The magnitude of the vector **S** gives the rate of energy flow, while the direction of the vector specifies the direction of the energy flow. The intensity of radiation (or rate of energy flow per unit area) is the time averaged magnitude of the Poynting vector.

$$\int d\omega\ I(\omega) \;=\; |\,\{\,\mathbf{S}\,\}\,| \;=\; \frac{c}{4\pi}\,|\,\{\,\mathbf{E}\times\mathbf{B}\,\}\,|. \tag{1.4-48}$$

Equation (1.4-48) provides the basic connection between radiation intensity (a macroscopically adjustable variable) and field vectors (employed in microscopic descriptions of the radiation-matter interaction). For a plane wave it leads to the result

$$I \;\equiv\; \int d\omega\ I(\omega) = \frac{c}{8\pi}\,\{\,\mathbf{E}^2 + \mathbf{B}^2\,\} = \frac{c}{4\pi}\,\{\mathbf{E}^2\}, \tag{1.4-49}$$

where the last expression uses the fact that (in cgs units) the **E** and **B** fields have equal magnitudes. For the plane waves of Eqn. (1.4-10) the intensity is

$$I = \frac{c}{8\pi}\,|\,\mathcal{E}\,|^2. \tag{1.4-50}$$

This expression employs Gaussian cgs units. For numerical evaluation one requires the conversion factors

$$1 \ \frac{\text{erg}}{\text{cm}^3} = \left[1 \ \frac{\text{statvolt}}{\text{cm}} \right]^2 = 8.982 \times 10^4 \ \frac{\text{volt}^2}{\text{cm}^2}. \qquad (1.4\text{-}51)$$

From these results we deduce the electric field that corresponds to given intensity:

$$\mathcal{E} \ [\text{volt / cm}] = 27.4682 \ \sqrt{I[\text{watts / cm}^2]}. \qquad (1.4\text{-}52)$$

Partial Polarization: The Statistical Matrix

Let us consider a nearly monochromatic plane wave propagating along the z direction, as described by the electric field of Eqn. (1.4-27). This electric field originates in radiation from a distribution of atoms in the radiation source. To the extent that the individual source atoms are independent, and are subject to random interruptions and motions, the field from the composite source will vary randomly with time. In particular, the two phases δ_i and the two real-valued amplitudes $\mathcal{E}_i$ may fluctuate randomly in time. We readily obtain the following results for time averages:

$$\{ \ \mathbf{E}^{(+)}(t).\mathbf{E}^{(+)}(t) \ \} = \{ \ \mathbf{E}^{(-)}(t).\mathbf{E}^{(-)}(t) \ \} = 0 \qquad (1.4\text{-}53)$$

$$2 \ \{ \ \mathbf{E}^{(-)}(t).\mathbf{E}^{(+)}(t) \ \} \qquad \{ \ \mathbf{E}(t)^2 \ \} = \tfrac{1}{2} \ \{ \ | \ \mathcal{E}_x \ |^2 + | \ \mathcal{E}_y \ |^2 \ \}. \qquad (1.4\text{-}54)$$

We are interested in specifying the way in which the field vectors at a given spatial point maintain their correlation with time. For that purpose we need to consider time averages of various combinations of Cartesian components of the electric field. Let us continue using the field description of Eqn. (1.4-27) and introduce the following array of time-averaged electric-field amplitude pairs (the denominator is introduced to supply the normalization $\rho_{xx} + \rho_{yy} = 1$):

$$\rho_{ij} = (\rho_{ji})^* = \{ \ \mathcal{E}_i(t) \ \mathcal{E}_j(t)^* \ \} \ / \ \{ \ 2\mathbf{E}^2 \ \}. \qquad (1.4\text{-}55)$$

The four complex numbers ρ_{ij} , termed the *coherence matrix*, embody basic statistical information about the polarisation properties of the radiation at a given location in space and time. They form the elements of a 2x2 Hermitian matrix, ρ, with unit trace. This matrix is an elementary example of a *statistical matrix* (or *density matrix*) . Table 1.4-1 exhibits the coherence matrix (in a Cartesian basis) for the several examples of polarization considered previously.

The Field Polarization Tensor

We defined the statistical matrix as an average of products of two electric field components. This definition is a special case of the more general coherence matrix [*]

[*] *footnote:* This expression places the fields in *normal order*, with positive frequency components to the right of negative frequency components, as is customary in quantum optics.

$$\rho_{\lambda,\lambda'} = \{\, e(\lambda').E^{(-)}(t)E^{(+)}(t).e(\lambda)^* \,\}\, 2/\{\, E(t)^2 \,\}, \tag{1.4-56}$$

where λ and λ' are any pairs of labels appropriate to unit vectors. The scalar products of the unit vectors and the fields can be written explicitly in terms of Cartesian components for the several vectors. The unit vectors are not affected by the averaging, and hence we can write

$$\rho_{\lambda,\lambda'} = \sum_{ij} e_i(\lambda')e_j(\lambda)^* \{\, E_i^{(-)}(t)E_j^{(+)}(t) \,\}\, 2/\{\, E(t)^2 \,\}. \tag{1.4-57}$$

This is an example of a scalar relationship between two second rank tensors, or dyadics. The unit vectors form a dyadic $\in \equiv e(\lambda')e(\lambda)^*$, with Cartesian components

$$\in_{ij}(\lambda',\lambda) = e_i(\lambda')e_j(\lambda)^*, \tag{1.4-58}$$

while the pair of field components form another dyadic. Thus we can write

$$\rho_{\lambda,\lambda'} = \sum_{ij} \in_{ij}(\lambda',\lambda) \{\, E_i^{(-)}(t)E_j^{(+)}(t) \,\}\, 2/\{\, E(t)^2 \,\}$$

$$= \in(\lambda',\lambda) : \{\mathbf{E}^{(-)}(t)\mathbf{E}^{(+)}(t) \}\ 2/\{\, E(t)^2 \,\}. \tag{1.4-59}$$

The colon in the second line is an abbreviation for the component summation of the first line. The connection between different representations is

$$\rho_{\lambda,\lambda'} = \sum_{ij} U_{\lambda i}{}^* \rho_{ij} (U^T)_{j\lambda'}, \quad \rho_{ij} = \sum_{\lambda\lambda'} (U^T)_{i\lambda} \rho_{\lambda\lambda'} U_{\lambda' j}{}^* \tag{1.4-60}$$

where U is the matrix that transforms the basis vectors, as in Eqn. (1.4-30):

$$e(\lambda) = \sum_i U_{\lambda i}\, e(i). \tag{1.4-61}$$

Representations of Polarization: The Stokes Parameters

The statistical matrix comprises four complex numbers. The numbers are not entirely independent, however. We can replace these numbers by four real-valued numbers. Let us introduce a normalizing constant $\mathcal{S}$ and define the four real-valued *Stokes parameters* s_0, s_1 and s_2, s_3 by the formulas (cf. ONeil, 1963)

$$s_0 = \{\, |\, \mathcal{E}_x\, |^2 + |\, \mathcal{E}_y\, |^2 \,\}\, \mathcal{S}/\, 2\{\mathbf{F}^2\} = (\rho_{xx} + \rho_{yy})\, \mathcal{S} \tag{1.4-62a}$$

$$s_1 = \{\, |\, \mathcal{E}_x\, |^2 - |\, \mathcal{E}_y\, |^2 \,\}\, \mathcal{S}/\, 2\{\mathbf{E}^2\} = (\rho_{xx} - \rho_{yy})\, \mathcal{S} \tag{1.4-62b}$$

$$s_2 + is_3 = 2\{ \mathcal{E}_y^* \mathcal{E}_x \} \mathcal{I} / 2\{\mathbf{E}^2\} = 2\rho_{xy} \mathcal{I}$$

$$= 2\{ |\mathcal{E}_x| \exp(-i\delta_x + i\delta_y) |\mathcal{E}_y| \} \mathcal{I} / 2\{\mathbf{E}^2\}. \quad (1.4\text{-}62c)$$

The definitions can also be expressed in terms of helicity components, as in the formulas

$$s_0 = \{ | \mathcal{E}_{-1} |^2 + | \mathcal{E}_{+1} |^2 \} \mathcal{I} / 2\{\mathbf{E}^2\} = (\rho_{+1,+1} + \rho_{-1,-1}) \mathcal{I} \quad (1.4\text{-}63a)$$

$$s_3 = \{ | \mathcal{E}_{-1} |^2 - | \mathcal{E}_{+1} |^2 \} \mathcal{I} / 2\{\mathbf{E}^2\} = (\rho_{-1,-1} - \rho_{+1,+1}) \mathcal{I}. \quad (1.4\text{-}63b)$$

The connection between Stokes parameters s_j and elements of the coherence matrix depends upon the choice of basis vectors. For the two choices discussed above it is

$$\mathcal{I}\begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} s_0+s_1 & s_2-is_3 \\ s_2+is_3 & s_0-s_1 \end{bmatrix}, \quad \mathcal{I}\begin{bmatrix} \rho_{+1,+1} & \rho_{+1,-1} \\ \rho_{-1,+1} & \rho_{-1,-1} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} s_0-s_3 & -s_1+is_2 \\ -s_1-is_2 & s_0+s_3 \end{bmatrix}. \quad (1.4\text{-}64)$$

The Stokes parameters can be regarded as the components of a four-dimensional *Stokes vector* s. Two choices for the normalization $\mathcal{I}$ prove useful. With $\mathcal{I} = 2\{\mathbf{E}^2\}$ $c/4\pi$ the Stokes parameter s_0 becomes the radiation intensity I. (With this normalization the four Stokes parameters s_0, s_1, s_2, and s_3 are sometimes denoted I, Q, U, V). An alternative normalization is $\mathcal{I} = 1$, which makes $s_0 = 1$, and corresponds to a ray of unit intensity.

Stokes Parameters as Observables

The Stokes parameters that describe a light ray may be determined by six simple measurements of the intensity of light that passes through two types of optical element (cf. ONeill). One element is a *polarizer* that passes radiation only with linear polarization along a direction that makes an angle θ with the x axis. The other element is a *phase compensator* that retards the phase of y polarization by δ relative to the x polarization. For example, a quarter-wave plate (a compensator with $\delta = \pi/2$) converts light that is polarized at 45° into right circular polarization. By measuring the intensity $I(\theta,\delta)$ that emerges from a combination of polarizer and compensator, one determines the Stokes parameters as (cf. Born and Wolf p551)

$$\begin{aligned} s_0 &= I = I(0°, 0) + I(90°,0) = \text{total} \\ s_1 &= Q = I(0°,0) - I(90°,0) = \text{excess } 0° \text{ linear} \\ s_2 &= U = I(45°,0) - I(135°,0) = \text{excess } 45° \text{ linear} \\ s_3 &= V = I(45°, \pi/2) - I(135°, \pi/2) = \text{excess right circ. (negative helicity).} \end{aligned} \quad (1.4\text{-}62)$$

Apart from s_0, each of these parameters expresses the excess of one sense of polarization over the orthogonal polarization. Similarly, light with specified Stokes parameters may be prepared by means of suitable polarizing filters and optical elements. Thus the Stokes parameters represent a complete set of observable polarization properties of

light rays [*].

Complete and Partial Polarization

For any choice of normalization the four Stokes parameters satisfy the constraining inequality

$$(s_0)^2 \geq (s_1)^2 + (s_2)^2 + (s_3)^2. \tag{1.4-63}$$

If the radiation maintains a fixed phase difference $\delta = \delta_x - \delta_y$, the radiation is completely polarized. The components of the Stokes vector then have the property

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 \tag{1.4-64}$$

Specific examples of completely polarized radiation, with the normalization $s_0 = 1$, are given in Table 1.4-1

Generally the wave trains that comprise a ray do not maintain a constant phase relationship as time progresses. As a consequence, the radiation is only partially polarized. The most extreme case occurs when the two squared electric-field amplitudes have equal averages, so that $s_1 = 0$, and concurrently the phases average to zero, so that $s_2 = s_3 = 0$. Such radiation is *completely unpolarized* light. It has Stokes vector $[s_0, 0, 0, 0]$ and statistical matrix (in any reference system)

$$[\rho(\text{unpol})] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.4-65}$$

To parameterize intermediate cases of partial polarization one introduces the *degree of polarization*, $\wp$, defined as

$$\wp = \frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{s_0}, \tag{1.4-66}$$

and with values in the range $0 \leq \wp \leq 1$. Any statistical matrix ρ describing the polarization properties of a light ray can be expressed as a multiple of the matrix ρ(unpol) and a matrix representing completely polarized light, ρ(pol):

$$\rho = (1-\wp)\,\rho(\text{unpol}) + \wp\,\rho(\text{pol}). \tag{1.4-67}$$

That is, any light ray may be regarded as a combination of completely polarized light and completely unpolarized light.

[*] *footnote:* The polarization properties are but one elementary type of correlation function that may be used to characterize waves. A more complete description of a light ray requires other variables to establish the temporal characteristics, such as bandwidth.

We can see that, with the normalization $s_0 = 1$, the three remaining Stokes parameters s_1, s_2, and s_3 form Cartesian components of vector of length p in a three-dimensional abstract space. For unpolarized light the vector has zero length, while for completely polarized light it has unit length. In this latter case, the three parameters s_1, s_2 and s_3 define Cartesian coordinates of a point on the Poincarè sphere. The angles θ and ϕ used above, in Eqn. (1.4-35), to locate a point on the Poincarè sphere, provide an alternative parameterization of polarization, based on spherical coordinates rather than Cartesian coordinates.

The Pauli Matrices

The Stokes parameters defined by Eqns. (1.4-57) and (1.4-60) can be expressed in two other forms, involving two-dimensional rather than three- or four-dimensional abstract spaces. Let us use the Cartesian representation and introduce the following four independent 2x2 Hermitian matrices

$$[\sigma(0)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad [\sigma(1)] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$[\sigma(2)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad [\sigma(3)] = -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{1.4-68}$$

Using these matrices we can rewrite Eqn. (1.4-61) as

$$\mathcal{J}\ \rho \ = \mathcal{J} \begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix} = \frac{1}{2} \sum_j s_j\ \sigma(j). \tag{1.4-69}$$

The matrix $\sigma(0)$ is the unit matrix. The triad of traceless 2x2 matrices $\sigma(1)$, $\sigma(2)$, and $-\sigma(3)$, known as *Pauli matrices*, are conventionally denoted σ_z, σ_x, and σ_y . The minus sign with $\sigma(3)$ occurs because we have followed the traditional definition of the Stokes parameter s_3 as the excess of negative helicity. Had we chosen s_3 to be excess positive helicity we could use the Pauli matrices as the basis matrices $\sigma(n)$. The matrices $\sigma(n)$ have the property

$$\sum_{ij} \sigma_{ij}(n)\sigma_{ji}(m) \equiv \mathcal{J}\,trace(\ \sigma(n)\sigma(m)\ ) = 2\delta(n,m) \tag{1.4-70}$$

and so we can express the Stokes parameter s_j as a normalizing factor times the trace of the matrix product $\rho\sigma(j)$:

$$s_j \ = \ \mathcal{J} \sum_{ik} \rho_{ik}\ \sigma_{ki}(j) \ \equiv \ \mathcal{J}\ \mathcal{J}\,trace(\ \rho\sigma(j)\ ). \tag{1.4-71}$$

Alternatively, we can introduce the Dirac notation

$$|E(t)\rangle \ = \ \mathbf{E}^{(+)}(t), \qquad \langle E(t)| \ \equiv \ (\mathbf{E}^{(+)}(t))^{\dagger} \tag{1.4-72}$$

for two-component vectors, and write the Stokes parameter s_j as the (normalized) time averaged scalar product of the row-vector $\langle E(t)|$ with the column-vector $\sigma(j)|E(t)\rangle$:

$$s_j \{E^2\} = 2\mathcal{S} \{ \langle E(t)| \sigma(j) |E(t)\rangle \}$$

$$\equiv 2\mathcal{S} \sum_{ik} \{ E^{(-)}_i(t) \sigma_{ik}(j) E^{(+)}_k(t) \} \qquad (1.4\text{-}73)$$

(Recall that $\mathcal{S}$ is the normalization factor for the Stokes parameters.)

Interference

Intensity, like wavelength and frequency, provides an important parameter with which to characterize radiation flowing along a ray. With the inclusion of polarization properties, the four Stokes parameters describe the elementary properties of a light ray. These parameters form the basis of traditional treatments of radiative transfer. Such theories deal with the geometric optics of independent light rays, and so they provide expressions for the change of the Stokes vector (or, more simply, the intensity) along a ray. Typically the theory takes the form

$$\frac{d}{dz} s = M s \qquad (1.4\text{-}74)$$

where s is the four-component Stokes vector and M is a 4x4 matrix, the *Mueller matrix* (cf. O'Neill, 1963). Such descriptions of radiation flow permit attenuation and growth of intensity, as well as changes in polarization, but they omit reference to the phases that are needed to describe the superposition of electric fields from separate rays. What is missing from the Stokes parameters is the possibililty of interference between separate rays.

Electric fields, like vectors in general, obey a linear superposition principal: if $\mathbf{E}_1$ and $\mathbf{E}_2$ are permissible electric fields, then the combination $a_1\mathbf{E}_1 + a_2\mathbf{E}_2$ is also a permissible field. (Permissible here means that each constituent field, as well as their superposition, obeys the Maxwell equations and boundary conditions that define electrodynamics.) Radiation detectors, such as photoelectric devices, respond to intensity rather than to the field amplitudes. This fact, taken with the superposition property, leads to the interference effects that evidence wave properties of radiation. The simplest example occurs when the field at a detector is a superposition of fields $\mathbf{E}_1$ and $\mathbf{E}_2$ (of intensities I_1 and I_2) that originate in the passage of a plane wave through two fixed slits. The intensity I at the detector depends not only upon the two intensity values I_1 and I_2 but also upon the relative phase of the two electric fields:

$$I = \frac{c}{8\pi} \{ (\mathbf{E}_1 + \mathbf{E}_2)^2 \} = I_1 + I_2 + 2\sqrt{I_1 I_2} \{ \cos(\xi_1 - \xi_2) \} \qquad (1.4\text{-}72)$$

Here $\xi_j = kz_j - \omega t$ is the relevant phase of the jth wave. If the individual intensities are equal, $I_1 = I_2$, then the detected intensity I may range between zero (completely destructive interference) and $4I_1$ (compleletly constructive interference). The observed

value will depend upon the time average of the cosine of a phase difference. If the two slits produce independent fluctuations, as will occur when they are separate incandescent lamps, then the time average will be zero. If the light passing through the two slits maintains a fixed phase relationship, as will occur when the light originates in a common source or in two stable laser sources, then the time average will be nonzero for appropriate spatial locations, and the intensity distribution exhibits an interference pattern of bright and dark bands.

Interference patterns are manifestations of coherence between waves. In the traditional two-slit optics demonstration, the coherence and the interference pattern are spatial. We shall be concerned with light arriving in time at a fixed position (an atom), so the coherence of interest is temporal. When the radiation is the result of random emission from uncorrelated atoms, as it is from incandescent vapors, the phases are incoherent and produce no temporal interference effects. Light from a laser originates with correlated emission from many atoms, amongst which there are well defined phases. Such ligh permits temporal coherence.

We conclude that a description of coherent excitation must deal with the electric field E(r,t) rather than with intensity, in order to retain the phase information that makes possible calculation of interference.

Fields as a Prototype of Quantum Mechanics

The fields discussed above are purely classical entities, i.e. they are not subject to any constraints of quantum theory. The quantum of action $\hbar$ that is the hallmark of quantum mechanics does not appear in the theory. The fields are wavelike entities, a consequence of their definition by a wave equation. They do not have the granularity of classical particles. That is, the fields are not localized in space or time, but fill finite volumes of space during an extended interval of time. This wavelike nature has the consequence that the basic macroscopic observable, the intensity, involves bilinear products of field amplitudes. More generally, the elementary observables are expressible by means of scalar products of abstract vectors. It is these vectors, rather than the observables, for which the basic electromagnetic theory provides equations -- the wave equation for the fields. These properties may be regarded as the essence of elementary wave properties: linear equations of motion for fields whose bilinear products are observable.

Quantum theory endows particles with wavelike properties. The theory is, for this reason, often termed wave mechanics. The mathematics of these waves parallels the mathematics presented here for classical fields. In place of the electric field amplitude, particle wave mechanics introduces a probability amplitude. Both the Dirac notation and the Pauli matrices, as well as the statistical matrix, find wide application in quantum mechanics. They appear above as tools in describing purely classical phenomena, the intensity and polarization properties of a light ray passing through a fixed point in space.

Table 1.4-1: Polarization Representations

$\|j\rangle$	$\|\theta,\phi\rangle$	Polarization state	ϵ	ρ	$[S_0, S_1, S_2, S_3]$
$\|x\rangle$	$\|\frac{\pi}{2}, 0\rangle$	linear, along x	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	[1, 1, 0, 0]
$\|y\rangle$	$\|\frac{\pi}{2}, \pi\rangle$	linear, along y	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	[1, -1, 0, 0]
	$\|\frac{\pi}{2}, \frac{\pi}{2}\rangle$	linear, 45° to x	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}$	[1, 0, 1, 0]
	$\|\frac{\pi}{2}, -\frac{\pi}{2}\rangle$	linear, 135° to x	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$	[1, 0,-1, 0]
$\|-1\rangle$ ($\|R\rangle$)	$\|0, \phi\rangle$	negative helicity (right circular)	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&i\\-i&1\end{bmatrix}$	[1, 0, 0, 1]
$\|+1\rangle$ ($\|L\rangle$)	$\|\pi, \phi\rangle$	positive helicity (left circular)	$\frac{-1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&-i\\i&1\end{bmatrix}$	[1, 0, 0,-1]

after O'Neil, *Statistical Optics* (1963). The conventions are

$$[\epsilon] = \begin{bmatrix}\epsilon_x\\ \epsilon_y\end{bmatrix}, \qquad [\rho] = \begin{bmatrix}\rho_{xx} & \rho_{xy}\\ \rho_{yx} & \rho_{yy}\end{bmatrix}, \qquad \rho_{ij} = \epsilon_i \epsilon_j^*$$