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NONLINEAR DE BROGLIE WAVES AND THE RELATION BETWEEN RELATIVISTIC AND NONRELATIVISTIC SOLITONS*

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1. Nonrelativistic Limit

There is no systematic method, up to now, of classifying an arbitrary nonlinear partial differential equation (NLPDE) as being C–integrable or S–integrable (Calogero, 1988). If a given NLPDE is integrable, then we have a number of systematic methods, such as the inverse scattering transform (IST), Bäcklund transform (BT), Darboux transform (DT), to construct exact solutions. When we are not in a position to judge the integrability then we can use other special techniques like Hirota's bilinear operator, similarity and symmetry analysis, to find the exact solutions. Most of the NLPDE are not integrable and so their solutions are difficult to find. Hirota's method will not always give exact solutions, for example, it fails for equation of motion of the $\lambda\phi^6$ Lagrangian field theory. Similarity analysis or group theoretical methods are also not always available. Even if we are able to find some symmetries, the reduced equations are nonlinear and difficult to solve in most cases.

The purpose of this work is to try to relate some nonintegrable class of equation to the well-known integrable class of NLPDEs using the nonrelativistic limiting procedure and so this method may give some information about the possible structure of the exact solutions for a nonintegrable class of equations if we know the respective integrable class of NLPDEs.

In this paper we show that the well-known envelope soliton and kink solutions of the nonlinear Schrödinger equations (NLSE) are the nonrelativistic (NR) limit of the corresponding solutions of the nonlinear Klein–Gordon equation (NLKGE). To our knowledge, the NR limit of the nonlinear field equations and their solutions have not been discussed before. As a result of this NR limiting process, we obtain new solutions for the 'double nonlinear Schrödinger equations (DNLSE)' and exact solutions for the perturbed NLSE or Generalized NLSE. We shall further show that the dispersion relations for the phase of the solution differs from the de Broglie relation by a negative internal energy ε_0 of the soliton solution. In the limit $\varepsilon_0 \to 0$, these solutions disappear and we have the usual plane waves. Conversely, we also conjecture that corresponding to all the envelope solutions of NLSEs, there are envelope solutions of NLKGEs, such that their NR limit exactly reduce to the solutions of the NLSEs. This method provides new exact solutions for the NLKGE family. We further discuss the role of the internal degrees of freedom on the de Broglie phase.

The NR limit (Barut and Raçzka, 1980; Barut and Van Huele, 1985) or the transformation for constructing an NR Schrödinger equation from an NLKGE of motion, is defined by

$$\lim_{c\to\infty} \exp\left(\frac{imc^2t}{\hbar}\right)\phi(\bar{x},t) \to \psi(\bar{x},t) \tag{1.1}$$

where $\phi(\bar{x},t)$ is the field variable of the NLKGE.

$$\Box\,\phi + m^2c^4\phi - ac^2\phi|\phi|^{2\sigma} - bc^2\phi|\phi|^{4\sigma} = 0 \tag{1.2}$$

and σ is any real number, a, b, are real parameters and $\psi(\bar{x},t)$ is the field variable for the NLSE and

$$\Box\,\phi \equiv \hbar^2(\phi_{tt} - c^2\phi_{xx})$$

Inserting Eq. (1.1) into Eq. (1.2) we get

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\Delta\psi + \frac{a}{2m}\psi|\psi|^{2\sigma} + \frac{b}{2m}\psi|\psi|^{4\sigma} = 0 \tag{1.3}$$

where $\Delta\psi = \psi_{xx} + \psi_{yy} + \psi_{zz}$. Eq. (1.3) is an NLSE but with two nonlinear terms, and we shall denote it as the double nonlinear Schrödinger equation (DNLSE).

Another interesting fact is that the perturbed DNLSE or Generalized nonlinear Schrödinger equation (GNLSE) (Cowan et al., 1986; Tussynski et al., 1984; Gagnon and Winternitz 1987 and 1988a,b)

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\Delta\psi + \frac{a}{2m}\psi|\psi|^{2\sigma} + \frac{b}{2m}\psi|\psi|^{4\sigma} = \tilde{\beta}\psi \tag{1.4}$$

where $\tilde{\beta}$ is the perturbation coefficient, is the NR limit of the same KGE (Eq. (1.2)) but with a slight modification in the coefficient of linear terms

$$\Box\,\phi + (m^2c^4 + 2mc^2\beta)\phi + ac^2\phi|\phi|^{2\sigma} + bc^2\phi|\phi|^{4\sigma} = 0 \tag{1.5}$$

The above correspondence assures us that we will be able to find exact solutions of DNLSE and GNLSE if we know the solutions of NLKGEs. However it is important to note that all known solutions of NLKGEs are without an envelope phase factor except for some recent results (Barut and Rusu, 1987; Baby and Barut, 1987 and 1988), whereas the NLSE has the well-known envelope type solutions! Moreover, if we take the NR limit of the known solutions of the NLKGs we get vacuum solutions of NLSEs. This suggests that we have to modify the structure of the solutions of the NLKGEs so that envelope type solutions are the relativistic form of the corresponding nonrelativistic envelope solutions of NLSEs. For this purpose we introduce a phase factor $\exp(i\theta')$ into the solutions of NLKGEs, so that when we take the modulus of the field variable the phase factor disappears. This property is analogous to L. de Broglie's earlier ideas on the 'double solutions of nonlinear equations' (de Broglie, 1960; Barut and Rusu, 1987)

2. Envelope Solutions of Nonlinear Klein–Gordon Equations

In this section we will describe the method of introducing de Broglie phase factor $\exp(i\theta')$ in the known solutions of NLKGEs. Throughout this paper our solutions will have a complex phase factor of the form

$$\exp(i\theta') = \exp\left[\frac{i\epsilon\gamma}{\hbar}\left(\frac{\beta}{c}x - t + \delta'\right)\right] \tag{2.1}$$

in a relativistic quantum mechanical notation of a 'de Broglie phase'. Here $\beta = v/c$, $\gamma = (1 - v^2/c^2)^{-1/2}$ and δ' is an arbitrary parameter and if we write $\theta' = k'(x - ut) + \delta'_0$, where $k' = \frac{\epsilon\beta\gamma}{\hbar c}$, then the phase velocity is u.

We introduce another exponential factor with a real exponent and with δ as constant

$$\exp(\theta) = \exp\left[\frac{\omega\gamma}{\hbar c}(x - vt) + \delta\right]. \tag{2.2}$$

Our soliton-like envelope solutions are expressible entirely in terms of the product of these two exponentials

$$g(x,t) = \exp(i\theta') \cdot \exp(\theta) \tag{2.3}$$

so that the envelope velocity is v, which is different from the phase velocity u. There is a relation between the parameter ω of the envelope with the parameter ϵ of the de Broglie phase

$$\omega^2 + \epsilon^2 = m^2c^4 \tag{2.4}$$

where m is a parameter fixed by the nonlinear equation, whereas the parameters ϵ and ω do not occur in the wave equations, they characterise the solutions. The known dispersion relations are obtained by assigning $\epsilon = 0$. We need also the combinations

$$gg^* = \exp(2\theta) \tag{2.5}$$

where g^* is the complex conjugate of $g(x,t)$.

The absolute valued $\gamma\phi^6$ Lagrangians gives the well-known NLKGE

$$\Box\phi + m^2c^4\phi - ac^2\phi|\phi|^{2\sigma} - bc^2\phi|\phi|^{4\sigma} = 0 \tag{2.6}$$

for $\sigma = 1$.

Let $\phi(x,t)$ be a complex-valued function and assume that Eq. (2.6) has a solution of the form

$$\phi(x,t) = \frac{h(x,t)}{\left\{\left[1 + \frac{ac^2(hh^*)^\sigma}{4(\sigma+1)\omega^2}\right]^2 + \frac{bc^2(hh^*)^{2\sigma}}{4(2\sigma+1)\omega^2}\right\}^{1/2\sigma}} \tag{2.7}$$

where $h(x,t)$ is also a complex valued function. Then it can be shown that $h(x,t)$ has to satisfy the following set of equations

$$\Box h = -m^2c^4h, \quad (\partial^\mu h)(\partial_\mu h) = -m^2c^4h^2$$

$$\Box h^* = -m^2c^4h^*, \quad (\partial^\mu h^*)(\partial_\mu h^*) = -m^2c^4h^{*2}$$

$$(\partial^\mu h)(\partial_\mu h^*) = (-\omega^2 + \epsilon^2)hh^*, \quad \mu = 0, 1 \tag{2.8}$$

where h^* is the complex conjugate of $h(x,t)$. All solutions of Eq. (2.8) are the solutions of Eq. (2.6) via Eq. (2.7). For real valued $h(x,t)$ and $\phi(x,t)$, this result is reported earlier (Burt, 1978). It can easily be verified that $h(x,t) = g(x,t)$ is a solution of Eq. (2.8). So the new exact envelope-type solution of the NLKGE (Eq. (2.6)) is given by (Baby and Barut, 1987)

$$\phi(x,t) = \frac{e^{i\theta'}e^{\theta}}{\left\{\left[1 + \frac{ac^2e^{2\sigma\theta}}{4(\sigma+1)\omega^2}\right]^2 + \frac{bc^2e^{4\sigma\theta}}{4(2\sigma+1)\omega^2}\right\}^{1/2\sigma}} \tag{2.9}$$

For $b = 0$, this solution is known (Barut and Rusu, 1987) and for $\sigma = 1$, it reduces to the soliton-like envelope solution of the well-known $\lambda\phi^4$-Lagrangian field theory. The solution (2.9) is reported earlier when $h(x,t)$ is a real valued function (Burt, 1978).

Of Eqs. (2.8) the first two are linear and so we may be able to use the linear superpositions of solutions of the form

$$h(x,t) = \sum_{j=1}^{N} \exp(\theta_j + i\theta_j') \tag{2.10}$$

where

$$\theta_j = (p_{1j}x - p_{0j}t + \delta_j), \quad \theta_j' = (v_{1j}x - v_{ij}t + \delta_j') \tag{2.11}$$

then in order to find an exact solution of Eq.(2.8) we need the following additional constraints

$$(p_{0j} + p_{0k} + v_{0j}v_{ok}) - c^2(p_{ij}p_{1k} + v_{ij}v_{1k}) = -\frac{m^2c^4}{\hbar^2}$$

$$p_{0j}v_{0ok} - c^2 p_{1j}q_{1k} = 0 \quad j,k = 1,2\dots N \tag{2.12}$$

But the set of Eq. (2.12) is an overdetermined system and it is valid only for $j = k = 1$. In (1+1) dimensions there exist only one independent wave vector. The dimensionality of space–time (d+1) and the maximum possible number of solutions N are related by $N \leq 2d - 1$. Moreover, these N–envelope solutions are highly restricted by the constraints (2.10) and therefore they are not of the type of the well-known N–soliton solutions of the integrable systems (Hirota, 1972) and so we call them 'constrained N–envelope soliton' like solutions.

3. Nonrelativistic reduction

In the first section we obtained, as a result of the NR limiting process, DNLSE and GNLSE from the NLKGEs. In this section we will develop exact solutions of the DNLSEs and GNLSEs using the NR limit from the solutions given in the previous sections. Before

that we will find the different values of the parameter combinations and their dispersions relations under this limiting process.

Since ε has the dimension of energy it can be written in the form

$$\varepsilon = mc^2\gamma + \frac{B}{2}mv^2 + \varepsilon_{NR}, \qquad B, \varepsilon_{NR} \quad \text{are parameters.} \tag{3.1}$$

Using Eq. (2.4), we get

$$\frac{\omega^2}{c^2} = m^2v^2 - \left\{\frac{B^2}{4c^2}m^2v^4 + \frac{Bm^2v^2}{\gamma} + \frac{Bmv^2}{c^2}\varepsilon_{NR} + \frac{2m}{\gamma}\varepsilon_{NR} + \frac{\varepsilon_{NR}}{c^2}\right\}. \tag{3.2}$$

We have the following limits

$$\underset{v\to\infty}{\ell t} \frac{\omega^2}{c^2} = (1-B)m^2v^2 - 2m\varepsilon_{NR} = \lambda^2 \tag{3.2}$$

$$\underset{c\to\infty}{\ell t} \frac{\varepsilon\gamma\beta}{\hbar c} = \frac{mv}{\hbar} = p \tag{3.3}$$

$$\underset{c\to\infty}{\ell t} \left(\frac{\varepsilon\gamma}{\hbar} - \frac{mc^2}{\hbar}\right) = \frac{1}{\hbar}\left(\frac{B}{2}mv^2 + \varepsilon_{NR}\right) = p_0 \tag{3.4}$$

$$\underset{c\to\infty}{\ell t}\, g(x,t) = g_0(x,t) \tag{3.5}$$

where

$$g_0(x,t) = \exp(\theta_0' + i\theta_0')$$

$$\theta_0 = (qx - q_0t + \delta_0), \quad \theta_0' = (px - p_0t + \delta_0') \tag{3.6}$$

$$q = \pm\lambda/\hbar, \quad q_0 = \pm\lambda v/\hbar \tag{3.7}$$

such that

$$pq = mq_0/\hbar \tag{3.8}$$

$$p^2 - q^2 = 2mq_0/\hbar, \quad q^2 = \lambda^2/\hbar^2. \tag{3.9}$$

Consequently by our limiting procedure we obtain the following new exact soliton like envelope solution of the DNLSE (Eq. (1.3))

$$\psi(x,t) = \pm \frac{e^{i\theta_0'} e^{\theta_0}}{\left\{ \left[1 + \frac{a e^{2\sigma\theta_0}}{4(\sigma+1)\lambda^2} \right]^2 + \frac{b e^{4\sigma\theta_0}}{4(2\sigma+1)\lambda^2} \right\}^{1/2\sigma}} \tag{3.10}$$

For $\epsilon_0 = \frac{-1}{2m}\lambda^2$, $b = 0$ this solution reduces to the well-known soliton envelope solutions (Zakharov and Shabat, 1972; Hirota, 1973)

$$\psi(x,t) = \pm \left[\frac{-2(\sigma+1)}{\alpha} \epsilon_0 \right]^{\frac{1}{2\sigma}} e^{\frac{i}{\hbar}[mvx - (mv^2/2+\epsilon_0)t]} \operatorname{sech}^{\frac{1}{\sigma}} \left[\frac{\sigma}{\hbar} \sqrt{-2m\epsilon_0}(x - vt) \right] \tag{3.11}$$

which satisfies the NLSE

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\psi_{xx} + \frac{\alpha}{2m}\psi|\psi|^{2\sigma} = 0. \tag{3.12}$$

When B = 1 and $\epsilon_{NR} = 0$, our solution (3.10) vanishes identically. Using the limiting process we can also find the exact solution of the GNLSE (Eq. (1.4)). The new dispersion relation is now

$$\omega^2 + \epsilon^2 = m^2c^4 + 2mc^2\tilde{\beta} \tag{3.13}$$

and the corresponding change of the Eq. (3.2) becomes

$$\lim_{c\to\infty} \frac{\omega^2}{c^2} = (1 - B)m^2v^2 - 2m(\epsilon_{NR} - \tilde{\beta}) = \lambda_0^2. \tag{3.14}$$

The new exact envelope solution of the GNLSE is

$$\psi(x,t) = \frac{\pm e^{i\theta_0'} e^{\theta_0}}{\left\{ \left[1 + \frac{a e^{2\sigma\theta_0}}{4(\sigma+1)\lambda_0^2} \right]^2 + \frac{b e^{4\sigma\theta_0}}{4(2\sigma+1)\lambda_0^2} \right\}^{1/2\sigma}} \tag{3.15}$$

This shows that the perturbation sustains the soliton envelope solution, or any other solution of the DNLSE, obtainable from the limiting process, provided that

$$\tilde{\beta} \neq \epsilon_{NR} = (1 - B)\frac{mv^2}{2}. \tag{3.16}$$

When equality holds, the solution (Eq. (3.15)) vanishes identically. If $\tilde{\beta}$ is sufficiently large and negative, the solution can even be purely imaginary. Another interesting fact is that the perturbation never influences the 'internal frequency' of the de Broglie phase, which is constant under all types of perturbations. When we assign b = 0 in Eq. (1.4), we get the well-known perturbed NLSE, which has been very extensively studied using approximate solutions and Inverse Scattering transforms (IST) perturbation methods (Zakharov and Synakh, 1976; Pereira and Stenflo, 1977; Davydov, 1979; Rabinovich and Fabrikant 1979; Hasse, 1980; Kodama and Ablowitz, 1980 and 1981; Scott, 1982; Tuszynski, Paul and Chatterjee, 1984; Kivshar and Malomed, 1986; Malomed, 1987a,b; see also Fushchich and Serov, 1987). Our results are in agreement with those earlier results. In addition to that we can find that if β is sufficiently large then $\lambda^2 < 0$ and in such situations amplitude of the soliton grows indefinitely and we get even imaginary solutions for some values of $\tilde{\beta}$ and ϵ_{NR}.

4. Conclusions

It is well-known that the NLSE in (1+1) dimensions is an integrable system and its IST is well studied, whereas, the DNLSE and GNLSE have no such property and so it is difficult to find exact solutions for those equations. The method of constructing NR limit of a relativistic equation has been used to find a number of solutions of the NR equations like DNLSE and GNLSE from the solutions of the corresponding NLKGEs. Then converse of this procedure, i.e. a systematic method of finding the solutions of relativistic equations like NLKGEs from the known solutions of NR equations like NLSEs is also interesting and will be reported separately.

For example, it is known that NLSE Eq. (3.12) has an envelope kink solutions

$$\psi(x,t) = \pm\sqrt{\frac{-2\lambda^2}{\alpha}}e^{i\theta_0'}tanh\left[\pm\frac{\lambda}{\sqrt{2}\hbar}(x-vt)\right] \tag{4.1}$$

The respective relativistic form of the kink envelope type solutions and the $\lambda\psi^4$ Lagrangian equations are respectively

$$\phi(x,t) = \pm\sqrt{\frac{-2\omega^2}{\alpha c^2}}e^{i\theta'}tanh\left[\pm\sqrt{\frac{\omega}{2c\hbar a}}(x-vt)\right] \tag{4.2}$$

and

$$\Box\phi + m^2c^4\phi - ac^2\phi|\phi|^2 = 0 \tag{4.3}$$

Note that the known kink solution of (4.3) was without the de Broglie phase (Rajaraman, 1975, Bullough and Caudrey, 1985).

Our study reveals thus that the envelope–type solutions of NLKGEs are the natural relativistic forms of the envelope solutions of the NR equations like NLSEs. In quantum theory free particles are described by plane waves solutions of the linear Schrödinger equation with the usual probability interpretation. On the other hand, appropriate nonlinear field equation could be used to desribe a localized single particle by their envelope soliton–like solutions (Klein, 1976; Ens, 1985 and 1986; Barut and Rusu, 1987). The present study continues the searches in this direction (Barut and Rusu, 1987; Baby and Barut, 1987; Barut and Baby, 1988). The final goal, which is also related to the original thoughts of L. de Broglie (de Broglie, 1960) is to find a satisfactory description of single events in quantum phenomena.

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