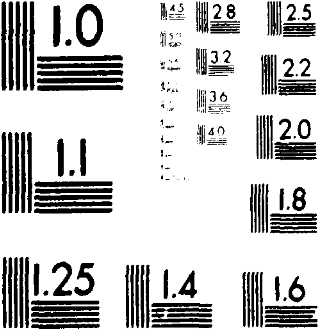
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W.A. Rodrigues Jr., M.A. Faria-Rosa, A. Maia Jr. and E. Recami:(*)

Phisico-Matematical approach to (generalized) monopoles without a string

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Phisico-Matematical approach to (generalized) monopoles without a string

W.A.Rodrigues Jr. (1)

Dipartimento di Matematica, Università degli Studi di Trento, 38050 Povo (Trento) Italia

M.A.Faria-Rosa, A. Maia Jr., E. Recami (2)

IMECC-UNICAMP, CP 6065, 13081, Campinas, SP, Brasil

(2) I.N.F.N., SEZIONE DI CATANIA, Catania, Italia

ABSTRACT: In this paper we present a theory of the generalized magnetic monopole without string, which is distinct from Dirac's original theory and also distinct from the topological theory of the monopole. Our theory is first formulated in the Clifford bundle formalism; and in the particular case of electrodynamics we deduce from Maxwell equations the generalized Lorentz force and the equations of motion of charges and monopoles. We discuss the conservation laws and the problem of the Lagrangian formalism. We obtain Dirac quantization condition in two different ways.

Finally, we present a principal fiber bundle formulation of our theory using the spliced-bundle concept with gauge group G×G, where G is the gauge group of the theory without monopoles.

1.INTRODUCTION

In this paper we present the theory of the magnetic monopole without string. By this we mean that in our theory the electromagnetic field generated by charges and monopoles is described by a generalized potential which is the sum (in the Clifford bundle) of a 1-form and a 3-form field, which are singular only at the location of the charges and monopoles.

Our approach contrasts with the one by Dirac where an unphysical singularity called string (where the potential is singular) is introduced in order to be possible to describe the electromagnetic field of charges and monopoles through a single potential which is a 1-form field (see §3).

Also it is worth-while to compare our theory, where no change in the topology of space-time occurs, with the topological monopole theory formulated as a principal fiber theory with group U(1). Indeed, in such a theory the monopole appears as a hole in space-time which has then the non trivial topology $R^2 \times S^2$. All this is described in §2, §3 and §4.

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(1) On leave on absence from IMECC, UNICAMP

(2) Permanent adress: Dipartimento di Fisica dell'Università di Catania, Corso Italia, 57 - 95129 Catania

(2) DEPARTAMENTO DE MATEMÁTICA APLICADA ← present address
IMECC - UNICAMP
C. P. 6065 - 13081 - Campinas, S. P. - Brasil

In our approach to the monopole problem using the Clifford bundle formalism (§5), we are able to deduce from Maxwell equations the conservation laws and the correct coupling of electric and magnetic particles to the electromagnetic field. We are consequently able to derive the motion equations for electric and magnetic particles. In §4 we discuss also Dirac's quantization condition using two different methods. In §6 we discuss the problem of the Lagrangian formalism when monopoles are present.

Although the Clifford bundle formalism seems to be a perfect mathematical design for the generalized electrodynamics with monopoles without strings, it is insufficient for introducing analogous monopoles for non abelian gauge theories. We then, produce in §7 a monopole theory without string in normal way. Indeed, we are able to associate with the two potentials of the theory of §5 a connection in an appropriate spliced bundle. In this way we obtain the geometrization of the theory as a principal fiber bundle with gauge group G×G and then use the full apparatus of these theories to obtain the field equation, etc.

In §8 we present our conclusions.

The paper contain Appendices A,B,C, that introduce respectively the Clifford bundle formalism and the necessary ingredients for the formulation of the theory of §7 as a principal fiber bundle theory.

2.STANDARD ELECTRODYNAMICS IN INSTRINSIC FORM

Let (M,h,∇) be a Lorentzian manifold[1], and J_e and F respectively a 1-form and a 2-form fields over M[i.e., J_e and F are sections of the Hodge Bundle (see Appendix A)]. The Maxwell equations in free space are

$$dF = 0, \quad \delta F = -J_e \tag{1}$$

where d is the exterior derivative (differential) and δ is the Hodge codifferential (see Appendix A for details). Let x^μ, $\mu = 0,1,2,3$, be a chart for $U \subset M$. Let $(\theta^0 = dx^0, \theta^1, \theta^2, \theta^3)$ be an orthonormal system in T^*U. Then, for $x \in U$ we have the identifications $[h = \eta_{\mu\nu}\,\theta^\mu \otimes^S \theta^\nu;\ \eta_{\mu\nu} = \mathrm{diag}\,(+1,-1,-1,-1)]$:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}; \quad (J_e)_\mu = (\rho_e, -J_{e_1}, -J_{e_2}, -J_{e_3}) \tag{2}$$

where $\vec{E}=(E_1,E_2,E_3)$ is the electric field, $\vec{B}=(B_1,B_2,B_3)$ is the magnetic field and ρ_e and j_e are respectively the charge and current densities. When space-time is flat, then there exist charts valid for all M where $h=\eta_{\mu\nu}dx^\mu \otimes^S dx$, and in this case eqs.(1) are equivalent to

$$\nabla.\vec{E} = \rho_e \qquad \nabla\times\vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}_e \qquad (a)$$

$$\nabla.\vec{B} = 0 \qquad -\nabla\times\vec{E} = \frac{\partial \vec{B}}{\partial t} \qquad (b) \qquad (3)$$

where $dF=0$ is equivalent to the homogeneous equations (3b) and $\delta F = -J_e$ is equivalent to the inhomogeneous equations (3a).

To complete the formulation of classical electrodynamics it is necessary to know the coupling between the electric currents and the fields.

This is done after we introduce the concept of a charged particle as a triple (m_e, e, γ), where $m_e>0$ is said to be the mass of the particle, $e\in \mathbf{R}$ is said to be the charge of the particle and $\gamma: \mathbf{R} \supset I \to M$ is a future-pointing time-like curve[1]. We now introduce the

LORENTZ POSTULATE: The equation of motion of charged particles is given by the equation

$$\frac{dp}{ds} = \frac{e}{m}\bar{F}(p,.) \qquad (4)$$

where $p=m\gamma_*$ is the momentum, $s\in I$ is the proper-time, γ_* is the velocity of the electric particle, $\bar{F}=F^\mu{}_\nu\, dx^\nu \otimes e_\mu$ and $e\bar{F}(p,)$ is called the Lorentz-force.

3.STANDARD ELECTRODYNAMICS AS A PRINCIPAL FIBER BUNDLE THEORY

The model of electrodynamics as a PFB is a follows(*).Let (M,h,∇) be a Lorentzian manifold and let $\pi:P\to M$ be a PFB with group $U(1)=\{e^{i\theta},\theta\in \mathbf{R}\}$ and with Lie algebra $\hat{U}(1)=\{\alpha i,\alpha\in \mathbf{R}\}$.

Let ω be a connection 1-form over P, i.e., $\omega\in \Lambda^1(P, \hat{U}(1))$, and let $U\subset M, \sigma_U:U\to P$ be a local section. The pullback of ω is the electromagnetic potential, and we write

$$\omega_U = \sigma_U^* \omega = -iA_U \qquad (5)$$

(*) See Appendix B for details.

The electromagnetic field $(\Omega^\omega = d\omega)$ relative to $\sigma_U:U\to P$ is

$$i\Omega_U = F_U = dA_U . \qquad (6)$$

If $\sigma_V:V\to P$ is another local section, then from the fact that $U(1)$ is abelian we have [see eq. (B18)] that $F_U=F_V$. It follows that the curvature of the connection is in this case a 2-form which is well defined in the base manifold M; in other words, the electromagnetic field is gauge invariant.

Then, fixing $U\subset M$, if $A\in\Lambda^1(U,\mathbf{R})$ is the potential (*), the field is $F=dA$ with $A=A_\mu dx^\mu$, and $F_{\mu\nu}=\partial_\mu A_\nu - \partial_\nu A_\mu$. From Bianchi identity it follows that

$$dF = 0 \quad \text{(homogeneous Maxwell equations)} \qquad (7)$$

As there are no reasons for δF to be null ,we put

$$\delta F = -J_e \quad \text{(inhomogeneous Maxwell equations)} \qquad (7')$$

where J_e is the current one-form here introduced in a purely "*phenomenological*" way. Eqs (7) are the Maxwell equations introduced in §1 [eqs(1)].

4.THE MONOPOLE

We put now a question: How can we modify Maxwell equations in order to describe also magnetic monopoles. In order to understand the problems associated with the existence of the monopole we consider here (M,h,∇) as being a Minkowski space-time. Then, there is a global coordinate system (x^μ) in M where Maxwell equations have the form of eqs (3). The natural guess for introducing monopoles is to generalize phenomenologically eqs (3) into

$$\nabla.\vec{E} = \rho_e \qquad \nabla\times\vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}_e \qquad (a)$$

$$\nabla.\vec{B} = \rho_m \qquad -\nabla\times\vec{E} = \frac{\partial \vec{B}}{\partial t} + \vec{j}_m \qquad (b) \qquad (8)$$

The magnetic particles (monopoles) are modelled as triples (m_g,g,σ) where

(*) See Appendix B for details.

$m_g \in \mathbb{R}^+$ is the mass of the monopole, $g \in \mathbb{R}$ is its magnetic charge and σ: $\mathbb{R} \supset I \to M$ is a future-pointing time like curve. The Lorentz force in this case is postulated to be

$$\vec{F}_g = -g\vec{B} + g\vec{v}_g \times \vec{E} \; ; \quad \sigma_* = (v_g^0, \gamma\vec{v}_g) \tag{9}$$

where γ is the Lorentz factor.

The intrinsic formulation of eqs (8) in the Hodge bundle read

$$\begin{aligned} dF &= -{}^*J_m && \text{(a)} \\ \delta F &= -J_e && \text{(b)} \end{aligned} \tag{10}$$

where J_e, already introduced (eq. (2)), is the *phenomenological* electric current 1-form and $J_m = (\rho_m, -j_m)$ is the *phenomenological* magnetic current 1-form.

Before we go on, it is necessary to emphasize that eqs (8) are invariant under the transformations

$$\begin{aligned} \vec{E} &\to \vec{E}'\cos\theta + \vec{B}'\sin\theta \\ \vec{B} &\to -\vec{E}'\sin\theta + \vec{B}'\cos\theta \\ \vec{j}_e &\to \vec{j}_e'\cos\theta + \vec{j}_m'\sin\theta \qquad \vec{j}_m \to -\vec{j}_e'\sin\theta + \vec{j}_m'\cos\theta \\ \rho_e &\to \rho_e'\cos\theta + \rho_m'\sin\theta \qquad \rho_m \to -\rho_e'\sin\theta + \rho_m'\cos\theta \end{aligned} \tag{11}$$

As a consequence of this fact, we see that if the ratio e/g of all particles in nature is a universal constant, ie, $J_e = kJ_m$, k a constant, then it is a always possible to choose an angle in eq (11) such that eqs (8) transform into eqs (3), i.e., the usual Maxwell equations. In that case the label electric charge or magnetic charge would be arbitrary. In what follows we suppose that $J_e \neq kJ_m$.

It is a well known fact that the existence of a *Lagrangian formalism* for the electrodynamics of *charged electric particles* rests on the fact that we can write F=dA, for all $x \in M$, since in this case the canonical momentum of a given charged electric particle e is

$$\Pi_\mu = p_\mu + eA_\mu = \frac{\partial L}{\partial \dot{x}^\mu} \quad ; \mu = 0,1,2,3 \tag{12}$$

where p_μ is the kinetic momentum and

$$L = \frac{1}{2m}(p_\mu + eA_\mu)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{13}$$

Now, a magnetic monopole g at the origin of the coordinate system $\langle x^\mu \rangle$ in M creates a magnetic field satisfying $\nabla\cdot\vec{B} = g\delta(\vec{x})$.

If the magnetic field is defined globally by a potential A, singular only at the origin $\vec{x}=0$, i.e., F=dA, (or $\vec{B} = \nabla\times\vec{A}$), then the magnetic flux through any closed surface S containing g must vanish. Indeed, $\partial S = 0$ and by Stokes theorem we have $\int_S F = \int_{\partial S} A = 0$ or

$$\int_S \vec{B}.d\vec{S} = \int_S (\nabla\times\vec{A}).d\vec{S} = \oint_\Gamma \vec{A} \,.\, d\vec{S} - \oint_\Gamma \vec{A} \,.\, d\vec{S} = 0 \tag{14}$$

where Γ is any closed curve in S. The moral of eq (14) is the following: If the field of a magnetic monopole is to be described by a *single potential* A, then at least one assumption used to deduce eq (14) must fail to hold. We have at least three possible solutions for the formulation of the monopole theory:

- **THE DIRAC STRING:** A way out of this dilemma was found by Dirac[2]. Imagine an infinitely thin solenoid extending form $-\infty$ to the origin in the x^3-axis of the $\langle x^\mu \rangle$ coordinate system. Its field $\vec{B}_{sol}$ satisfies $\nabla\cdot\vec{B}_{sol} = 0$ and is given by

$$\vec{B}_{sol} = \frac{g}{4\pi r^3}\vec{r} + g\theta(-z)\delta(x)\delta(y)c \tag{15}$$

where $\hat{z}$ is the unitary vector in the x^3 direction and $\theta(\xi)$ is the Heaviside function. This magnetic field differs from the field of the monopole $\vec{B} = g\vec{r}/4\pi r^3$ only by the contribution $g\theta(-z)\delta(x)\delta(y)z$ due to the solenoid. Writing $\vec{B}_{sol} = \nabla\times\vec{A}$, we have

$$\frac{g}{4\pi r^3}\vec{r} = \nabla\times\vec{A} - g\theta(-z)\delta(x)\delta(y)\hat{z} \tag{16}$$

The line occupied by the solenoid is called the Dirac string. We can solve eq (16) for $\vec{A}$ obtaining

$$\vec{A}(\vec{r}) = \frac{g}{4\pi r} \cdot \frac{1-\cos\theta}{\sin\theta}\hat{\phi} \qquad (17)$$

where $\hat{\phi}$ is the *unitary vector* in the φ-direction of the spherical coordinate system in $\mathbf{R}^3$. Eq.(17) shows very clearly the singularity in the negative z-axis ($\theta=\pi$). Using this potential invalidates the deduction of eq (14).

It is fundamental to observe that the line occupied by the string can be changed into another arbitrary line in $\mathbf{R}^3$ starting at the origin, by a gauge transformation $A \to A - \frac{g}{4\pi}\nabla\Omega$ where $\Omega: \mathbf{R}^3 \to \mathbf{R}$ is a differentiable function.

This fact shows that the *Dirac string* is a non physical object. Dirac in 1931[2] and then in 1948[3] developed, using the potential given by eq (17) a Lagrangian theory for the motion of a charged particle in the field of a monopole. In particular in 1931 studying the quantum version of the theory he found the famous quantization condition(+)

$$\frac{eg}{4\pi} = n\,\frac{\hbar}{2};\, n \in \mathbf{Z} \qquad (18)$$

We will obtain this condition in our theory (§5) using a procedure different from the one used by Dirac.

To end, we must observe that in the case of the quantum formulation of the motion of charges and monopoles with string there are non-trivial problemas which have not been solved in a satisfactory way [4,5].

- **THE TOPOLOGICAL MONOPOLE:** We saw in §3 that, interpreting the electromagnetic connection as a 1-form defined globally over a principal U(1) bundle over M, provides an alternative description for the electrodynamics of charged particles. *In this case eqs (3) hold good instead of eqs (10) and we meet the question:* Can Maxwell equations [eqs (3)] accomodate the existence of magnetic monopoles?[7]

The answer is *yes* : all we need is a situation where does not exist a global defined potential such that $F=dA$. This happens if the PFB $\pi: P \to M$ with group U(1) is non trivial, since the existence of a global section (gauge) σ_U would provide a way to define the potential $iA = \sigma_U^* \omega$ globally over space-time M . As all fibre bundle over a contractible paracompat base space are trivializable[7], we must choose as base of our PFB a *noncontractible* space-time. This can be done by deleting the world line of the magnetic monopole from Minkowski space. We choose then $M = \mathbf{R}^4$ - (pole world-line)

In this theory the magnetic monopole is then of topological origin: it is a hole in Minkowski space-time!

(+) In general we use units such that $\hbar=1$

Since $M = \mathbf{R}^2 \times S^2$ and $\mathbf{R}^2$ is contractible, the classification of the PFB's $\pi: P \to M$ with group U(1) reduces to the classification of the PFB's $\pi: P \to S^2$ with group U(1). The classification is given by the first group of homotopy $\pi_1(U(1)) = \pi_1(S^1) = \mathbf{Z}$. The integer n, coresponding to the element of $\pi_1(S^1)$, is obtained by calculating the *first Chern class* of P , $c_1(P) \in H^2(S^2, \mathbf{R})$ over S^2 [see Milnor-Stasheff[6]]. Quantity $c_1(P)$ is given by

$$c_1(P) = -\frac{\Omega}{2\pi i} \qquad (19)$$

where Ω is the curvature of the electromagnetic field. The number

$$C_1 = 2\pi \int_{S^2} c_1(P) \qquad (20)$$

is an integer called the first Chern-number and it classifies all nonequivalent PFB $\pi: P \to S^2$ with fiber S^1 *and then it also classifies all solutions of Maxwell equation in* $\mathbf{R}^3 - \{0\} \simeq S^2$. This integer n is called the *magnetic charge* of the monopole.

Let then S^2 be described by two open sets H_+ and H_- and let $0 \le \theta \le \pi$; $0 \le \phi \le 2\pi$ be the coordinates of H_+ and H_- .

Let U(1) S^1 be described by the coordinates $e^{i\psi}, 0 \le \psi \le 2\pi$.

$H_+ \cap H_-$ is a thin band around the equator parametrized by ψ . The PFB$\pi: P \to S^2$ with group U(1) is then decomposable into two local trivializations.

$H_+ \times U(1)$, coordinates $(\theta, \varphi, e^{i\psi_+})$

$H_- \times U(1)$, coordinates $(\theta, \varphi, e^{i\psi_-})$

The transition functions $g_{H_+H_-}: H_+ \cap H_- \to U(1)$ are functions of φ and therefore are elements of U(1) . We then relate H_+ with H_- by

$$e^{i\psi_-} = e^{in\varphi}\, e^{i\psi_+} \qquad (21)$$

For the resulting structure to be a manifold, n must be an integer. This means that the fibers must be indentified when we complete a turn around the equator. This is essentially the topological version of Dirac quantization condition [7].

When n=0 , we have a trivial PFB;$P(n=0) = S^2 \times S^1$. When n=1 we have the famous Hopf fibration of S^3,$P(n=1) = S^3$, which descibes a monopole with charge n=1. For a general $n \in \mathbf{Z}$ we have a PFB corresponding to a monopole with charge n . As we already said n corresponds to the first Chern class and is given by eq (20). Let us do the explicit calculations. Consider then a connection ω defined globally over the PFB $\pi: P \to S^2$ whith group U(1) such that the "pull-backs" for two local trivializations $H_\pm$ are

$$-i\sigma^*_{H_\pm}\omega = \begin{cases} A_+ + \frac{1}{2\pi} d\psi_+ \text{ over } H_+ \\ A_- + \frac{1}{2\pi} d\psi_- \text{ over } H_- \end{cases} \quad (22)$$

The choice of the transition function $e^{i\psi_+}=e^{in\phi}e^{in\psi_-}$ [eq.(19)] implies the gauge transformation $A_+=A_- + \frac{n}{2\pi} d\phi$.

The *potentials* that satisfy Maxwell equations $[\nabla\cdot\vec{B}=0]$ in $S^2 \simeq R^3 - \{0\}$ and are regular in H_+ and H_- are given by

$$A_\pm = \frac{n}{4\pi}(\cos\theta - 1)\, d\varphi = \frac{n}{4\pi}\frac{xdy - ydx}{z \pm r} \quad (23)$$

The electromagnetic field on $H_+ \cup H_-$ is given by

$$F = dA_\pm = \frac{n}{4\pi}\sin\theta\, d\theta \wedge d\varphi = -\frac{n}{4\pi r^3}(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy) \quad (24)$$

Using eq (20) we get

$$C_1 = 2\pi \int_{S^2} c_1(P) = -\int_{S^2} F$$
$$= -\int_{H_+} F_+ - \int_{H_-} F_- = \int_{S^1} (A_+ - A_+) = \int_{S^1} \frac{n}{2\pi} d\varphi \quad (20')$$

We see from eq (23) that $A_\pm$ are singular along the "strings" - z and +z ,respectively, where they are not defined. In the Dirac formulation local charts have not been defined, A_+ being used for all $R^3 - \{0\}$: this is the source of the (fictitious) "string" singularities.

We end this "resume" about the topological magnetic monopole with the observation that in the U(1) gauge theory of electromagnetism the discreteness of the unitary representations of U(1) implies *also* the quantization of electric charge [7]. To see this, consider a one-dimensional unitary representation of U(1) on C, i.e.,

$$\rho: U(1) \to U(1)\,; \quad \exp(it) \to \exp(iat)$$

The condition $a(t+2\pi)=at+2\pi n$, where $n \in Z$, gives $a=n$. Assuming a minimal coupling of the electromagnetic potential A with the matter field, of the type (ρ, C),[see App. C] we have

$$D_\mu\Psi = \partial_\mu\Psi - \rho_*(0)eA_\mu\Psi = \partial_\mu\Psi - ineA_\mu\Psi \quad (25)$$

Since the electromagnetic potential couples to all charged fields, e is fixed!

We mentioned at the begining of §4 that there are three possible ways to formulate a monopole theory. We already examined: (i) the Dirac monopole with string, and (ii) the topological monopole.

In (i) we have unphysical singularities, and in (ii) [which refers to an extremely beautiful theory] we need to change the topology of the space-time manifold.

- We now present in §5 a theory of *magnetic monopoles without string*, with a generalized potential, which is formulated in the Clifford bundle over space-time (for definitions see Appendix A). The resulting theory rivalizes in mathematical beauty with the topological monopole theory although it desviates from the main stream of present theoretical physics which gives emphasis to the Lagrangian formulation. In § 6 we present a generalization of the theory of the monopole theory without string as a PFB theory with gauge group G×G using the spliced bundle concept and where G is a gauge group of standard gauge theory.

5.THEORY OF MAGNETIC MONOPOLES WITHOUT STRING IN MINKOWSKI SPACE.

We learned in Appendix A that, among others, we can give the structure of a Clifford algebra to $\oplus\Lambda^p(T_x^* M)$ and, then, we can define the Clifford bundle over space-time M , C(M). We know that in C(M) the natural derivative operator is the Dirac operator

$$\gamma^\mu \nabla_{e_\mu} \quad (26)$$

where $\gamma^\mu = dx^\mu$ in the case of a natural basis and ∇_{e_μ} is the usual covariant derivative.

In what follows we restrict ourselves (for simplicity) to the case where M is the Minkowski space-time and then

$$\partial = \gamma^\mu \nabla_{e_\mu} = \gamma^\mu \partial/\partial x^\mu \quad (27)$$

where $\gamma^\mu \cdot \gamma_\nu = \delta_\nu{}^\mu$.

We know that

$$\partial = d - \delta \quad (28)$$

and

$$*f_p = (-1)^i \gamma^5 f_p\,; \; f_p \in \Lambda^p\tau^* M \quad (29)$$

We then can write equations (10 a) and (10 b) describing the phenomenological theory of monopoles and charges as

$$\partial F = J_e - {}^*J_m = J_e + \gamma^5 J_m \qquad (30)$$

We now define *generalized potential* [8] the quantity

$$\omega = \alpha + \gamma^5\alpha' \;;\; \alpha,\ \alpha' \in \quad \sec \Lambda^1\tau^*M \qquad (31)$$

Applying the operator ∂ to ω we get the sum of a 0-form, two 2-forms and 4-form, i.e.

$$F = \partial\cdot\alpha + \partial\wedge\alpha + \partial\cdot(\gamma^5\alpha') + \partial\wedge(\gamma^5\alpha')$$

and imposing the Lorentz gauge $\partial\cdot\alpha = 0 \Leftrightarrow \partial^\mu\alpha_\mu = 0, \partial\wedge(\gamma^5\alpha') = 0 \Leftrightarrow \partial^\mu\alpha'_\mu = 0$ we obtain F as a 2-form, i.e.

$$F = \partial\wedge\alpha + \partial.(\gamma^5\alpha') \qquad (32)$$

We get also that

$$\Box\alpha = J_e \quad ; \quad \Box\alpha' = J_m \quad ; \quad \Box = \partial\partial = (\partial_0)^2 - \nabla^2 \qquad (33)$$

To see the power of the Clifford-formalism, we now deduce the conservation-laws and the couplings of the electromagnetic field to electric charge and magnetic monopole.

CONSERVATION LAWS AND GENERALIZED LORENTZ FORCE:

Let us observe that from eq (30), by applying the anti-automorphism + (reversion), we get the equation

$$F^+\partial^\wedge = J_e + J_m\gamma^5 \qquad (36)$$

where the symbol $\partial^\wedge$ refers to the fact that the Dirac operator acts on the right; i.e.: $F^+\partial^\wedge = -\partial_\alpha(F_{\mu\nu})\gamma^\mu\gamma^\nu\gamma^\alpha$.

Multiplying eq (30) by F^+ on the left and eq (3.7) by F on the right, and summing both equations, we get

$$\frac{1}{2}(F^+\partial F + F^+\partial^\wedge F) = \frac{1}{2}(J_e F - F J_e) + \frac{1}{2}(J_m\gamma^5 F - \gamma^5 F J_m) \qquad (37)$$

Defining moreover

$$S^\mu = -\frac{1}{2}F^+\gamma^\mu F \qquad (38)$$

eq (37) can be written as

$$\partial_\mu S^\mu = F.J_e - {}^*F.J_m \qquad (39)$$

Now, from eq. (38), we get immediately that $S^{\mu+} = S^\mu$ and $S^\mu = \cdot S^\mu$,where the bar indicatey the inversion, i.e., the main automorphism defined in Appendix A. The unique objects in the Clifford algebra of differential forms that satisfy these equations are the 1-forms. We call the quantities S^μ the *energy-momentum* 1-forms. The reason for such name is that $E^{\mu\nu} = S^\mu \cdot \gamma^\nu$ are the components of the *symmetric energy momentum* tensor of the electromagnetic field, as we show below.

In particular $S^0 = -\frac{1}{2}F F\gamma^0$, $F = \gamma^0 F\gamma^0$ and, writing $F = \vec{E} - \gamma^5\vec{B}$, we get by projecting into the Pauli-Algebra (see Appendix A) the following splitting into two quantities

$$S^0\gamma^0 = U + \vec{S}^0 \quad ; U = \frac{1}{2}(\vec{E}^2 + \vec{B}^2); \quad \vec{S}^0 = \vec{E}\times\vec{B} \qquad (40)$$

which we recognize as the energy-density and the Poynting vector of the electromagnetic field, respectively.

More generally we have

$$E^{\mu\nu} = -\langle\frac{1}{2}F\gamma^\mu F\gamma^\nu\rangle = -\langle(F.\gamma^\mu)F\gamma^\nu\rangle - \frac{1}{2}\langle\gamma^\mu F^2\gamma^\nu\rangle$$

$$= -(F.\gamma^\mu).(F.\gamma^\nu) - \frac{1}{2}(F.F)\gamma^\mu.\gamma^\nu \qquad (41)$$

$$= F^{\mu\alpha}F^{\lambda\nu}\eta_{\alpha\lambda} + \frac{1}{2}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

Writing ,

$$K_e = F\cdot J_e \quad \text{and} \quad K_m = -{}^*F\cdot J_m,$$

$$K_e = F^{\mu\nu}J_e{}^\nu \qquad K_m = -{}^*F^{\mu\nu}J_m{}^\nu \qquad (42)$$

and projecting K_e and K_m on the Pauli-Algebra we get

$$K_e\gamma^0 = \vec{J_e}.\vec{E} + (\rho_e\vec{E} + \vec{J_e}\times\vec{B}) \qquad (a)$$

$$K_m\gamma^0 = -\vec{J_m}.\vec{B} + (-\rho_m\vec{B} + \vec{J_m}\times\vec{E}) \qquad (b) \qquad (43)$$

We see, then, that K_e and K_m represent the Lorentz forces that act on the electric charges and the magnetic monopoles, respectively. As this result has been derived only from the Maxwell equations, we arrive at the conclusion that the Lorentz forces (electric and magnetic) need not be postulated, as is usually done (see § 2).

We note that, due to the symmetry $E^{\mu\nu}=E^{\nu\mu}$, we can write $\partial_\mu E^{\mu\nu}=\partial_\mu E^{\nu\mu}=\partial_\mu(S^\nu\cdot\gamma^\mu)=\partial\cdot S^\nu$. Then eq (39) can be written

$$\partial.S^\nu = Q^\nu \quad ; \quad Q^\nu = (F.J_e).\gamma^\nu - (^*F.J_m).\gamma^\nu \qquad (39')$$

The interpretation of eq (39) is now clear. The equation

$$\partial_\mu E^{\mu\nu} = F^\mu{}_\nu J_e{}^\nu - {}^*F^\mu{}_\nu J_m{}^\nu \qquad (39'')$$

expresses very clearly the fact that the energy momentum of the field is not conserved, $\partial_\mu E^{\mu\nu}\neq 0$ when mater (described by J_e, J_m) is present. Actually, one expects that only the total energy momentum of field and currents be conserved.

If we write the r.h.s. of eq (39') as $-\partial_\mu M^{\mu\nu}$, with

$$\partial_\mu M^{\mu\nu} = \gamma^\nu.K_e + \gamma^\nu.K_m \qquad (44)$$

then eq (39) assumes the structure of a global conservation equation

$$\partial_\mu(E^{\mu\nu}+M^{\mu\nu}) = 0 \qquad (45)$$

where $M^{\mu\nu}$ plays the role of the *symmetric* energy momentum of matter (i.e. of the currents).

THE MOTION EQUATIONS DERIVED FROM MAXWELL EQUATIONS[9]: In analogy to what happens in general relativity, the identification of $M^{\mu\nu}$ with the actual energy -momentum tensor of the matter currents *leads* directly to the motion equations.

Let us show this in the simple, but easily generalizable, case in which the field F is generated by a single electric charge e and a single magnetic charge g. Be in fact, the electric and magnetic matter represented by the triples (m_e,e,γ) and (m_g,g,σ) so as in §3 and §4, respectively. The most general *symmetric* tensor that we can write to represent matter is then[20]

$$M = M^{\mu\nu}\gamma^\mu \otimes^s \gamma^\nu$$

$$= -m_e \int ds\, \delta(x-\gamma(s))\, \gamma_* \otimes^s \gamma_* \; - m_g \int ds'\, \delta(x-\sigma(s'))\, \sigma_* \otimes^s \sigma_* \qquad (46)$$

In components, writing $x^\mu(\gamma(s))=z^\mu(s); x^\mu(\sigma(s'))=y^\mu(s')$, we have

$$M^{\mu\nu} = -m_e \int ds\, \delta(x^\alpha - z^\alpha)\frac{dz^\mu}{ds}\frac{dz^\nu}{ds} - m_g \int ds'\, \delta(x^\alpha - y^\alpha)\frac{dy^\mu}{ds'}\frac{dy^\nu}{ds'} \qquad (46')$$

Incidentally, it is $J_e = \int ds\,\delta(x-\gamma(s))\gamma_*$, $J_m = \int ds\,\delta(x-\sigma(s'))\sigma_*$,

wherefrom it follows $\partial\cdot J_e = \partial\cdot J_m = 0$.

Now, comparing eq (44) with eq (40) and recalling eqs (43), it is immediately seen that

$$m_e\ddot{z}_i = \rho_e E_i + (\vec{z}\times\vec{B})_i \;\; ; m_g\ddot{y}_i = -\rho_m B_i + (\vec{y}\times\vec{E})_i \qquad (47)$$

which are the correct *equations of the motion* of electric and magnetic charges.

DIRAC QUANTIZATION CONDITION: In studying the motion of an electric charge in the field of a monopole (neglecting radiative effects), the kinetic energy of the charge is conserved. Taking in what follows x^μ as the relative coordinates and m as the reduced mass, we can adopt as quantum Hamiltonian the quantity $\mathcal{H}=\frac{1}{2m}p^2$, $p=dr/ds$ where $s\rightarrow\gamma(s)=(x^0(s),x^i(s))$ with the ordinary postulates $[c=\hbar=1]$

$$[x_i,x_j]=0; \quad [x_i,p_j] = i\,\delta_{ij} \qquad (48a)$$

$$[p_i,p_j] = i\epsilon_{ijk}B^k \qquad (48b)$$

The quantization rules given by eqs (48) are satisfactory, since, e.g., the angular momentum $\vec{J}=\vec{r}\times\vec{p}-eg\vec{r}/4\pi$ does commute with $\mathcal{H}$, and $[J_i,J_j]=i\epsilon_{ijk}J_k$. Moreover, from the Ehrenfest relation $\dot{\vec{r}}=i[\vec{r},\mathcal{H}]$, it follows the Lorentz-force expression

$$\ddot{\vec{r}} = \frac{e}{m}(\dot{\vec{r}}\times\vec{B} - \vec{B}\times\dot{\vec{r}})$$

We would like to stress here that the Jacobi identity is violated for the p_i's, since

$$\Sigma_{ciclic}[p_i, [p_j, p_k]] = e\,\nabla\cdot\vec{B} \qquad (49)$$

so that the conjugated momenta p_i do not close a Lie algebra under the commutator product whenever $\nabla\cdot\vec{B}\neq 0$ (i.e., when monopoles are present). Incidentally let us recall that $\nabla\cdot\vec{B}\neq 0$ corresponds also to a violation of Bianchi's identity in the standard U(1) theory of electromagnetism as PFB (§4). This has as a consequence that a PFB for electromagnetism with our monopoles can be implemented only by making resource to "spliced" $U(1)\times U(1)$ bundles: as we shall show explicity in §7.

Eq (49) does imply that the momenta p_i (conjugated to the coordinates x_i) cannot play the role of canonical momenta in any (local) Lagrangian whatsover. In fact, if such a Lagrangian existed, then the Jacobi equation would be automatically satisfied(*). This then justifies the well known fact that there does not exist using the classical tensor calculus a Lagrangian which yield simultaneously the field equations and the equations of motion of charged particles and magnetic monopoles[10].

Happily enough, this is not a problem when we use the Clifford formalism, since knowing the field equations does already mean knowing also the equations of motion of electric and magnetic charges.

We now show that eq (48) implies the Dirac quantization condition. Indeed, if the operators $U(\vec{a}) \equiv \exp[i\,\vec{a}\cdot\vec{p}]$, with $\vec{a}$, any euclidian vector ($\vec{a}\in R^3$), have to yield a projective representation of the translation group in the Hilbert space of our charge-monopole system, the associative law, in particular, must be satisfied

$$[\,U(\vec{a})U(\vec{b})\,]U(\vec{c}) = U(\vec{a})\,[U(\vec{b})U(\vec{c})\,] \qquad (50)$$

Explicit calculations, using eqs (48) and (49), then yield:

$$[\,U(\vec{a})U(\vec{b})\,]U(\vec{c}) = e^{ie\Phi}U(\vec{a})\,[U(\vec{b})U(\vec{c})\,] \qquad (50')$$

where Φ is the magnetic field flux crossing the surface of the tetrahedron individuated by the three vectors $\vec{a},\vec{b},\vec{c}$. The compatibility of eq. (49) and eq. (50) requires $e\Phi=2\pi n$. If one monopole only is localized inside the tetrahedron, then:

(*) Note that a canonical momentum, as the one present in eq (12), makes no sense here, because now we have two potentials α and $*\alpha'$

$$\frac{eg}{4\pi} = \frac{n}{2}\ ,\ n\in \mathbb{Z} \qquad (51)$$

Note that eq. (51) implies that $\vec{J}\cdot\vec{r} = \frac{n}{2}$, which means that (even in the case when the electric charge and the monopole are bosons) they can be in an half-integral spin state.

GENERALIZATION OF MANDELSTAM APPROACH[18]: Let $\phi(x,\Gamma)$ be the Mandestam path dependent wave function[19] for a charged particle in an ordinary electromagnetic field $F_e=\partial\alpha=d\alpha$. If $\varphi(x)$ is the usual wave function of the particle and $\mathcal{A}_i = -e\int_\Gamma^x \alpha$ is the classical interaction action, we have

$$\phi(x,\Gamma) = \varphi(x)\exp(\int_\Gamma -ie\alpha) \qquad (52)$$

where Γ is the arbitrary path from $\infty\rightarrow x$. If we choose two paths Γ and Γ' differing only for a finite part, we get (using Stokes theorem)

$$\phi(x,\Gamma') = \phi(x,\Gamma)\exp(\int_S^x -ied\alpha) \qquad (53)$$

where S is an arbitrary surface such that $\partial S=\Gamma'-\Gamma$.

How to generalize eq (53) for the case when the charge e interacts with the electromagnetic potential $\omega=\alpha - *\alpha'$? Let us introduce the following

Interaction postulate (IP): The introduction of an electric charge e, represented by the path dependent wave-function $\phi(x,\Gamma)$, with the generalized electromagnetic field $F=\partial\omega$ is given by

$$\phi(x,\Gamma') = \phi(x,\Gamma)\exp(\int_S -ieF) \qquad (54)$$

We now show that the IP implies the Dirac quantization condition, being then compatible with the commutation relations given by eqs. (48).

If eq. (54) is to be independent of the surface S, so that $\partial S=\Gamma'-\Gamma$, we must have

$$\exp(\int_{S_0} -ie(d\alpha - *d\alpha')) = 1 \qquad (55)$$

where S_0 is a closed surface. By Stokes theorem we can write eq (54) as

$$\exp\left(\int_V ie*\delta d\alpha' \right) = 1 \qquad (56)$$

Supposing, now, that we have one monopole inside V and that $J_m=(g\delta(\vec{r}),0,0,0)$, and taking into account that $\Box\alpha'=-(d\delta+\delta d)\alpha'=-\delta d\alpha'=J_m$, we have

$$\exp\left(\int_V ie*J_m \right) = \exp(-ieg) = 1 \quad \Rightarrow \quad \frac{eg}{4\pi} = \frac{n}{2}, n \in Z \qquad (57)$$

6.LAGRANGIAN FORMALISM:

It is well known[4,10,12] that there does not exist a local lagrangian which gives simultaneously the equation of motion of electric charges and magnetic monopoles, as well as the field equations of the generalized electromagnetic field. The recent claim by Fryberger[13] that this is possible within the Clifford algebra formalism is *non-sequitur* as we now show.

We start by considering the classical action $\mathcal{A}^C=\mathcal{A}_e^C+\mathcal{A}_i^C$ for the interaction of a charged particle with the electromagnetic field $\alpha=\alpha_\mu dx^\mu$. We have

$$\mathcal{A}^C=\mathcal{A}_e^C+\mathcal{A}_i^C = -\int_a^b (m_e ds - e\alpha_\mu dx^\mu) \qquad (58)$$

Now, the variation of $\mathcal{A}_i^C$ can be written in two independent ways:

$$\delta_x \mathcal{A}_i^C = -e \int_{\partial S} \alpha = -e \int_S d\alpha = -\frac{e}{2} \int_S (Fe)_{\mu\nu} dx^\mu \wedge dx^\nu \qquad (59a)$$

$$\delta_x \mathcal{A}_i^C = -e \int_a^b \delta_x (\alpha_\mu dx^\mu) = -e \int_a^b (Fe)_{\mu\nu} dx^\mu \, \delta x^\nu \qquad (59b)$$

while the variation of $\mathcal{A}_e^C$ is

$$\delta_x \mathcal{A}_e^C = \int_a^b m_e du_\mu \, \delta x^\mu \qquad (60)$$

where δ_x indicates that Γ, the equilibrium path of integration from a to b has been changed into Γ' by an arbitrary infinitesimal function δx^μ, such that $\delta x^\mu(a)=\delta x^\mu(b)=0$. Moreover $\partial S=\Gamma'-\Gamma$; etc...

Comparison of eqs (59 a) and (59 b) suggested to Fryberger[13] the following identification $(d\sigma^{\mu\nu} = dx^\mu \wedge dx^\nu)$:

$$\frac{1}{2} \int_S d\sigma^{\mu\nu} \quad \Leftrightarrow \quad \int_a^b dx^\mu \, \delta x^\nu \qquad (61)$$

He interpreted eq.(61) as having the following geometrical meaning: the surface S, spanning the loop formed by Γ and Γ'; has an infinitesimal width, δx^μ, thus reducing the surface integral to a line integral.

The IP suggests the possible existence of a generalized action $\mathcal{A}=\mathcal{A}_e+\mathcal{A}_i$, since eq (54) suggests that

$$\delta_x \mathcal{A}_i = -e \int_S F = -e \int_S (F_e - \gamma^5 F_m) \qquad (62)$$

where $F = \partial\omega = \partial\alpha - \gamma^5\partial\alpha' = F_e - \gamma^5 F_m$.

To proceed with our analysis, we are going to use below (in the remaining part of this section) the Clifford bundle $C(\tau M)$ instead of $C(\tau^* M)$. In other words, we are going to use *multivectors* instead of multiforms. Obviously there is a canonical isomorphism (see Appendix A) between multivectors and multiforms, and in the following we represent the multivector corresponding to a given multiform by the same letter. Letting $e_\mu(\mu=0,1,2,3)$ be an orthonormal basis of $T_x M$ and $e^\mu=\eta^{\mu\nu}e_\nu$ the reciprocal basis, the Dirac operator when acting on multivectors is then $\partial=e^\mu\partial/\partial x^\mu$. The fundamental pseudo-scalar is now $e_5=e_0e_1e_2e_3$. By using multivectors, eq (59 a) reads

$$\delta_x \mathcal{A}_i^C = -\int_{\partial S} \alpha.dx = -e \int_S F.d\sigma \qquad (59a')$$

where $\alpha=\alpha_\mu(x)e^\mu$, $dx=dx^\mu e_\mu$ and $d\sigma = (dx^\mu \wedge dx^\nu)e_\mu e_\nu$.

In ref[8] we considered the following generalized action

$$\mathcal{A}_i = -\int_a^b (\alpha.dx + e_5\alpha'.dx) \qquad (63)$$

which is non-conventional, since it is the sum of scalar and pseudo-scalar terms.

Fryberger considers, instead of $\mathcal{A}_i$, the action

$$\mathcal{A}_i = \int_a^b \langle \alpha.dx + e_5\alpha'.dx \rangle \qquad (64)$$

where the brackets mean the scalar part.

Now the variation of $\mathcal{A}_i$ is

$$\delta_x \mathcal{A}_i = -e \int_a^b (Fe)_{\mu\nu} dx^\mu \, \delta x^\nu - e \int_a^b e_5 (Fm)_{\mu\nu} dx^\mu \, \delta x^\nu \qquad (65)$$

It is quite obvious that the second term on the right hand side of eq (65) cannot be combined with $\delta\mathcal{A}_e$, that does not contains a pseudo-scalar part.Now, Fryberger writes the sequence of "identities"

$$\int_a^b e_5(F_m)_{\mu\nu}dx^\mu\,\delta x^\nu \overset{(61)}{=} \frac{1}{2}\int_S e_5(F_m)_{\mu\nu}dx^\mu\wedge dx^\nu \tag{66}$$

$$\overset{\text{Fryberger}}{=} \int_S {}^*F_m = \int_a^b ({}^*F_m)_{\mu\nu}dx^\mu\,\delta x^\nu \tag{66'}$$

While eq (66) is *correct*, it does not *imply* however eq (66'), since eq. (66) forwards a pseudo-scalar and eq (66') a scalar.

The situation is even more confusing in ref[13], since that author uses $\bar{\mathcal{A}}_i$ instead of $\mathcal{A}_i$ and in that case the term $e_5\alpha'\cdot dx$ make obviously no contribution to $\delta_x\mathcal{A}_i$.

Having discussed the fact that we cannot write a generalized action even in the Clifford formalism that yield the equations of motion, we only mention here that it is possible to write an action $\mathcal{A}_t=\mathcal{A}_f+\mathcal{A}_i$ which yields the generalized Maxwell equations $\partial F=J_e+\gamma^5 J_m=J$.

This can be done by writing

$$\bar{\mathcal{A}}_t = -\frac{1}{2}\int \langle F\,F\rangle\, d^4x - \int\langle J\omega\rangle\, d^4x \tag{67}$$

and varying independently α and α'; or by making use of $\mathcal{A}_t=\mathcal{A}_f+\mathcal{A}_i$:

$$\mathcal{A}_t = \int d^4x\,(-\frac{1}{2}F.F - J\,\omega) = \mathcal{A}_f+\mathcal{A}_i \tag{68}$$

and treating $\omega=\alpha+\gamma_5\alpha'$ as the canonical coordinate (and deriving formally with respect to ω and $\partial\omega$ = F,when doing the variation). Such a procedure, even if not totally justified, gives the correct generalized Maxwell equations,as has been proved in ref[8].

To conclude,let us observe that in the ordinary field theories the Lagrangians are postulatedin such a way to yield the field and motion equation and conservation laws. On the contrary, all these things are automatically obtained by our Clifford bundle formalism, once the field equation are known; so that in our more economical approach it is redundant to look for Lagrangians.

7.SPLICED BUNDLE FORMULATION OF THE THEORY OF THE GENERALIZED MONOPOLES WITHOUT STRING[14]

In what follows we shall present an extension of the theory of electrodynamics with monopoles using a generalized potential in an arbitrary gauge theory. We emphasize once more that in our theory space-time have no holes.

The mathematical structure we shall use in our gauge theory with monopoles is the spliced bundle π_{12}:P o P→M with group G×G (see Appendix C) obtained from two identical PFB each one describying an ordinary gauge theory.(π:P→M with group G and base the space-time). We also observe that in what follows M may be a general Lorentzian manifold with non zero curvature.

In our theory we associate the gauge potential with a connection ω in π_{12}:P o P → M ; i.e., given a choice of the gauge in the PFB, T_U: $\pi_{12}^{-1}(U)\to U\times G\times G$ with the associated local section σ_U:U→P o P ,we define $\omega_U=\sigma_U^*\,\omega$ the gauge potential associated with the chosen gauge.

We observe that there exist (see Appendix C) two connections ω_1 and ω_2 in π:P→M such that $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$. It is fundamentally different to use, for describing the gauge potential,

(a): a connection $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$. $\in\Lambda^1$(P o P, $\hat{G}\oplus\hat{G}$) in the spliced bundle, or

(b): two connections $\omega_1,\omega_2\in\Lambda^1(P,\hat{G})$ in the original PFB of the theory without monopoles.

Let us consider first the case (a): Let then $T_U:\pi_{12}^{-1}(U)\to U\times(G\times G)$ and T_V : $\pi_{12}^{-1}(V)\to V\times G\times G$ be two gauges in π_{12}: P o P→M and such that $U\cap V\neq\emptyset$, and let be σ_U:U→P o P and σ_V :V→P o P the associated local sections.
The transference functions $g_{UV}:U\cap V\to G\times G$ are such that $g_{UV}(x)=((g_1)_{UV}(x),(g_2)_{UV}(x))$ with $x\in U\cap V$. Since

$$\omega = \pi^{1*}\omega_1\oplus\pi^{2*}\omega_2 \tag{69}$$

takes its values in $\hat{G}\oplus\hat{G}$, the gauge transformation between the gauge potentials ω_U and ω_V can be written as the two relations

$$(\omega_i)_V = ((g_i)_{UV})^{-1}d(g_i)_{UV} + ((g_i)_{UV})^{-1}(\omega_i)_U(g_i)_{UV}\ ,\ i=1,2. \tag{70}$$

In the case of standard electrodynamics, we can write

$$(\omega_1)_V = (\omega_1)_U + id\chi_{UV}\quad ;\quad (\omega_2)_V = (\omega_2)_U + id\psi_{UV} \tag{71}$$

with $(g_1)_{UV}(x) = \exp i\chi_{UV}$ and $(g_2)_{UV}(x)=\exp i\psi_{UV}$, and $\chi_{UV},\psi_{UV}:U\cap V\to R$.

In that way a gauge transformation of $\omega_U \in \Lambda^1(U,\hat{G}\oplus\hat{G})$ corresponds to *two independent gauge transformations* of $(\omega_1)_U$ and $(\omega_2)_U \in \Lambda^1(U,\hat{G})$.

In the case (b) we have two connections ω_1 and ω_2 in the PFB $\pi:P\to M$ with group G. Let $T_U:\pi^{-1}(U)\to U\times G$ and $T_V:\pi^{-1}(V)\to V\times G$ be two gauges an let $U\cap V\neq\varnothing$. Let moreover $\sigma_U:U\to P$ and $\sigma_V:V\to P$ be the associated local sections. The transition function now is $g_{UV}:U\cap V\to G$, and we have

$$(\omega_i)_V = (g_{UV})^{-1}dg_{UV} + (g_{UV})^{-1}(\omega_i)_U g_{UV}, \quad i=1,2.$$

In the case of standard electrodynamics we have

$$(\omega_i)_V = (\omega_i)_U + id\chi_{UV} \; ; \; \chi_{UV}: U\cap V\to R$$

These relations show that both potentials $(\omega_1)_U$ and $(\omega_2)_U$ change by effects of the same gauge transformation in case (b). This makes clear the difference between (a) and (b). In what follows we adopt the choice (a).

To show the necessity of such a choice, let us consider first the case of electrodynamics *with* monopoles described by two potentials: one related to the electric charges and the other to the magnetic charges as we did in § 5.

Notice that,in the case of standard electrodynamics without monopoles, it is $G=U(1)$ and the gauge potential takes its valued in $iR=\hat{G}$,i.e., $(\omega_1)_U = -iA_\mu dx^\mu \in \Lambda^1(U,iR)$. The gauge field is then

$$i(\Omega_1)_U = \tfrac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu \in \Lambda^2(U,iR) \qquad (72)$$

Observe that the field is invariant under the gauge transformation

$$(\omega_1)_U \to (\omega_1)_U + id\chi \qquad (73)$$

This information is interpreted geometrically as a choice of gauge, or local trivialization, and the associated transition function is $g = \exp i\chi : U\cap V\to G$.

As we already know (§ 2), we can write

$$\begin{aligned} i(\Omega_1)_U &= \tfrac{1}{2}(F_{0k}dx^0\wedge dx^k + F_{lm}dx^l\wedge dx^m) \\ &= (E_k dx^0\wedge dx^k + B_k {}^*(dx^0\wedge dx^k)) \; ; \quad k,l,m = 1,2,3, \end{aligned} \qquad (74)$$

and, since $(^*)^2 = -1$ when applied to 2-forms, the Hodge star operator changes $\vec{E}$ into $-\vec{B}$ and $\vec{B}$ into $\vec{E}$. Let us now consider the electrodynamics with charges and monopoles. The field generated only by electric charges can be described by the usual potential

$$(\Omega_1)_U = d(\omega_1)_U \qquad (75a)$$

The field generated by the magnetic charges is a dual field:

$${}^*(\Omega_2)_U = {}^*d(\omega_2)_U \qquad (75b)$$

The total field generated by electric and magnetic charges will be given by

$$-iF_U = (\Omega_1)_U + {}^*(\Omega_2)_U \qquad (76)$$

In this way one of the potential describes the field generated by electric, and the other the field generated by magnetic charges.

We observe that, if we make two independent gauge transformations

$$(\omega_1)_U \longrightarrow (\omega_1)_U + id\chi \; ; \; (\omega_2)_U \longrightarrow (\omega_2)_U + id\psi$$

the field F_U does not change. If we interpret the above transformations as changes in the local trivialization of a PFB, we must use a spliced bundle (due to the independence of χ and ψ). Once we justified our choice (a), we now go on with the theory.

Let us observe that the spliced bundle of two copies of the PFB $\pi:P\to M$ has each point $p\in P \circ P$ associated with two points of P *over* the same fiber. This permits us to understand that a gauge transofrmation in $\pi_{12}:P\circ P\to M$ corresponds to two gauge transformations in $\pi:P\to M$. Indeed, $\sigma_U:U\to P\circ P$ corresponds to $\sigma^1{}_U=\pi^1\circ\sigma_U:U\to P$ and also to $\sigma^2{}_U=\pi^2\circ\sigma_U:U\to P$. In this way we can associate to a given connection ω in P two gauge potentials $\omega_U=\sigma_U{}^{1*}\omega$ and $\omega_U=\sigma^2{}_U{}^*\omega$. Observe that

$$\omega_U = (\pi^1\circ\sigma_U)^*\omega = \sigma_U^*(\pi^{1*}\omega) = \sigma_U^*(\pi^{1*}\omega\oplus 0) \qquad (77a)$$

and

$$\tilde{\omega}_U = (\pi^2 \circ \sigma_U)^*\omega = \sigma_U^*(\pi^{2*}\omega) = \sigma_U^*(0\oplus\pi^{2*}\omega) \qquad (77b)$$

where ω_U and $\tilde{\omega}_U$ correspond to gauge potentials associated with the 1-forms $\pi^{1*}\omega\oplus 0$ and $0\oplus\pi^{2*}\omega$, which are possible extensions of ω to the spliced bundle. This shows that, given two connections ω_1 and ω_2 in π:P→M, we can associate with them two distinct connections $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$ and $\tilde{\omega}=\pi^{1*}\omega_2\oplus\pi^{2*}\omega_1$ in π_{12}:P o P →M. We show in Appendix C that, given a connection ω on the spliced bundle, we have two connections ω_1 and ω_2 well defined on the original fiber bundles.

We see now that, when both PFB are equal, the connections ω_1 and ω_2 can generate another connection $\tilde{\omega}$ on the spliced bundle. We call $\tilde{\omega}$ the connection dual to ω.

Observe that we have two curvatures $\Omega^{\omega} = D^{\omega}\omega$ and $\Omega^{\tilde{\omega}} = D^{\tilde{\omega}}\tilde{\omega}$ associated to the connections ω and $\tilde{\omega}$. These curvatures must, by the Bianchi identities, satisfy $D^{\omega}\omega=0$ and $D^{\tilde{\omega}}\tilde{\omega}=0$. Before we analyse these identities, we must understand some of the properties of the horizontal forms in a spliced bundle.

If $\tau\in\overline{\Lambda}^k(P_1oP_2, \hat{G}_1\oplus\hat{G}_2)$ with the adjoint representation $G_1\times G_2\to\hat{G}_1\oplus\hat{G}_2$, then:

(a) we can write $\tau=\pi^{1*}\tau_1+\pi^{2*}\tau_2$ with $\tau_1\in\overline{\Lambda}^k(P_1,\hat{G}_1)$ and $\tau_2\in\overline{\Lambda}^k(P_2,\hat{G}_2)$, where we use the adjoint representations Ad: $G_i\to\hat{G}_i$, i = 1,2 ;

(b) $$D^{\omega}\tau=\pi^{1*}D^{\omega_1}\tau_1+\pi^{2*}D^{\omega_2}\tau_2 \ ; \ \text{for } \omega=\pi^{1*}\omega_1+\pi^{2*}\omega_2 \ ; \qquad (78)$$

(c) Let $\overline{*}_{12}: \overline{\Lambda}^k(P_1oP_2,\hat{G}_1\oplus\hat{G}_2)\to\overline{\Lambda}^{n-k}(P_1oP_2,\hat{G}_1\oplus\hat{G}_2)$ be the Hodge operator for horizontal forms in P_1oP_2 (n=dimension of M), and $\overline{*}_1:\overline{\Lambda}^k(P_1,\hat{G}_1)\to\overline{\Lambda}^{n-k}(P_1,\hat{G}_1)$; $\overline{*}_2:\overline{\Lambda}^k(P_2,\hat{G}_2)\to\overline{\Lambda}^{n-k}(P_2,\hat{G}_2)$ be the Hodge operators for horizontal forms in the original PFB. Then $\overline{*}_{12}\tau = \pi^{1*}(\overline{*}_1\tau_1)\oplus\pi^{2*}(\overline{*}_2\tau_2)$.

We employ the relations (a), (b) and (c) in the following way. Returning to the curvature $\Omega^{\omega}\in\overline{\Lambda}^2(P_1oP_2,\hat{G}_1\oplus\hat{G}_2)$ we can write $\Omega^{\omega}=\pi^{1*}\Omega_1\oplus\pi^{2*}\Omega_2$, where $\Omega_1,\Omega_2\in\overline{\Lambda}^k(P,\hat{G})$ are well defined. We now prove that, if $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$, we have $\Omega_1=\Omega^{\omega_1}=D^{\omega_1}\omega_1$ and $\Omega_2=\Omega^{\omega_2}=D^{\omega_2}\omega_2$.

Indeed:

$$\Omega^{\omega} = d\omega + \frac{1}{2}[\omega,\omega] = d(\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2) + \frac{1}{2}[\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2, \pi^{1*}\omega_1\oplus\pi^{2*}\omega_2]$$
$$= \pi^{1*}D^{\omega_1}\omega_1\oplus\pi^{2*}D^{\omega_2}\omega_2 = \pi^{1*}\Omega^{\omega_1}\oplus\pi^{2*}\Omega^{\omega_2} .$$

Moreover, we have $\Omega^{\tilde{\omega}} = \pi^{1*}\Omega^{\omega_2}\oplus\pi^{2*}\Omega^{\omega_1}$.

In this way, the Bianchi identities $D^{\omega}\Omega^{\omega}=0$ and $D^{\tilde{\omega}}\Omega^{\tilde{\omega}}=0$ according to (b) are equivalent, and correspond to $D^{\omega_1}\omega_1=D^{\omega_2}\omega_2=0$, which are the Bianchi identities associated to ω_1 and ω_2 in π: P→M . In electrodynamics, these equations imply that $d(\Omega_1)_U=0$ and $d(\Omega_2)_U=0$, which *physically* mean that both the magnetic field of electric origin and the electric field of magnetic origin have null divergences.

In what follows we are going to generalize the gauge principle[15] for a gauge theory with our monopoles.

Let be π:P o P→M a spliced bundle with group G×G and let be G×G→GL(V) a representation of G×G . We remember that the space of 1-jets of the mappings from P to V is:

$$J(PoP,V)\equiv\{(p,v,\theta) \mid p\in P\text{ o }P,\ v\in V \text{ and } \theta:T_pP\text{ o }P \text{ is linear}\}$$

We call a Lagrangian the mapping L: J(P o P,V)→R such that, for all (p,v,θ)∈ J(P o P,V) and g∈ G×G , we have

$$L(pg,g^{-1}v,g^{-1}\theta\circ R_{g^{-1}*})=L(p,v,\theta)$$

If $L(p,g^{-1}v,g\cdot\theta)=L(p,v,\theta)$ then L is said to be G×G -invariant, and in what follows we suppose L to have this property.

Now given a Lagrangian L: J(P oP ,V)→R , let $\mathcal{C}$ be the space of the connections in P o P . Define the action density by

$$\mathcal{L}:C(P,V)\times\mathcal{C}\to C^{\infty}(M) \qquad (79)$$

where $C^{\infty}(M)$ are the set of the C^{∞} functions on M.
We have

$$\mathcal{L}(\Psi,\omega)(x) = L(p,\Psi(p),D^{\omega}\Psi(p)) \qquad (80)$$

where x∈ M, p∈ $\pi^{-1}(x)$ and *the generalized wave function (matter field) describing the* electric and the magnetic particles is $\Psi\in\Lambda^0$ (P o P,V) ≡ C(P o P,V).
Then $\mathcal{L}$ *is not only well defined but is also gauge invariant in the sense that, for* all f∈ GA(P o P) , we have, $\mathcal{L}(f^*\Psi,f^*\omega)=\mathcal{L}(\Psi,\omega)$. GA (P o P) is the gauge algebra of the spliced bundle; more precisely, it is the space C(Po P,$\hat{G}\oplus\hat{G}$) with the adjoint representation G×G→GL($\hat{G}\oplus\hat{G}$) ; g→Ad_g .

If we impose that $\mathcal{L}^{\omega}(\Psi)$ is stationary with respect to Ψ, we obtain the Euler-Lagrange equations[15]. We show now that, if we add an appropriate term S(ω) to

$\mathcal{L}^{\omega}(\Psi)$, obtaining then the total action $(\mathcal{L}+S)(\Psi,\omega)$, this density will generate not only the Euler-Lagrange equations for Ψ but also the non-homogeneous field equations. More precisely, these results follow once we impose that $(\mathcal{L}+S)(\Psi,\omega)$ is stationary with respect to the pair (Ψ,ω). We will see that the non-homogeneous equations obtained in this way correspond in the case of electrodynamics to Maxwell equations with monopoles, like in § 5.

We define the autoaction by

$$S(\omega) = -\frac{1}{4}\hbar k_{12}(\mathcal{F}_U,\mathcal{F}_U)\,;\quad \mathcal{F} = \Omega^{\omega} + {\mp}\Omega^{\bar{\omega}}\,;\quad \mathcal{F}_U = \sigma_U^{*}\,\mathcal{F} \tag{81}$$

where

$$\hbar k_{12}\colon \overline{\Lambda}^{k}(P \circ P,\, \hat{G}\oplus\hat{G}) \times \overline{\Lambda}^{k}(P \circ P,\, \hat{G}\oplus\hat{G}) \to \mathbb{R}$$

is the metric for horizontal forms in $(\hat{G}\oplus\hat{G})$ (with the adjoint representation). We observe that k_{12} is the Killing-Cartan metric in $(\hat{G}\oplus\hat{G})$ and that $k_{12}(A_1\oplus A_2, B_1\oplus B_2) = k(A_1,B_1)+k(A_2,B_2)$, where k is the Killing-Cartan metric in $\hat{G}$ (see Appendix B).

Let us observe that, as $S(\omega)$ is F-equivariant, it is gauge invariant as required for the autoaction term[15]. Let us observe also that, had we constructed the autoaction term as $\frac{1}{2}\hbar k_{12}(\Omega^{\omega},\Omega^{\omega})$, there would be no interaction between charges and monopoles. Indeed, take the case of electrodynamics where $\Omega^{\omega} = \pi^{1*}\omega_1 \oplus \pi^{2*}\omega_2$, then Ω_1 and Ω_2 correspond to the fields generated by charges and monopoles:

$$\hbar k_{12}(\Omega^{\omega},\Omega^{\omega}) = \hbar k(\Omega_U^{\omega 1},\Omega_U^{\omega 1}) + \hbar k(\Omega_U^{\omega 2},\Omega_U^{\omega 2}) \tag{82}$$

and there are not, in this expression, interaction terms between the fields Ω_1 and Ω_2. For $S(\omega)$, instead, we have

$$-\frac{1}{4}\hbar k_{12}(\mathcal{F}_U,\mathcal{F}_U) = -\frac{1}{2}\hbar k(\Omega_U^{\omega 1},\Omega_U^{\omega 1}) - \frac{1}{2}\hbar k(\Omega_U^{\omega 2},\Omega_U^{\omega 2}) - \hbar k(\Omega_U^{\omega 1},\, {*}\Omega_U^{\omega 2}) \tag{83}$$

where the interaction term appears explicitly.

Before we apply the variational principle to the total action, let us remember the definition of the current in terms of the Lagrangian

$$\frac{d}{dt}\mathcal{L}(\Psi,\omega+t\sigma)\big|_{t=0} = \hbar k_{12}(J^{\omega}(\Psi),\sigma)$$

$\forall\sigma\in\overline{\Lambda}^{1}(P\circ P,\, \hat{G}\oplus\hat{G})$. In this case $J^{\omega}(\Psi)\in\overline{\Lambda}^{1}(P\circ P,\hat{G}\oplus\hat{G})$, and we can write

$J^{\omega}(\Psi) = -\pi^{1*}J_1 \oplus \pi^{2*}J_2$; so that we can asociate J_1 and $J_2 \in \overline{\Lambda}^{1}(P,\hat{G})$ to the "electric" and "magnetic" currents, respectively.

Effecting the variation at $t=0$ $(\sigma = \pi^{1*}\sigma_1 \oplus \pi^{2*}\sigma_2)$ we get

$$\begin{aligned}
&\frac{d}{dt}\int_U (\mathcal{L}+S)(\Psi+t\tau,\omega+t\tau)\mu \\
&= \int_U \mathcal{L}(\Psi+t\tau,\omega)\mu + \frac{d}{dt}\int_U S(\Psi,\omega+t\tau)\mu \\
&= \frac{d}{dt}\int_U \frac{1}{2}\hbar k(\Omega_U^{\omega 1+t\sigma_1},\Omega_U^{\omega 1+t\sigma_1})\mu \\
&- \frac{d}{dt}\int_U \frac{1}{2}\hbar k(\Omega_U^{\omega 2+t\sigma_2},\Omega_U^{\omega 2+t\sigma_2})\mu \\
&- \frac{d}{dt}\int_U \frac{1}{2}\hbar k(\Omega_U^{\omega 1+t\sigma_1},\, {-\mp}\Omega_U^{\omega 2+t\sigma_2})\mu
\end{aligned} \tag{84}$$

We have at $t=0$ for the four first terms in eq (84):

$$\frac{d}{dt}\int_U \mathcal{L}(\Psi+t\tau,\omega)\mu = \int_U \hbar\Big(\delta^{\omega}\frac{\partial L}{\partial(D^{\omega}\Psi)} + \frac{\partial L}{\partial\Psi},\tau\Big)\mu$$

$$\frac{d}{dt}\int_U \mathcal{L}(\Psi,\omega+t\tau)\mu = \int_U \hbar k_{12}(J^{\omega}(\Psi),\tau)\mu$$

$$\frac{d}{dt}\int_U \frac{1}{2}\hbar k(\Omega_U^{\omega 1+t\sigma_1},\Omega_U^{\omega 1+t\sigma_1})\mu = -\int_U \hbar k(\delta^{\omega 1}\Omega^{\omega 1},\sigma_1)\mu$$

$$\frac{d}{dt}\int_U \frac{1}{2}\hbar k(\Omega_U^{\omega 2+t\sigma_2},\Omega_U^{\omega 2+t\sigma_2})\mu = -\int_U \hbar k(\delta^{\omega 2}\Omega^{\omega 2},\sigma_2)\mu$$

Moreover:

$$\begin{aligned}
&\frac{d}{dt}\hbar k(\Omega^{\omega 1+t\sigma_1},{\mp}\Omega^{\omega 2+t\sigma_2}) \\
&= \frac{d}{dt}\hbar k(\Omega^{\omega 1+t\sigma_1},{\mp}\Omega^{\omega 2}) + \frac{d}{dt}\hbar k(\Omega^{\omega 1},{\mp}\Omega^{\omega 2+t\sigma_2})
\end{aligned}$$

and, at $t=0$, we have $\frac{d}{dt}\Omega^{\omega+t\sigma} = D^{\omega}\sigma$[15]. Then

$$\frac{d}{dt}\, \hbar k(\Omega^{\omega 1}+t\sigma_1, \overline{\star}\,\Omega^{\omega 2}+t\sigma_2)$$
$$= \hbar k(D^{\omega 1}\sigma_1 , \overline{\star}\,\Omega^{\omega 2}+t\sigma_2) + \hbar k(\Omega^{\omega 1} , \overline{\star}\, D^{\omega 2}\sigma_2)$$
$$= \hbar k(\delta^{\omega 1}(\overline{\star}\,\Omega^{\omega 2})\cdot\sigma_1) + \hbar k(\delta^{\omega 2}(\overline{\star}\,\Omega^{\omega 1}),\sigma_2)$$

and we obtain for the last term in eq (84)

$$\frac{d}{dt} \int_U \hbar k(\Omega^{\omega 1}+t\sigma_1, \overline{\star}\,\Omega^{\omega 2}+t\sigma_2)$$

$$= \int_U (\hbar k(\delta^{\omega 1}(\overline{\star}\,\Omega^{\omega 2})\cdot\sigma_1) + \hbar k(\delta^{\omega 2}(\overline{\star}\,\Omega^{\omega 1}),\sigma_2))\mu$$

Now, summing all the terms obtained and taking into account that t, σ_1 and σ_2 are all independent, and also that

$$\hbar k_{12}(J^{\omega}(\Psi),\sigma) = -\hbar k(J_1,\sigma_1) + \hbar k(J_2,\sigma_2) \ , \qquad (85)$$

we get the equations

$$\delta^{\omega} \frac{\partial L}{\partial(D^{\omega}\Psi)} + \frac{\partial L}{\partial \Psi} \qquad (86)$$

$$\delta^{\omega 1}\Omega^{\omega 1} + \delta^{\omega 1}(\overline{\star}\,\Omega^{\omega 2}) = -J_1 \qquad (87)$$

$$\delta^{\omega 2}\Omega^{\omega 2} + \delta^{\omega 2}(\overline{\star}\,\Omega^{\omega 1}) = J_2 \qquad (88)$$

Eq (86) corresponds to the Euler-Lagrange equation, which gives the equation of the generalized field describing the motion in P o P of charges and monopoles. We are not going to investigate in this paper the nature of $\mathcal{L}^{\omega}(\Psi)$.

Eqs (87) and (88) can be written, putting $\Omega=\Omega^{\omega_1} + {}^{\star}\Omega^{\omega_2}$, as

$$\delta^{\omega 1}\Omega = -J_1 \qquad (89)$$

$$\delta^{\omega 2}(\overline{\star}\,\Omega) = J_2 \ \Leftrightarrow\ D^{\omega 2}\,\Omega = -\,{}^{\star}J_2 \qquad (90)$$

which are the non-homogeneous equations of the theory.

In the case of electrodynamcics these equations reduce to

$$\delta\Omega = -J_1 \qquad (91a)$$

$$d\Omega = -\,{}^{\star}J_2 \qquad (91b)$$

which we recognize as the Maxwell equations for the electromagnetic field $\Omega=\Omega^{\omega_1} + {}^{\star}\Omega^{\omega_2}$ generated by electric and magnetic charges. We have for Ω^{ω_1} and Ω^{ω_2} the equations:

$$\delta\Omega^{\omega_1} = -J_1 \ ; \ \delta\Omega^{\omega_2} = -J_2 \qquad (92)$$

since $d\Omega^{\omega_1} = d\Omega^{\omega_2}=0$. Also, since $\Omega^{\omega_1}=d\omega_1$, $\Omega^{\omega_2}=d\omega_2$, we have

$$\Box\omega_1 = J_1 \ ; \ \Box\,\omega_2 = J_2 \qquad (93)$$

where $\Box = -(d\delta+\delta d)$.

8.CONCLUSIONS

We presented in this paper a theory of magnetic monopoles without strings. In order to show the crucial difference between our theory and the usual presentations of the subject, we described briefly the string theory by Dirac and the topological monopole theory, where the monopole appears associated with a change in the topology of the world manifold (§ 2,3,4).

In order to express conveniently our theory of the generalized potential, we use the Clifford bundle formalism described in Appendix A. In our approach, we show that Maxwell equations imply the correct coupling of the electromagnetic field to electric charges and magnetic monopoles; i.e. we deduce the form of the generalized Lorentz force (§ 5). From that we deduce the motion equations of charges and monopoles. Moreover we derive, from the quantum version of the theory the Dirac quantization condition in two different ways.

In § 6 we discussed the impossibility of constructing a local Lagrangian which gives simultaneously the motion equations of particles and monopoles and the field equations. We arrive at the conclusion that, contrary to recent claims[13], even in the Clifford bundle formalism this is not possible.

Finally in § 7 we present a generalization of our theory in § 6 to generalized monopoles associated with an arbitrary gauge group G. In other words, we succeed in giving to our theory a principal fiber bundle structure: a spliced bundle with group G×G. We obtain the equation of the generalized field in our theory using in the spliced bundle a generalization of the Principle of the Stationary Action. We postulate the existence of a Lagrangian density $\mathcal{L}^{\omega}(\Psi)$ for the generalized field, that describes in the spliced bundle the "electric" and "magnetic" matter; but we do not use explicitly any Lagrangian to deduce the equations of the generalized matter field. Indeed, $\mathcal{L}^{\omega}(\Psi)$ is used only to produce the currents. The question of the existence of $\mathcal{L}^{\omega}(\Psi)$ and its form in this formalism will be investigated in another paper.

To conclude, we observe that the approach here developed shows explicity an interesting interplay of several different branches of modern mathematics, which conspire together in order to shed new light on various physical problems.

APPENDIX A

A 1. SOME ALGEBRAS AND THEIR RELATIONS

In A_1, A_2 and A_3 we follow the presentation of Graf[16].

Let V be a n-dimensional vector space. In this subsection we introduce some algebras that will be usefull to derive the equations of motion for electrical and magnetical charges in the field of magnetic monopoles and electric charges.

The tensor algebra T(V) over **R** is the **R**-vector space of the direct sum of the powers $\otimes^p V$ together with the usual tensor product $\otimes$ of its elements. Then we have:

$$T(V)=\left(\oplus_{p=0}^{\infty}\otimes^p V,\otimes\right) \tag{A.1.1}$$

is **Z**-graded: $(\otimes^p V)\otimes(\otimes^q V)\subset\otimes^{p+q}V$ and infinite-dimensional if $n\geq 1$. As V is finite-dimensional we can identify V with its image $\otimes^1 V$ in T(V) and we also define $\otimes^0 V=\mathbf{R}$.

On T(V) there are two important involutive morphisms (both being linear automorphisms of $\oplus_{p=0}^{\infty}\otimes^p V$:

(i) the main automorphism α

$$\alpha(A\otimes B)=\alpha(A)\otimes\alpha(B)\ ,\ A,B\in T(V) \tag{A.1.2}$$

$$\alpha(A)=A \text{ if } A\in\otimes^0 V \text{ and } \alpha(A)=-A \text{ if } A\in\otimes^1 V\ ; \tag{A.1.3}$$

(ii)the main anti-automorphism β,

$$\beta(A\otimes B)=\beta(A)\otimes\beta(B)\ ;\ A,B\in T(V) \tag{A.1.4}$$

$$\beta(A)=A \text{ if } A\in\otimes^0 V+\otimes^1 V\ . \tag{A.1.5}$$

The exterior Algebra $\Lambda(V)$ over the **R**-vector space V can be defined as the quotient algebra $T(V)/J$ of T(V) over the two sided ideal $J\subset T(V)$ generated by the element with the form $a\otimes a$, where $a\in V$.

As usual, we denote the exterior multiplication by the sign $\wedge$. Since J is homogeneous in the **Z**-graduation of T(V), also $\Lambda(V)$ is **Z**-graded: $\Lambda(V)=\oplus\Lambda^p(V)$, with $\Lambda^p(V)\wedge\Lambda^q(V)\subset\Lambda^{p+q}(V)$. As before, we make the identifications $\Lambda^1(V)=V$ and $\Lambda^0(V)=\mathbf{R}$.

The subspaces $\Lambda^p(V)$ are $\binom{n}{p}$–dimensional and $\Lambda(V)$ is 2^n-dimensional. For the elements $A\in\Lambda^p(V)$ and $B\in\Lambda^q(V)$, the exterior product is commutative or anticommutative:

$$A\wedge B=(-1)^{pq}B\wedge A \tag{A.1.6}$$

The morphisms α and β of T(V) pass to the quotient $\Lambda(V)$. Denoting them by the same symbols α and β, we have:

$$\alpha(A\wedge B)=\alpha(A)\wedge\alpha(B) \tag{A.1.7}$$

$$A,B\in T(V)$$

$$\beta(A\wedge B)=\beta(A)\wedge\beta(B) \tag{A.1.8}$$

$$\text{If } A\in\Lambda^p(V)\rightarrow\alpha(A)=(-1)^p A \text{ and } \beta(A)=(-1)^{p(p-2)/2}A \tag{A.1.9}$$

We define as Grassman-Algebra $\Lambda(V,Q)$ the pair $(\Lambda(V),Q)$, consisting of the exterior algebra $\Lambda(V)$ together with the inner product $(\ ,\ )_Q:\Lambda(V)\times\Lambda(V)\rightarrow\mathbf{R}$ induced in $\Lambda(V)$ by a quadratic form Q over V as follows:

- If $A\in\Lambda^p(V)$ and $B\in\Lambda^q(V)$ with $p\neq q$, then $(A,B)_Q=0$;
- If $A=a_1\wedge a_2\wedge\ldots\wedge a_p$ and $B=b_1\wedge b_2\wedge\ldots\wedge b_p$ with $a_i,b_i\in\Lambda^1(V)$, then $(A,B)_Q=\det(B(a_i,b_j))$, where B is the bilinear form associated to Q by

$$2B(x,y)=Q(x+y)-Q(x)-Q(y)\ ; \tag{A.1.10}$$

- The case of general $a,b\in\Lambda(V)$ can then be reduced, due to the linearity, to (i) and (ii).

The Clifford Algebra C(V,Q) of the real vector space V with quadratic form Q is defined as the quotient algebra $T(V)/J'$, where the two-sided ideal J' is generated by elements of the form $a\otimes a - Q(a).1$ with $a\in V$. As before, we can identify V with its image in C(V,Q). Denoting the Clifford-multiplication by a simple justaposition, for a, b∈ V we have

$$ab + ba = 2B(a,b) \qquad (A.1.11)$$

with the bilinear form B defined in (A.1.10).

The ideal J',being inhomogeneous of even degree in T(V), induces a Z_2 gradation of the Clifford Algebra, $C(V,Q)=C^+(V,Q)\oplus C^-(V,Q)$, where $C^+(V,Q)$ is the image of the elements of even degree in T(V). Since $\alpha(J')=J'$, the morphisms α and β induce morphisms (designated by the same symbols) in C(V,Q); For all A ,B ∈ C(V,Q):

$$\alpha(AB) = \alpha(A)\alpha(B) \qquad (A.1.12)$$

$$A,B\in T(V)$$

$$\beta(AB) = \beta(A)\beta(B) \qquad (A.1.13)$$

$$\alpha(A) = \beta(A) = A \text{ for } A\in \mathbf{R}\,; -\alpha(A) = \beta(A) = A \text{ for } A\in V \qquad (A.1.14)$$

In particular, for $A\in C^+(V,Q)$ it is $\alpha(A)=A$, and for $a\in C^-(V,Q)$ it is $\alpha(A)=-A$.

A.2.STRUCTURE OF THE CLIFFORD ALGEBRA

In this sub-section we study the structure of the Clifford Algebras and their relations with the Grassmann Algebras.

First, for any element X of the dual vector space V* let us define the *contraction** of an element of T(V) with X∈ V* as the (V*,T(V)) -bilinear map $V^*\times T(V)\to T(V)$ of degree −1 with

$X\rfloor 1 = 0$

$a = X(a)$, if $a\in V\subset T(V)$

$(a\otimes b) = (X\rfloor a)\otimes b + \alpha(a)\otimes(X\rfloor b)$; a ,b ∈ V

(In particular, $X\rfloor X\rfloor$ will annihilate any element of T(V)).

Since $X\rfloor J=J$ and $X\rfloor J'=J'$, the contraction also passes to the quotients Λ(V) and C(V,Q) and to Λ(V,Q), and we have:

if A , B∈ Λ(V) or Λ(V,Q)

$$X\rfloor(A\wedge B) = (X\rfloor A)\wedge B + \alpha(A)\, A\wedge(X\rfloor B) \qquad (A.2.1)$$

And

$$X\rfloor(A\,B) = (X\rfloor A)\,B + \alpha(A)\,A\,(X\rfloor B)\ ,\ \text{if } A,B\in C(V,Q) \qquad (A.2.2)$$

with

$$X\rfloor a = 0 \text{ if } a\in \mathbf{R}\subset \Lambda(V)\ ,\ \Lambda(V,Q),\ C(V,Q) \qquad (A.2.3)$$

Now, for a∈ V, define the Q-adjoint to a∈ V⊂ T(V),Λ(V), (V,Q), C(V,Q) to be the element ǎ∈ V* such that for any a∈ V ⊂ Λ(V,Q) and b∈ Λ(V,Q) their product a∇b is defined as

$$a\nabla b = a\wedge b + \check{a}\rfloor b \qquad (A.2.5)$$

Then

$$a\nabla a = Q(a) \qquad (A.2.6)$$

By the theorem on the universality of Clifford Algebras[21], the ∇-Algebra generated by this relation on the elements of Λ(V,Q) is the Clifford-Algebra C(V,Q) with the Clifford product (designated by mere justaposition) replaced by ∇.

Conversely, if for a Clifford-Algebra C(V,Q) we define a Δ product of a∈ V⊂ C(V,Q) with B∈ C(V,Q) as

$$a\,\Delta\, B = a\,B + \check{a}\rfloor B\ , \qquad (A.2.7)$$

then we get

$$a\Delta a = 0 \qquad (A.2.8)$$

which is the defining relation of the exterior algebra. Since in the Clifford-Algebra $aa=Q(a)$, this exterior algebra can be made a Grassmann-Algebra.

This correspondence of Clifford and Grassmann-Algebras does not depend on Q being non-degenerated or not. In particular, if Q=0, the Q-adjoint vanishes and $C(V,0)=\Lambda(V,0)=\Lambda(V)$.

Another important observation is that the Grasmann-Algebra Λ(V,Q) and the Clifford-Algebra C(V,Q) are isomorphic as vector spaces over **R**. Then, the generators of Λ(V,Q) are the generators of C(V,Q) and vice-versa.

Then if $\{e_1 \ldots, e_n\}$ is a basis of V, the set of the p-vectors, $p = 0,1,2,\ldots,n$:

$\{ e_0 = 1, e_1 \ldots, e_n\ , e_1\wedge e_2\ , e_1\wedge e_3, \ldots\ , e_1\wedge e_2\wedge e_3, \ldots, e_1\wedge e_2 \ldots\wedge e_n \}$

generates Λ(V,Q) and also C(V,Q). Both are 2^n-dimensional algebras. Any element A∈ Λ(V,Q) or C(V,Q) can be written as

$$A = \sum_{p=0}^{n} A_p = \sum_{p=0}^{n} \langle A\rangle_p\ ,\ \text{where } A_p = \langle A\rangle_p \in \Lambda^p(V) \qquad (A.2.9)$$

So we have that, over the direct sum $\oplus\Lambda^p(V)$ of the linear spaces $\Lambda^p(V)$, we can impose the structure of a Grassmann Algebra by means of $\wedge$ and Q, as well as the structure of a Clifford Algebra; and each of the two products can be reduced to the other, as seen above.

For general elements, Clifford and exterior product are related as follows:

$$AB = \sum_p \frac{(-1)^{p(p-2)/2}}{p!} h^{i_1 j_1} \ldots h^{i_p j_p}\, \alpha^P(e_{i_1} \rfloor \ldots \rfloor e_{i_p} \rfloor A) \wedge (e_{j_1} \rfloor \ldots \rfloor e_{j_p} \rfloor B) \qquad (A.2.10)$$

$$A \wedge B = \sum_p \frac{(-1)^{p(p-2)/2}}{p!} h^{i_1 j_1} \ldots h^{i_p j_p} [(\alpha^P(e_{i_1} \rfloor \ldots \rfloor e_{i_p} \rfloor \alpha(A))][(e_{j_1} \rfloor \ldots \rfloor e_{j_p} \rfloor B)] \qquad (A.2.11)$$

where $h^{ik} := B(\gamma^i, \gamma^k)$; e_i is the dual basis to the basis $\gamma^i \in \Lambda^1(V)$; and the product of the elements inside the brackets in (A.2.11) is the Clifford -product.

The formulas (A.2.10) and (A.2.11) for the special cases that a or $b \in \Lambda^1(V)$ reduce to:

$$a\psi = a \wedge \psi + a \rfloor \psi \qquad (A.2.12)$$

$$\phi b = b \wedge \alpha(\phi) - b \rfloor \alpha(\phi) \qquad (A.2.13)$$

A.3.SOME VECTOR BUNDLES RELATED TO THE COTANGENT BUNDLE

Since all the algebraic structures considered above possess a $\mathbf{R}$-linear structure inherited form the vector space V, for their generalization to manifolds we will use the formalism of the vector bundles (with additional algebraic structures). Here M will be a real n-dimensional C^∞-manifold. Moreover, bundles, cross sections and maps will be C^∞. Quantity τM denotes the Tangent Bundle associated to M .

The basic bundle for our constructions will be the *Cotangent Bundle* τ^*M of the manifold M. Moreover , cross sections $c \in \mathrm{Sec}(\tau^*M)$ will be called 1-form fields. Given a cross secion $h \in \mathrm{Sec}(\tau^*M \times \tau^*M)$,let be $\bar{h} \in \mathrm{Sec}(\tau M \times \tau M)$ such that $h_{ij}\bar{h}^{jk} = \delta^k_i$ In each fiber $\pi^{-1}(x)$, quantity $\bar{h}_x$ will be a quadratic form over the cotangent space $T_x{}^*M$.Let us denote the pair $(\tau^*M, \bar{h})$ a *Riemannian (or Lorentzian) vector bundle*.

We denote the vector-bundle, whose fibers $\Lambda T_x{}^*M$ are the exterior algebras over $V = T_x{}^*M$, the *Cartan-bundle* of exterior differential forms on M . As is well known, on a Cartan-bundle the *exterior derivative* d can be uniquely characterized by the following conditions:

$$d(A+B) = dA + dB$$

$$d(A \wedge B) = dA \wedge B + \alpha(A) \wedge dB$$

$$d^2 = 0$$

$$x \rfloor (df) = X(f)$$

for any $A, B \in \mathrm{Sec}\,\Lambda\tau^*M$, $f \in \Lambda^0\tau^*M$ and $X \in \mathrm{Sec}\,\tau M$.

In particular, d will be homogeneous of degree +1 in the $\mathbf{Z}$-gradation of the ring of cross sections of $\Lambda\tau^*M = (\oplus\Lambda^p\tau^*M, \wedge)$.

The pair $(\Lambda\tau^*M, \bar{h})$, where each fiber $(\Lambda T_x{}^*M, \bar{h}_x)$ is a Grassmann Algebra, will be called *Hodge-bundle* on M with metric h .

If for any $x \in M$, quantity h_x is nondegenerate, in addition to d there is the *divergence* δ , which is the formally h-adjoint operator of d , defined [7] by:

$$\delta\omega_p = (-1)^p *^{-1} d * \omega_p \qquad (A.3.1)$$

where the operator * (Hodge star operator) is defined as the linear isomorphism:

$$*: \Lambda^p\tau^*M \to \Lambda^{n-p}\tau^*M \ , \ \phi \to *\phi$$

$$\sigma \wedge *\phi = (\sigma, \phi)\mu$$

for all p-forms $\sigma, \phi \in \mathrm{Sec}\,\Lambda^p\tau^*M$, where μ is the volume n-form

$$\mu = \frac{1}{n!}\mu_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n} = \sqrt{|h|}\, dx^1 \wedge \ldots \wedge dx^n \qquad (A.3\ 3)$$

If $(\theta^1, \theta^2, \ldots, \theta^n)$ is a orthonormal basis, then μ can be written as: $\mu = \theta^1 \wedge \ldots \wedge \theta^n$

Because $d^2 = \delta^2 = 0$, the laplacian for differential forms $\Box = -(d\delta + \delta d)$ can be written also as a square

$$\Box = (d - \delta)^2 \qquad (A.3.4)$$

A vector bundle is called a Clifford-bundle $C(\tau^*M, \bar{h})$ if each fiber is a Clifford-algebra $C(T^*_xM, \bar{h}_x)$. We can prove that $C(\tau^*M, \bar{h})$ is a vector bundle associate to the PFB $P_{O(1,3)}(\tau^*M)$, i.e., $C(\tau^*M, \bar{h}) = P_{O(1,3)}(\tau^*M) \times_{O(1,3)} R_{1,3}$.

If h is non-degenerate, there is a particular differential operator ∂ called the *Dirac-operator* odd in the $\mathbf{Z}_2$-gradation of $C(T^*_xM, \bar{h}_x)$ defined as follows:

For any $\iota^* \in \mathrm{Sec}\,\tau^*M \subset \mathrm{Sec}\,C(\tau^*M, h)$ and any $\iota \in \mathrm{Sec}\,\tau M$, consider the bilinear tensorial map of type (1,1) given by

$$\psi \to \iota^* \nabla_\iota \psi \qquad (A.3.5)$$

where ψ is any element of Sec $C(\tau^*M,\bar{\eta})$ and ∇_t is the covariant derivative of ψ, considered as element of the tensor-bundle, in the direction of t. Because $\nabla_t J' \subset J'$, ∇_t passes to the quotient bundle $C(\tau^*M,\bar{\eta})$. Then ∂ is defined as the tensorial trace of the map:

$$\partial = \mathrm{Tr}\,(\iota^*\nabla_\iota) \qquad \text{(A.3.6.)}$$

In terms of a local basis $\{\gamma^i\}$ of 1-form fields and its dual basis $\{e_j\}$ of vector fields, we can also write

$$\partial = \gamma^i \nabla_{e_i} \qquad \text{(A.3.7)}$$

In particular, taking a local coordiante basis $\{dx^\mu\}$ we have

$$\partial = dx^\mu \nabla_\mu \qquad \text{(A.3.8)}$$

The Dirac Operator can be reformulated as follows. Take any local neighbourhood $U \subset M$ with coordinate basis $\{dx^\mu\}$. Then in U, the quantity $\partial = dx^\mu \nabla_\mu$ can be written when acting on $\psi \in$ sec $(\Lambda^p \tau^* M,\bar{\eta})$ as:

$$\partial\psi = dx^\mu \wedge (\nabla_\mu \psi) + \partial_\mu \lrcorner (\nabla_\mu \psi) \qquad \text{(A.3.9)}$$

We get for $\psi = \psi_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n}$.

$$dx^\mu \wedge (\nabla_\mu \psi) = \frac{1}{p!} \nabla_{[\mu} \psi_{i_1 \ldots i_n]} dx^\mu \wedge \ldots \wedge dx^{i_n} = d\psi$$

$$\partial_\mu \lrcorner (\nabla_\mu \psi) = \frac{1}{(p-1)!} \nabla_\mu \psi^{\mu}{}_{i_2 \ldots i_n} dx^{i_2} \wedge \ldots \wedge dx^{i_n} = -\delta\psi$$

As the forms at the right-hand sides of the two last formulas are independent of any basis, we get

$$\partial = d - \delta \qquad \text{(A.3.10)}$$

A.4.THE GEOMETRIC CALCULUS OF THE CLIFFORD-ALGEBRA

In this section we show how to do some calcululations in the Clifford-Algebra C(V,Q). This is particularly important in order to obtain the results of §5 and §6.

Here we follows Hestenes[17], that yields a geometric interpretation for the elements of the Clifford Algebra (and this is the reason why we designate this section by Geometric Calculus).

We have seen (Sec.A.2) that the Grassmann Algebra is isomorphic, as vector space, to a Clifford Algebra. Then, any $A \in C(V,Q)$ can be written (eq.(A.2.9)):

$$A = \sum_{r=0}^{n} \langle A \rangle_r = \langle A \rangle_0 + \langle A \rangle_1 + \ldots\ldots + \langle A \rangle_n \qquad \text{(A.4.1)}$$

where $\langle A \rangle_r$ is the component of A in $\Lambda^r(V)$. Due to the above decomposition, the elements of C(V,Q) will be called *multivectors*, (or multiforms, depending on V).

If $A = \langle A \rangle_r$ for some integer $0 \le r \le n$, then we say that A is *homogeneous* at the r-grade. In this case we will write A as $A_r \in \Lambda^r V$.

The elements of $\Lambda^r V$ will be called, as usual, r-vectors. Now, we must introduce the following products in C(V,Q):

The *inner product* of homogeneous multivectors is

$$A_r . B_s = \langle A_r B_s \rangle_{|r-s|}, \text{ if } r,s > 0 \qquad \text{(A 4.2)}$$
$$A_r . B_s = 0 \text{ , if } r=0 \text{ or } s=0$$

The *inner product of arbitrary multivectors is then defined by:*

$$A \cdot B = \sum_r \langle A \rangle_r . B = \sum_s A . \langle B \rangle_s = \sum_{r,s} \langle A \rangle_r . \langle B \rangle_s \qquad \text{(A.4.3)}$$

The equivalence of the three expressions on the right side of (A.4.3) is an obvious consequence of the distributivity of the Clifford product.

The *outer product* (or exterior produt) of homogeneous multivectors is

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s} \qquad \text{(A.4.4)}$$

Note that, in contrast to the inner product, we have $A_r \wedge \lambda = \lambda \wedge A_r = \lambda A_r$ if $\lambda = \langle \lambda \rangle_0 \in \mathbb{R}$.

The outer product of arbitrary multivectors is defined by:

$$A \wedge B = \sum_r \langle A \rangle_r \wedge B = \sum_s A \wedge \langle B \rangle_s = \sum_{r,s} \langle A \rangle_r \wedge \langle B \rangle_s \qquad \text{(A.4.5)}$$

Now, in the following, let's designate the anti-automorphism in C(V,Q) (Sec. A.1) by $+$, and call it *reversion*.

We have, with $A,B \in C(V,Q)$:

$$(AB)^+ = B^+ A^+ \qquad \text{(A.4.6)}$$
$$(A + B)^+ = A^+ + B^+ \qquad \text{(A.4.7)}$$
$$\langle A^+ \rangle_0 = \langle A \rangle_0 \qquad \text{(A.4.8)}$$
$$a^+ = a \text{ , if } a = \langle a \rangle_1 \qquad \text{(A.4.9)}$$

It follows immediately that the *reversion* of a Clifford-product of vectors is

$$(a_1 a_2 \ldots\ldots a_r)^+ = a_r \ldots\ldots a_2 a_1 \qquad \text{(A.4.10)}$$

Moreover, we have

$$\langle A^+\rangle_r = \langle A\rangle_r^+ = (-1)^{r(r-1)/2}\langle A\rangle_r \qquad (A.4.11)$$

Using the above relations, we get also the relations:

$$\langle AB\rangle_r = (-1)^{r(r-1)/2}\langle B^+A^+\rangle_r \qquad (A.4.12)$$

$$\langle A_rB_s\rangle_t = \langle B_s^+A_r\rangle_t = (-1)^{r(r-1)/2}\langle B_sA_r\rangle_t \qquad (A.4.13)$$

$$\langle AB_rC\rangle_s = \langle C^+B_rA^+\rangle \qquad (A.4.14)$$

Using (A.4.11) and (A.4.12), we find, for the inner and outer product defined above, the following reordering rules:

$$A_r.B_s = (-1)^{r(s-1)}B_s.A_r \ ,r\le s \qquad (A.4.15)$$

$$A_r\wedge B_s = (-1)^{rs}B_s\wedge A_r\ . \qquad (A.4.16)$$

The inner and outer product are defined obviously by means of the Clifford-product in the Clifford Algebra.

Note that using (A.4.15) and (A.4.16) we can show easily the relations:

$$a.A_r = \frac{1}{2}(aA_r - (-1)^r A_ra) \qquad (A.4.17)$$

$$a\wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_ra) \qquad (A.4.18)$$

and then

$$aA_r = a.A_r + a\wedge A_r \qquad (A.4.19)$$

For our applications, the following are important:

- The ***identity***

$$(a_1a_2........a_r) = \sum_{k=1}^{r}(-1)^{k+1}a.a_k(a_1...\check{a}_k...a_r) \qquad (A.4.20)$$

where the $\check{a}_k$ means that the kth vector is omitted from the product;

- The ***Fundamental Formula***

$$A_rB_s = \langle A_rB_s\rangle_{|r-s|} + \langle A_rB_s\rangle_{|r-s|+2} + ... + \langle A_rB_s\rangle_{r+s} = \sum_{k=0}^{m}\langle A_rB_s\rangle_{|r-s|+2k} \qquad (A.4.21)$$

where $m = \frac{1}{2}(r+s-|r-s|)$.

Observations:

Obs. 1. Note that the equation (A.4.19)

$$aA_r = a\cdot A_r + a\wedge A_r$$

is the same as the equation (A.2.5) when the outer product in the Clifford Algebra is identified with the usual outer product and $a\cdot A_r \equiv a\rfloor A_r$.

Obs. 2. Note that the Clifford inner product between multivectors with the same grade is given by the Grassmann inner product, that is:

$$A_r\cdot B_r = (A_r,B_r)_Q.$$

But observe that, although $(\ ,\ )_Q$ is defined only for elements with the same grade, the Clifford inner product $(\ .\ )$ is defined for any elements of the Algebra $C(V,Q)$.

Obs.3. Using the Clifford inner product we can introduce a very important isomorphism for our calculus, between $\Lambda(V)$ and $\Lambda(V^*)$, as follows:

$$\Lambda^r(V^*)\ni\alpha_r \rightarrow A_r\in\Lambda^r(V)\ ,\alpha_r(a_1,a_2,...a_r) = A_r^+.(a_1\wedge a_2\wedge...\wedge a_r) \qquad (A.4.22)$$

for $a_1, a_2...a_r\in V$

We use this isomorphism in §6. It is necessary also in order to translate our present approach into the one in references [8,9].

Obs. 4. If V is n-dimensional vector space with a metric tensor h with signature (p,q), then the algebra $C(V,Q)$, where $Q(a)=h(a,a)$, is usually designated by $R_{p,q}$.

Obs.5. The Clifford Algebra $R_{1,3}$ (called Minkowski Algebra or Space Time Algebra) of the forms $\{dx^\mu\}$ is isomorphic to $H(2)$ quaternionic matrices. The Dirac complex matrices Algebra $\{\gamma_\mu\}$, with $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2h_{\mu\nu}\ I$, is the algebra $C(4)\cong R_{4,1}$.

We have the inclusion $R^+_{4,1}\cong R_{1,3}$, where $R^+_{4,1}$ is the even part of $R_{4,1}$. For completeness, we note that $R_{3,1}\cong R(4)$ is called the Majorana Algebra.

Obs. 6 We have the following relation between $\gamma^5=dx^0...dx^3$ and the Hodge star operator:

If $f_p\in\Lambda^p(V)$, with $V=T_x^*M$, then

$$*f_p = (-1)^t\gamma^5 f_p \qquad (A.4.23)$$

where the index t depends an the signature and on the grade of f_p.

In the paticular case of $R_{1,3}$ we have $* = -\gamma^5$ for $p=1,2,3$ and $* = \gamma^5$ for 0,4.

The product at the right hand side of (A.4.23) is the Clifford product.

A.5.THE PAULI ALGEBRA

The Pauli Algebra is defined as the even sub-algebra of the space-time algebra $R_{1,3}$. Truly, it is the Clifford Algebra $R_{3,0}$ of the euclidean tridimensional space R^3. Then, we have the isomorphism $R^+_{1,3}\cong R_{3,0}$. This isomorphism is given by the linear extension of

$$\gamma_k \rightarrow \sigma_k = \gamma_k\gamma_0 \qquad (A.5.1)$$

where $\gamma_k\in R_{1,3}$ and γ_0 is time-like $(\gamma_0^2=1)$.

The Pauli Algebra $R_{3,0}$ is isomorphic to C_2 (complex 2x2 matrices); and this is the reason why physicits can use the representation of σ_i in terms of the very well known Pauli matrices

The vectors σ_k satisfy the multiplication rule:

$$\sigma_j\cdot\sigma_k = \frac{1}{2}(\sigma_j\sigma_k + \sigma_k\sigma_j) \qquad (A.5.2)$$

The products of σ_k generate a tensorial basis for $R_{3,0}$, which is a vector-space with dimension $2^3=8$. Other elements of the basis are the bivectors $\sigma_{jk} = \sigma_j\wedge\sigma_k = \frac{1}{2}(\sigma_j\sigma_k - \sigma_k\sigma_j)$ and the pseudoscalar is $I = \sigma_1\sigma_2\sigma_3$.

The element I commute with all the others elements and $I^2=-1$.

From the isomorphism (A.5.1) it follows that:

$$\sigma_{jk} = -\gamma_{jk} \qquad (A.5.3)$$

and

$$I = \gamma_5 \qquad (A.5.4)$$

We see that for each choice of the arbitrary time-like vector γ_0 we obtain the *Pauli Algebra of* γ_0 by the isomorphism (A.5.1).

Given a vector $p\in R_{1,3}$, it can be represented by:

$$p\gamma_0 = p_0 + \vec{p} \qquad (A.5.5)$$

where

$$p_0 = p\cdot\gamma_0 \text{ and } \vec{p} = p\wedge\gamma_0 \qquad (A.5.6)$$

The Dirac operator $\partial = \gamma^\mu\partial_\mu$ is a vector operator. We have then:

$$\gamma_0\partial = \partial_0 + \nabla \qquad (A.5.7)$$

where

$$\partial_0 = \gamma_0\cdot\partial \text{ and } \nabla = \gamma_0\wedge\partial \qquad (A.5.8)$$

Obs.: If $A\in R_{1,3}$, then

$$\partial A = \partial\cdot A + \partial\wedge A \qquad (A.5.9)$$

Quantity $\partial\cdot A$ is called *divergence* of A and $\partial\wedge A$ is called *curl* of A.

APPENDIX B

B1. GENERAL GAUGE THEORY. THE PRINCIPAL FIBER BUNDLE APPROACH.

Let $(M,h\nabla)$ be a Lorentzian manifold[1]. Let $\pi: P\to M$ be a principal fiber bundle PFB with group G (and Lie algebra $\hat{G}$). The following conditions must hold:

(i) Given $g\in G$, there exists a mapping (diffeomorphism)

$$R_g: P\to P; p\to R_g(p) = pg$$

(ii) $\pi: P\to M$ is onto. If $x=\pi(p)\in M$, the orbit of G through $p, \pi^{-1}(x)=\pi^{-1}(gx) = pg$, $g\in G$ is called the fibre over $x=\pi(p)$.

In this way, given $p\in\pi^{-1}(x)$, there exists a diffeomorphism (non-canonical) $G\to\pi^{-1}(x)$; $g\to pg$.

(iii) P is locally trivial, i.e. for each $x\in M$, there exist an open set $U\subset M$ with $x\in U$ and a diffeomorphism $T_U: \pi^{-1}(U)\to U\times G$; $T_U(p) = (\pi(p), S_U(p))$ where $S_U: \pi^{-1}(U)\to G$ has the property $S_U(pg) = S_U(p)g$, $\forall g\in G, \forall p\in\pi^{-1}(U)$.

T_U is called a local trivialization (LT), or a choice of gauge.

We will need also the concept of transition function. Given the PFB $\pi: P\to M$ with group G and two LT, $T_U: \pi^{-1}(U)\to U\times G$ and $T_V: \pi^{-1}(V)\to V\times G$, we define the transition from T_U to T_V as the mapping $g_{UV}: U\cap V\to G$ where,

$$g_{UV}(x) = S_U(p)(S_U(p))^{-1};\ x=\pi(p)\in U\cap V \qquad (B1)$$

$g_{UV}(x)$ is well defined since $S_U(pg)(S_V(pg))^{-1} = S_U(p)gS_V(p)g^{-1} = S_U(p)gg^{-1}(S_V(p))^{-1} = S_U(p)(S_V(p))^{-1}$, and the following properties are true

$g_{UU}(x) = e$, $\forall x\in U$; (ii) $g_{UV}(x) = (g_{UV}(x))^{-1}$, $\forall x\in U\cap V$

$g_{UV}(x)g_{UW}(x)g_{WU}(x) = e$, $\forall x\in U\cap V\cap W$

In order to link this abstract theory with the theory in §2, we need the concept of a local section of a PFB $\pi: P\to M$ with group G, i.e. the mapping

$$M\supset U\to P;\ \pi\circ\sigma = Id_U \qquad (B2)$$

where Id_U is the identity in U.

It can be shown that there exists a natural correspondence between local sections and local trivializations. To analyse the monopole theory, we shall need also the following[6,3]

THEOREM: A PFB $\pi: P\to M$ with group G is trivial if and only if it has a continuous (global) cross section.

A trivial PFB is one where $P=M\times G$

We give now three equivalent definitions of a *connection* in a PFB. These three definitions contain necessary ingredients for the formulation of our monopole theory as a PFB theory.

(C1) A connection is a way to associate with $p\in P$ a subspace $H_p\subset T_pP$ such that:

$$R_{g*}(H_p) = H_{pg} \qquad (B3)$$

The mapping $p \to H_p$ is C^∞. H_p is called the *horizontal* subspace and V_p is called the *vertical* subspace, such that $T_pP = H_p \oplus V_p$.

(C2) A connection is a C^∞ 1-form ω over P which takes values in $\hat{G}$, the Lie Algebra of G, and such that:

(a) given $A \in \hat{G}$ and A^* a vector field defined over P by

$$A^* = \frac{d}{dt}(p\exp tA)_{t=0} \quad \Rightarrow \quad \omega_p(A^*{}_p) = A$$

A^* is called the *fundamental field*;

(b) given $g \in G$, it is $\omega_{pg}(R_{g*}X) = Ad_{g^{-1}}\omega_p(x)$, $\forall g \in G$, $p \in P$

We can write this equation as

$$R_{g*}(\omega) = Ad_{g^{-1}}\omega(\omega)$$

where ω is called the connection 1-form.

It is important to observe that in general G appears in the physical theories as a matrix group through its adjoint representation

$$G \to GL(\hat{G});\ g \to Ad_g.$$

In such a case, if $A \in GL(\hat{G})$ and $B \in \hat{G}$, we have

$$Ad_A B = \frac{d}{dt} Ad_A(\exp tB)_{t=0} = \frac{d}{dt}(A^{-1}(\exp tB)A)_{t=0} = A^{-1}BA \qquad (B5)$$

Assuming then that G is a matrix group we have

(C3) A connection associates for each local trivialization (i.e., a choise of gauge) $T_U: \pi^{-1}(U) \to U \times G$ a 1-form ω_U over U, with values in $\hat{G}$, with the following compatibility condition: If $g_{UV}: U \cap V \to G$ is the transition function from T_U to T_V we impose

$$\omega_V = (g_{UV})^{-1}dg_{UV} + (g_{UV})^{-1}\omega_U\, g_{UV} \qquad (B6)$$

which we sometimes write in a more compact notation as

$$\omega_V = g^{-1}dg + Ad_{g^{-1}}\,\omega_U \qquad (B6')$$

We now defien the *local gauge potential*. Given a connection over P, a local section $\sigma: U \to P$, the "pull-back"

$$\omega_U = \sigma_U^*\,\omega \qquad (B7)$$

is called the *local gauge potential*.

Now, given a connection 1-form ω over a PFB $\pi: P \to M$ with group G, we can write each $X \in T_pP$ as $X = X^V + X^H$, where X^V is *vertical* (i.e., $\pi_*(X^V) = 0$) and X^H is *horizontal* (i.e. $\omega(X^H) = 0$).

If $\Lambda^k(P,\hat{G})$ is the set of all the k-forms over P with values in $\hat{G}$, then if

$\varphi \in \Lambda(P,\hat{G})$ we define $\varphi^H \in \Lambda^k(P,\hat{G})$ by

$$\varphi^H(X_1,\ldots,X_k) = \varphi(X_1^H,\ldots,X_k^H) \qquad (B8)$$

Notice that $\varphi^H(X_1,\ldots,X_k) = 0$ if one of the X_i is vertical. Moreover, the exterior covariant derivative D^ω of $\varphi \in \Lambda^k(P,\hat{G})$ is

$$D^\omega\varphi = (d\varphi)^H \in \Lambda^{k+1}(P,\hat{G}) \qquad (B9)$$

We define now the *curvature form* of the connection $\omega \in \Lambda^1(P,\hat{G})$ as

$$\Omega^\omega = D^\omega\omega \in \Lambda^2(P,\hat{G}) \qquad (B10)$$

We can show that the curvatre satisfies the Cartan structural equation[2,3]

$$\Omega^\omega = D^\omega\omega = d\omega + \tfrac{1}{2}[\omega,\omega] \qquad (B11)$$

The meaning of the commutator in eq (15) is as follows: Let $\varphi \in \Lambda^i(P,\hat{G})$ and $\psi \in \Lambda^j(P,\hat{G})$. Then $[\varphi,\psi] \in \Lambda^{i+j}(P,\hat{G})$, in such a way that

$$[\varphi,\psi](X_1,\ldots,X_{i+j}) = \frac{1}{i!\,j!}\sum_\sigma(-1)^\sigma[\varphi(X_{\sigma(1)},\ldots X_{\sigma(i)}),\psi(X_{\sigma(i+1)},\ldots X_{\sigma(i+j)})] \qquad (B12)$$

where $\sigma \in$ Perm $(1,2,\ldots,i+j)$ and $(-)^\sigma = \pm$ is the sign of the permutation.

The brackets in the r.h.s. of (16) are the Lie brackets in $\hat{G}$ and $X_n \in T_pP$, $n = 1, \ldots, i+j$.

The curvature satisfies also certain integrability condition called Bianchi idenbty (which in the Physics literature are called the homogeneous field equations):

$$D^\omega\,\Omega^\omega = 0 \qquad (B13)$$

Once we choose a local section $\sigma_U: U \to P(U \subset M)$, we have a gauge potential ω_U associated with the connection via the pull-back (C3).

Now, the pull-back of Ω^ω is called the *force field* associated to ω_U. We have

$$\Omega_U = \sigma_U^*\,\Omega^\omega \qquad (B14)$$

The Cartan structural equation is also valid in U, i.e. we have

$$\Omega = D^\omega\omega_U = d\omega_U + \tfrac{1}{2}[\omega_U,\omega_U] \qquad (B15)$$

It is important to observe that, when (B6) is a matrix group, we have that, if $\varphi \in \Lambda^i(P,\hat{G})$ and $\Psi \in \Lambda^j(P,\hat{G})$, then

$$[\varphi,\Psi] = \varphi\wedge\Psi - (-1)^{ij}\Psi\wedge\varphi \qquad (B16)$$

where φ and Ψ are considered as matrices of forms with values in R, and $\varphi\wedge\Psi$ is the usual matrix multiplication (where the elements of the matrices are multiplied via the edge operator $\wedge$). Then, when G is a matrix group, eq (B11) and eq (B15) can be written

$$\Omega^\omega = d\omega + \omega\wedge\omega \qquad (B11')$$

$$\Omega_U = d\omega_U + \omega_U\wedge\omega_U \qquad (B15')$$

Eq.(10)givesthe rule for the transformation of the potential under a gauge transformation. For the *force field* we have the following rule[7,15].

$$g_{UV}: U\cap V \to G \Rightarrow \Omega_V = Ad_{(g_{UV})^{-1}}\Omega_U \tag{B17}$$

and in the case when G is a matrix group we have

$$\Omega_V = (g_{UV})^{-1}\Omega_U\, g_{UV} \tag{B18}$$

B2. EQUIVARIANT HORIZONTAL FORMS

Let $G\to GL(V)$, $g\to g.v$ for $g\in G$ and $v\in V$ be as usual a representation of G in a vector space V. By definition, an equivariant horizontal k-form (or horizontal k-form, for short) in P which assumes values in V satisfies the properties:

$$(h_1) \qquad \phi(X_1,\ldots\ldots X_k) = 0 \tag{B.19}$$

if at least one of the X_i is vertical, i.e.,$\pi_* X_i = 0$

$$(h_2) \qquad R_g^*\phi = g^{-1}\phi \tag{B.20}$$

Let us recall the space of the k-horizontal forms $\overline{\Lambda}^k(P,V)\subset\Lambda^k(P,V)$. Observe that , although a connection $\omega\in\Lambda^1(P,\hat{G})$ has the property (h_2) when we use the adjoint representation $Ad: G\to GL(V)$, it does not give zero when applied to vertical vectors and then does not belong to $\overline{\Lambda}^1(P,\hat{G})$. The difference between two connections $\tau=\omega_1-\omega_2\in\overline{\Lambda}^1(P,\hat{G})$ since satisfies (h_2) and (h_1). Indeed, ω_1 and ω_2 map a given vertical vector A^*_p on its (unique) generator $A=\omega_1(A^*)=\omega_2(A^*)=A\in\hat{G}$

Conversely, if $\tau\in\overline{\Lambda}^1(P,\hat{G})$, also $\omega+\tau$ is a connection, and if we fix a connection ω we have a 1-1 correspondence between the elements of $\overline{\Lambda}^1(P,\hat{G})$ and the connections in $\pi:P\to M$ .In this way, if ω is a connection and $\tau\in\overline{\Lambda}^1(P,\hat{G})$, then $\omega+t\tau$ is a cuve in $\mathcal{C}$,(the space of connections) which is equal to ω in t=0.

We can then characterizes $\overline{\Lambda}^1(P,\hat{G}) = T_\omega\mathcal{C}$ as the tangent space to $\mathcal{C}$.We also observe here that $\Omega^\omega\in\Lambda^2(P,\hat{G})$ with the adjoint representation.

We call horizontal functions or particle fields the maps $\Psi: P\to V$ which satisfy only (h_2) $(\Psi\in\overline{\Lambda}^0(P,V) = C(P,V))$. Such functions are associated to the quantum fields of the particles.

The covariant derivative maps horizontal forms into horizontal forms, i.e. $D^\omega:\overline{\Lambda}^k(P,V)\to\overline{\Lambda}^{k+1}(P,V)$. Indeed, $D^\omega\varphi=(d\varphi)^H$ gives zero on vertical vectors and $R_g^*\varphi = g^{-1}\varphi$. $R_g^* D^\omega\varphi = R_g(d\varphi)^H = (dR_g^*\varphi)^H = g^{-1}D^\omega\varphi$ (Observe that the pull-back commutes with the differential and that R_g^*changes only the vertical component of the vectors).

We can show[15] that for $\tau\in\overline{\Lambda}^k(P,V)$ we have

$$(h_3) \qquad D^\omega\tau = d\omega + \omega\dot{\wedge}\tau \tag{B.21}$$

The symbol $\dot{\wedge}$ in (h_3) carries not only the usual exterior product but also the action of G in V. Such an action comes from the representation of G in V,$v\to gv$ given by

$$A.v = \frac{d}{dt}\,((\exp(A).v)|_{t=0} \tag{B.22}$$

When $V=\hat{G}$ with the adjoint representation, (h_3) reduces to $D^\omega\tau=d\tau+[\omega,\tau]$. When $\tau=\Omega^\omega$ this is the Bianchi identity.

B3. METRICS AND THE HODGE STAR OPERATOR IN A PFB

Let $\pi:P\to M$ with group G be a PFB. For each $p\in P$ we can choose $H_p\subset T_pP$ such that $T_pP=H_p\oplus V_p$. We observe that $\pi_{*|H_p}: H_p\to T_xM$ $(x=\pi(p))$ is an isomorphism between H_p and T_xM . μ

The metric h of the space-time M can then be transported to H_p, which then will have a metric $h_p = \pi_p^* h_x$. We can also define a volume element $\overline{\mu}$ associated with the volume element μ in T_xM. This permits us to define a Hodge star operator in H_p.

$$*_p:\Lambda^k(H_p,V) \to \Lambda^{n-k}(H_p,V) \tag{B.23}$$

where $n=\dim H_p=\dim M$.

Due to the partition $T_pP=H_p\oplus V_p$ and due to the fact that horizontal forms give zero when applied to vertical vectors, we can define

$$\bar{*}:\overline{\Lambda}^k(P,V)\to\overline{\Lambda}^{n-k}(P,V) \tag{B.24}$$

For $\varphi\in\overline{\Lambda}^k(P,V)$ and $p\in P$ we define $(\bar{*}\varphi)_p$ as the unique extension of $*_p\varphi_{|H_p}$ to a (n-k) form in P with values in V which gives zero when acting on vertical vectors. We have for each local section $\sigma_U:U\to P$

$$(\bar{*}\varphi) = {*}\sigma_U^*(\varphi) \tag{B.25}$$

where $*: \Lambda^k(U,V)\to\Lambda^{n-k}(U,V)$ is the usual Hodge star operator $*$ defined by the metric and the volume element in M. (see eq (A.3.2))

Given $\bar{*}$ we can define a codifferential associated to a connection $\delta^\omega:\overline{\Lambda}^k(P,V)\to\overline{\Lambda}^{n-k}(P,V)$ for horizontal forms by (compare with eq (A.3.1)):

$$\delta^\omega A_p = (-1)^p\,\bar{*}^{-1}D^\omega\bar{*}A_p\ ,\forall A_p\in\overline{\Lambda}^k(P,V) \tag{B.26}$$

By using $\bar{*}$ we can also define a metric for the horizontal p-forms $\varphi,\Psi\in\overline{\Lambda}^k(P,R)$ by

$$\hbar : \overline{\Lambda^k}(P,R)\times\overline{\Lambda^k}(P,R)\to R \qquad \text{(B.27)}$$

with $\varphi\wedge\bar{*}\psi=\hbar(\varphi,\psi)\bar{\mu}$, being $\bar{\mu}$ the volume element in H_p.

This metric can be extended also to horizontal forms $\varphi,\psi\in\overline{\Lambda^k}(P,V)$ which take values in a vector space V with metric $\hat{h}$. Let $\{E_\alpha\}$ be a basis of V, $\hat{h}_{\alpha\beta}=\hat{h}(E_\alpha,E_\beta)$ and write $\varphi=\varphi^\alpha E_\alpha$, $\psi=\psi^\alpha E_\alpha$ with $\varphi^\alpha,\psi^\alpha\in\overline{\Lambda^k}(P,R)$.

We define

$$\hbar\,\hat{h} : \overline{\Lambda^k}(P,V)\times\overline{\Lambda^k}(P,V)\to R$$
$$\hbar\,\hat{h}\,(\varphi,\psi) = \sum_{\alpha\beta}\hbar(\varphi^\alpha,\psi^\beta)\hat{h}_{\alpha\beta} \qquad \text{(B.28)}$$

In the case in which $V=\hat{G}$ we always use $\hat{h}=h$, the bi invariant Killing-Cartan metric in $\hat{G}$ [15] and we have

$$\hbar\,\hat{h} : \overline{\Lambda^k}(P,\hat{G})\times\overline{\Lambda^k}(P,\hat{G})\to R$$

for all $\varphi\in\overline{\Lambda^k}(P,\hat{G})$.

APPENDIX C

SPLICED BUNDLES AND CONNECTIONS

Let $\pi_1:P_1\to M$ and $\pi_2:P_2\to M$ be two PFB with group G_1 and G_2 and the same base space M. We define the set $P_1oP_2=\{(p_1,p_2)\in P_1\times P_2 \mid \pi_1(p_1)=\pi_2(p_2)\}$ and observe that $G_1\times G_2$ acts freely on the right of P_1oP_2, $(p_1,p_2)(g_1,g_2)=(p_1g_1,p_2g_2)$. . It is easy to see that $\pi_{12}:P_1oP_2\to M$ with $\pi_{12}(p_1,p_2)=\pi_1(p)=\pi_2(p)$ is a PFB with group $G_1\times G_2$. We call such a PFB a spliced bundle associated to the two PFB which enters the definition.

We can also introduce two other PFB, $\pi^1: P_1oP_2\to P_1$ $(\pi^1(p_1,p_2)=p_1)$ with group G_2 and $\pi^2:P_1oP_2\to P_2(\pi^2(p_1,p_2)=p_2$ whith group G_1.

We have the structure of five PFB,represented in Figure 1

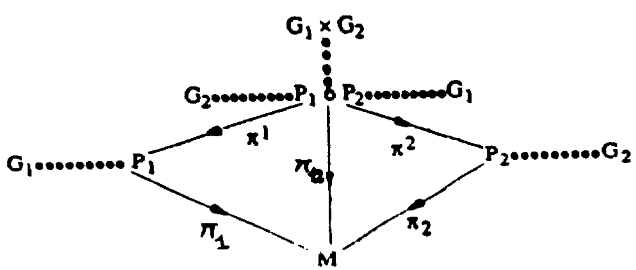

Fig 1 - The spliced bundle P_1oP_2 and other PFBs

We can show[15] that, if $\omega_1\in\Lambda^1(P_1,\hat{G}_1)$ and $\omega_2\in\Lambda^2(P_2,\hat{G}_2)$ are connections defined on the PFB $\pi_1:P_1\to M$ and $\pi_2:P_2\to M$, then

$$\omega = \pi^{1*}\omega_1 \oplus \pi^{2*}\omega_2 \qquad \text{(C1)}$$

is a connection on the PFB $\pi_{12}:P_1oP_2\to M$. We are going to show here that, if ω is a connection on the spliced bundle, there are connections ω_1 and ω_2 on the original PFB such that

$$\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2 \qquad \text{(C2)}$$

In order to prove our statement let us use the identifications $\hat{G}_1=\hat{G}_1\oplus O$, $\hat{G}_2=O\oplus\hat{G}_2$ in $\hat{G}_1\oplus\hat{G}_2$, and consider the projections $p^i:G_1\oplus G_2\to G_i(i=1,2)$. Observe that we can always write $\omega=p^1(\omega)\oplus p^2(\omega)$, and then, if we define ω_1 and ω_2 through the relations $\pi^{i*}\omega_i=p^i(\omega)$, i 1,2, we obtain $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$.

It remains to be proved that ω_1 and ω_2 are well defined and are connections. We have, for $\omega_1$for example, that

$$(\omega_1)_{p_1}(X_1) = p^1(\omega_p(X_p))$$

where $p\in P_1oP_2$, $\pi_1(p)=p_1$, $X_p\in T_p(P_1oP_2)$ and $(X)_{p_1}=\pi^1{}_*(X_p)\in T_{p_1}(P_1)$

We must show that this definition does not depend on the choice of p and X_p. Indeed, for p fixed, if $X_p,X'_p\in T_p(P_1oP_2)$ and $\pi^1{}_{*p}(X_p)=\pi^1{}_{*p}(X'_p)=X_{p_1}$, then

$$\pi^1{}_*(X_p-X'_p)=0 \qquad \text{(C3)}$$

and then $X_p-X'_p$ is a vertical vector in the PFB $\pi^1:P_1oP_2\to P_1$. The group associated to this PFB is G_2 and we can write

$$X_p-X'_p = \frac{d}{dt}(p\ \exp tA_2)|_{t=0} \qquad \text{(C4)}$$

for $A\in\hat{G}_2$. Making the identification $A_2=0\oplus A_2\in\hat{G}_1\oplus\hat{G}_2$ we obtain that

$$\omega_p(X_p-X'_p)=0\oplus A_2 \qquad \text{(C5)}$$

and then $p^1(\omega_p(X_p-X'_p))=0$ and $p^1(\omega_p(X_p))=p^1(\omega_p(X'_p))$

In the case $\pi^1(p)=\pi^1(p')=p_1$, we have $p'=pg_2$ and $X'_p=R^*_{g_2}X_p$, and then

$$\pi^1{}_*(X'_p)=\pi^1{}_*(X_p)$$

Moreover

$$\omega_{p'}(X_{p'}) = Ad_{(g_2)^{-1}}\omega_p(X_p) \quad\text{and}\quad p^1(\omega_{p'}(X_{p'}))=p^1(\omega_p(X_p))$$

It follows that ω_1 is well defined and we must now show that it is a connection. The properties (i) and (ii) of the definition of a connection are clearly satisfied. We observe that, from the point of view of the PFB, $\pi^1 : P_1 o P_2 \to R_{g_2}$ acts so as $R_{(e,g_2)}$ from the point of view of the spliced bundle:

$$\omega_1 (R_{g_1} * X_{p_1}) = p^1(\omega(R_{(g_1,e)} * X_p) = p^1 (Ad_{(g,e)^{-1}} \omega(X_p)) =$$
$$= Ad_{(g_1)^{-1}} p^1(\omega(X_p)) = Ad_{(g_1)^{-1}} \omega_1(X_{p_1}) \qquad (C6)$$

In this way it is proved that, for each connection ω on the spliced bundle $\pi_{12}: P_1 o P_2 \to M$, there are two connections ω_1 on P_1 and ω_2 on P_2 such that $\omega = \pi^{1*}\omega_1 \oplus \pi^{2*}\omega_2$

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REFERENCES

[1] R.K.SACHS and H. WU., *General Relativity for Mathematicians*, Springer-Verlag, N. York (1977).

[2] P.A.M.D.DIRAC, Proc. R. Soc. *A 133*, (1931).

[3] P.A.M.D.DIRAC, Phys. Rev. *74*, 817 (1948).

[4] D.ROSENBAUN, Phys. Rev. *140B*, 804 (1965).

[5] A.FRENKEL, preprint KFKI, 1986-12/A Hungarian Academy of Sciences, Central Institute for Physics, Budapest.

[6] J.W.MILNOR and J.D. STASHEFF, *Characteristic Classes*, Princeton Univ. Press (1974).

[7] Y.CHOQUET-BRUHAT, C. DEWITT-MORETTE, M. DILLAD-BLEICK, *Analysis, Manifolds and Physics*, second revised edition, Norh-Holland (1982).

[8] M.A.FARIA-ROSA, E. RECAMI, W.A. RODRIGUES Jr., Phys. Letters *B173*, 233 (1986); erratum *B188*, 511 (1987).

[9] W.A.RODRIGUES Jr., E. RECAMI, M.A. FARIA ROSA, A. MAIA Jr., preprint R.T. 31/87, IMECC-UNICAMP.

[10] F.RORLICH, Phys. Rev. *150B*, 1104 (1966).

[11] A.MAIA Jr. and W.A. RODRIGUES Jr., preprint R.T. 32/87 IMECC-UNICAMP.

[12] E.F.CARTER and H. COHEN, Am. J. Phys. *41*, 994 (1973).

[13] D.FRYBERGER, preprint SLAC-PUB-4237, Feb. 1987.

[14] M.A.FARIA-ROSA and W.A. RODRIGUES Jr., sub. for publication (1988).

[15] D.D.BLEECKER, *Gauge Theory and Variational Principles*, Addison-Wesley Reading, Mass. (1981).

[16] W.GRAF., Ann. Inst. Henri Poincaré *A29* ,85 (1978).

[17] D.HESTENES and G. SOBCZYK, *Clifford Algebra to Geometric Calculus*, Reidel, Dordrecht (1984).

[18] N.CABIBBO and E. FERRARI, N. Cimento *23*, 1147 (1962).

[19] S.MANDELSTAN, Ann. Phys. *19*, 1(1962).

[20] A.O.BARUT:*Electrodynamics and Classical Theory of Fields and Particles* p. 138, Mac Millan, N. York (1968).

[21] M. F. ATIYAH, R. BOTT and A. SHAPIRO,Topology *3*, *suppl*, 3 (1964).