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ABSTRACT

The future null infinity for all non-contracting Robertson-Walker space times is studied systematically. A theorem is proved which establishes the expected relation between the nature of $\mathcal{J}^+$ and the appearance or absence of cosmic event horizons.

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1 INTRODUCTION

In the standard model of cosmology it is assumed that, on the largest scale, the Universe can be reasonably represented by a Robertson-Walker (R-W) spacetime.

When we observe the Universe we usually obtain information encoded in the electromagnetic radiation that arrives from the particular observed object. As we try to study more distant objects we are forced to direct our attention to earlier times. Thus when we are studying the largest scale of our Universe we are, for all practical purposes, at future null infinity of the observed region.

It is then necessary in the study of the largest scale of the Universe to have a clear picture of future null infinity of the Robertson-Walker spacetimes.

In this article we make a systematic study of future null infinity for non-contracting R-W spacetimes.

Some particular examples have been extensively discussed in the literature[1,2,3] already. We here however develop the techniques that allow the study of all non-contracting models. In any case we review the Minkowski and de Sitter examples in the second section, and the dust Friedmann models in the third section.

In section 4 the general method is described, and is applied to characteristic asymptotic behavior.

Section 5 contains some closing remarks.

In the remainder of this introductory section we will mainly present the notation to be used.

Let us consider non-contracting Robertson-Walker models. Their line element can be given by

$$ds^2 = dt^2 - A(t)^2 \, dL_K^2 \quad , \tag{1.1}$$

where dL_K can be expressed in several equivalent ways:

$$dL_K^2 = \frac{d\hat{r}^2}{(1 - K\,\hat{r}^2)} + \hat{r}^2 \, d\Sigma^2 \quad , \tag{1.2}$$

$$dL_K^2 = \frac{d\tilde{r}^2 + \tilde{r}^2 \, d\Sigma^2}{\left(1 + \frac{K\,\tilde{r}^2}{4}\right)^2} \quad , \tag{1.3}$$

$$dL_K^2 = d\chi^2 + f_K^2(\chi) \, d\Sigma^2 \quad , \tag{1.4}$$

where $d\Sigma^2$ is the line element of the unit sphere, which can also be expressed in a variety of ways, for example

$$d\Sigma^2 = d\theta^2 + \sin(\theta)^2\, d\phi^2 \tag{1.5}$$

$$= \frac{4\, d\zeta\, d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} = \frac{d\zeta\, d\bar{\zeta}}{P_o^2} \quad , \tag{1.6}$$

with

$$P_o = \frac{(1+\zeta\bar{\zeta})}{2} \quad ; \tag{1.7}$$

finally the function f_K is defined by

$$f_K(\chi) = \begin{cases} \sinh(\chi) & \text{for } K = -1 \quad 0 \le \chi < \infty \\ \chi & \text{for } K = 0 \quad 0 \le \chi < \infty \\ \sin(\chi) & \text{for } K = 1 \quad 0 \le \chi \le \pi \end{cases} \quad . \tag{1.8}$$

The range of the coordinate t is associated with the behavior of the function A(t). When there is an initial singularity followed by a continuing expansion one takes $0 < t < \infty$.

By a non-contracting R-W space we mean that the scalar A(t) must satisfy

$$\frac{\partial A}{\partial t} \ge 0 \quad . \tag{1.9}$$

In the study of future null infinity of asymptotically flat spacetimes, the use of null coordinates and/or null tetrads adapted to scri has proved useful. It is also useful in our case to introduce a null coordinate u by the equation

$$du = \frac{dt}{A(t)} - d\chi \quad . \tag{1.10}$$

Using this relation to replace dt in the line element one obtains

$$ds^2 = A^2 \left(du^2 + 2\, du\, d\chi - f_K^2\, d\Sigma^2 \right) \quad . \tag{1.11}$$

The range of the coordinate u is determined by the asymptotic behavior of the scalar A(t); that is, for example, if

$$\lim_{t \to \infty} \int_{t_o}^{t} \frac{dt'}{A(t')} = \infty \tag{1.12}$$

then the function u is unbounded from above and its range in the cases $K = -1$ and $K = 0$ is $-\infty < u < \infty$. Instead if the limit is finite then the coordinate u is bounded from above, let us say $u < u_o$; this can be thought of as the manifestation of the appearance of cosmic event horizons.

We can also define the coordinate r by the expression

$$r = \int_{t_o}^{t(u,\chi)} A(t')\, dt' - r_o(u) \quad ; \tag{1.13}$$

which satisfies

$$\frac{\partial r}{\partial \chi} = \left.\frac{\partial r}{\partial \chi}\right|_u = \frac{\partial r}{\partial t}\frac{\partial t}{\partial \chi} = A\frac{\partial t}{\partial \chi} = A^2 \quad , \tag{1.14}$$

so that

$$dr = (A^2 - \dot{r}_o)\, du + A^2\, d\chi \quad ; \tag{1.15}$$

where we are using the notation $\dot{r}_o = \frac{dr_o}{du}$.

Then using r instead of the coordinate χ, the Robertson-Walker line element is given by

$$ds^2 = (2\dot{r}_o - A^2)\, du^2 + 2\, du\, dr - A^2 f_K^2\, d\Sigma^2 \quad . \tag{1.16}$$

Note that the line element is spherically symmetric, so we do not change the angular coordinates. The relation between the original non-angular coordinates and the new ones can be expressed by the following differential equations

$$d\chi = \left(\frac{\dot{r}_o}{A^2} - 1 \right) du + \frac{1}{A^2}\, dr \quad , \tag{1.17}$$

$$dt = \frac{\dot{r}_o}{A}\, du + \frac{1}{A}\, dr \quad . \tag{1.18}$$

Let us now introduce a null tetrad adapted to this new coordinate system. We can deduce from the line element that $u = \text{const.}$ are null hypersurfaces, so we take the null vector l to be

$$l = du \quad . \tag{1.19}$$

Then we have

$$g^{ab}\, l_a\, l_b = 0 \quad , \tag{1.20}$$

$$l^a \nabla_a l^b = 0 \quad , \tag{1.21}$$

and

$$l^a = g^{ab} l_b = \left(\frac{\partial}{\partial r}\right)^a \quad ; \tag{1.22}$$

that is r is an affine parameter of the future directed null geodesics in the null hypersurface $u = \text{const.}$.

Complex null vectors m^a and $\bar{m}^a$ are taken such that

$$m^a = \frac{1}{A f_k} m_o^a \qquad \text{with} \qquad m_o^a = \sqrt{2}\, P_o \frac{\partial}{\partial \zeta} \quad . \tag{1.23}$$

The last null vector n is defined by

$$n^a = \left(\frac{\partial}{\partial u} + U \frac{\partial}{\partial r} \right)^a \qquad \text{with} \qquad U = -\frac{1}{2} \left(A^2 - 2 \dot{r}_o\right) \quad . \tag{1.24}$$

This null tetrad satisfies the usual contraction relations

$$n^a l_a = - m^a \bar{m}_a = 1 \quad ; \tag{1.25}$$

while all other contractions are zero.

Although now the expansion parameter A is a function of u and r, it is convenient to retain the original functional dependance of A(t). Therefore we will use ′ to denote derivatives with respect to t; that is

$$A' = \frac{\partial A(t)}{\partial t} \quad , \tag{1.26}$$

and evaluate coordinate derivatives of A in terms of A′. They are:

$$\frac{\partial A}{\partial u} = A' \frac{\dot{r}_o}{A} \quad , \tag{1.27}$$

and

$$\frac{\partial A}{\partial r} = \frac{A'}{A} \quad . \tag{1.28}$$

We obtain similar expressions evaluating coordinate derivatives of A′:

$$\frac{\partial A'}{\partial u} = \frac{A''}{A} \dot{r}_o \quad , \tag{1.29}$$

and

$$\frac{\partial A'}{\partial r} = \frac{A''}{A} \quad . \tag{1.30}$$

Using the fact that

$$\frac{\partial f_K}{\partial \chi} = \sqrt{1 - K f_K^2} \quad , \tag{1.31}$$

the coordinate derivatives of f_K can be expressed by

$$\frac{\partial f_K}{\partial u} = \frac{\partial f_K}{\partial \chi} \frac{\partial \chi}{\partial u} = \sqrt{1 - K f_K^2} \left(\frac{\dot{r}_o}{A^2} - 1 \right) \quad , \tag{1.32}$$

and

$$\frac{\partial f_K}{\partial r} = \frac{\partial f_K}{\partial \chi} \frac{\partial \chi}{\partial r} = \sqrt{1 - K f_K^2} \frac{1}{A^2} \quad . \tag{1.33}$$

In terms of the G.H.P. notation[4] the only components of the curvature tensor which are different from zero are

$$\Phi_{22} = \frac{K + A'^2 - A A''}{4} \quad , \tag{1.34}$$

$$\Phi_{11} = \frac{K + A'^2 - A A''}{4 A^2} \quad , \qquad \Lambda = \frac{K + A'^2 + A A''}{4 A^2} \quad , \tag{1.35a),(1.35b}$$

$$\Phi_{00} = \frac{K + A'^2 - A A''}{A^4} \quad . \tag{1.36}$$

At this stage it is natural to try to follow some of the techniques that were used in the definitions of future asymptotically flat spacetimes[5]. These techniques include the use of a conformal factor Ω which will bring infinity to a finite distance in the conformally related manifold $\tilde{M}$ with metric

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \quad . \tag{1.37}$$

We have constructed our null tetrad and coordinate system out of a family of null hypersurfaces which define for us a null congruence reaching the asymptotic region under study. Then since the function r is an affine parameter along these null geodesics, we know that taking Ω proportional to 1/r will bring future null infinity to a finite affine distance.

We will study the consequences of taking $\Omega = 1/r$ in the third and following sections. But let us first review two of the most celebrated isotropic cosmological models.

2 TWO EXCEPTIONAL CASES, MINKOWSKI AND DE SITTER

Among the R-W family, two spacetimes deserve special treatment; they are Minkowski and de Sitter space. These are the only[6] metrics that can be expressed in more than one of the forms

$$ds^2 = dt^2 - A(t)^2 \, dL_K^2 \quad ; \qquad (1.1)$$

that is Minkowski can be represented by a line element of this form with K = 0 and K = -1, while de Sitter can be represented by the three possible values of K.

MINKOWSKI SPACE

In Minkowski space we have

$$0 = \frac{K + A'^2 - A A''}{4 A^2} \quad , \qquad (2.1)$$

and

$$0 = \frac{K + A'^2 + A A''}{4 A^2} \quad ; \qquad (2.2)$$

therefore

$$A' = \sqrt{-K} \quad , \qquad (2.3)$$

with solutions

$$A = t \qquad \text{for} \qquad K = -1 \qquad (2.4)$$

and

$$A = \text{constant} = 1 \qquad \text{for} \qquad K = 0 \quad ; \qquad (2.5)$$

where without loss of generality we have chosen the arbitrary constant to be 1. So we can express A(t) by

$$A(t) = -K\,t + (1 + K) \quad . \qquad (2.6)$$

Minkowski space is particularly special among the R-W modes since it is flat; and obviously asymptotically flat. When K = -1 one refers to it as the Milne model[7].

It has been customary to represent the null infinity of Minkowski space by its conformal map into the Einstein universe, which is obtained from the K = 0 form of the metric through the transformation

$$t' = \arctan(t + \chi) + \arctan(t - \chi) \qquad (2.7)$$

$$\chi' = \arctan(t + \chi) - \arctan(t - \chi) \quad , \qquad (2.8)$$

with the coordinate range $-\pi < t' + \chi' < \pi$ and $-\pi < t' - \chi' < \pi$.

The Minkowski line element can then be expressed by

$$ds^2 = dt^2 - d\chi^2 - \chi^2 \, d\Sigma^2$$

$$= \left[\frac{\sec(\frac{t' + \chi'}{2})\sec(\frac{t' - \chi'}{2})}{2}\right]^2 (dt'^2 - d\chi'^2 - \sin^2\chi' \, d\Sigma^2) \quad ; \qquad (2.9)$$

where from the last line one can see the conformal relation between the Minkowski metric and that of the Einstein spacetime. Null infinity agrees with the region where the conformal factor is zero; and as is well known it is formed by two null hypersurfaces in the Einstein space; they correspond to future and past null infinity respectively.

Alternatively one could also study future null infinity of this space without making any reference to the Einstein universe.

We here end our short characterization of Minkowski space, and go on to consider other nontrivial cases.

DE SITTER SPACE

If we only require the trace free part of the Ricci tensor to be zero we obtain

$$0 = K + A'^2 - A A'' \quad ; \qquad (2.10)$$

and due to the contracted Bianchi identities

$$\Lambda = \frac{K + A'^2 + A A''}{4 A^2} = \frac{\mathbb{C}}{2} \quad , \qquad (2.11)$$

where $\mathbb{C}$ is a constant. From these equations we deduce

$$\frac{K + A'^2}{A^2} = \mathbb{C} \quad , \qquad (2.12)$$

and

$$\frac{A A''}{A^2} = \mathbb{C} \quad . \qquad (2.13)$$

The solutions of which, for $\mathbb{C} > 0$, are

$$t = \mathbb{C}^{-1/2} \, ln\left(A \, \mathbb{C}^{1/2} + (A^2 \, \mathbb{C} - K)^{1/2}\right) \quad , \qquad (2.14)$$

or in terms of A(t)

$$A(t) = \begin{cases} \frac{1}{\sqrt{\mathbb{C}}} \sinh(t\,\mathbb{C}^{1/2}) & \text{for} \quad K = -1 \\ \frac{1}{2\sqrt{\mathbb{C}}} e^{(t\,\mathbb{C}^{1/2})} & \text{for} \quad K = 0 \quad ; \\ \frac{1}{\sqrt{\mathbb{C}}} \cosh(t\,\mathbb{C}^{1/2}) & \text{for} \quad K = 1 \end{cases} \tag{2.15}$$

which can also be expressed in one line by

$$A(t) = \frac{1}{\sqrt{\mathbb{C}}} \frac{e^{(t\,\mathbb{C}^{1/2})} + K\,e^{(-t\,\mathbb{C}^{1/2})}}{2} \quad . \tag{2.16}$$

Defining the function

$$t' = \text{arc}\tan(\sinh(t\,\mathbb{C}^{1/2})) \quad , \tag{2.17}$$

the de Sitter metric can be expressed by

$$ds^2 = A(t)^2 \left[dt'^2 - d\chi^2 - \sin(\chi)^2\, d\Sigma^2 \right] \quad . \tag{2.18}$$

We see that this space is conformal to a portion of the static Einstein universe with the range of the time coordinate given by $-\pi/2 < t' < \pi/2$. One then takes the conformal factor Ω to be given by A^{-1}. By doing so one finds that $\mathcal{I}^+$ turns out to be a spacelike hypersurface in the static Einstein universe.

As we have observed these two exceptional examples are easily related to the Einstein space. This suggests we should refer every R-W model to its conformal image in the Einstein universe since "it is a kind of maximal universal conformally flat spacetime"[3].

Instead of carrying out this program, in section 4 we will try to follow closely the techniques used in the study of asymptotically flat spacetimes. Note that de Sitter space is particularly easily related to the Einstein universe because it can be expressed as a line element with $K = 1$.

3 IS THE FRIEDMANN MODEL ASYMPTOTICALLY FLAT?

One of the key features of the definition of GeFAF spacetimes[5] is the flatness condition on the Riemann tensor; which in the regular case is

$$R_{abc}{}^{d} = \Omega\, \hat{R}_{abc}{}^{d} + \delta R_{abc}{}^{d} \quad , \tag{3.1}$$

where $\hat{R}_{abc}{}^{d}$ is a regular tensor at future null infinity and $\delta R_{abc}{}^{d}$ goes to zero faster than Ω.

In our case, for the Robertson-Walker spacetimes, we expect to have a generalization of this behavior, that will look like

$$R_{abc}{}^{d} = h(\Omega)\, \hat{R}_{abc}{}^{d} + \delta R_{abc}{}^{d} \quad ; \tag{3.2}$$

where now $h(\Omega)$ is some function of Ω that might diverge for $\Omega \to 0$.

It is amusing in any case to compare equation (3.1) with equations (1.34-36). Observing equations (3.81),(3.83), (3.85) and (3.86) of reference [5] we see that this comparison will imply the following relations

$$\Phi_{22} = \frac{K + A'^2 - A\,A''}{4} = \omega\, \hat{\Phi}_{22} + \delta\Phi_{22} \tag{3.3}$$

$$\Phi_{11} = \frac{K + A'^2 - A\,A''}{4\,A^2} = \omega^3\, \hat{\Phi}_{11} + \delta\Phi_{11} \tag{3.4}$$

$$\Phi_{00} = \frac{K + A'^2 - A\,A''}{A^4} = \omega^5\, \hat{\Phi}_{00} + \delta\Phi_{00} \tag{3.5}$$

$$\Lambda = \frac{K + A'^2 + A\,A''}{4\,A^2} = \omega^3\, \hat{\Lambda} + \delta\Lambda \quad ; \tag{3.6}$$

where we are using little ω in order to differentiate it from the true conformal factor Ω. It is observed then that this comparison establishes the following proportional expressions

$$(K + A'^2 - A\,A'') \propto \omega \tag{3.7}$$

$$\frac{1}{A} \propto \omega \tag{3.8}$$

$$(K + A'^2 + A\,A'') \propto \omega \quad . \tag{3.9}$$

Therefore one should have

$$(K + A'^2 - A\,A'') \propto \frac{1}{A} \tag{3.10}$$

and

$$(K + A'^2 + A\,A'') \propto \frac{1}{A} \quad ; \tag{3.11}$$

which implies

$$K + A'^2 = \frac{Q}{A} \quad , \tag{3.12}$$

and

$$A\,A'' = \frac{Q'}{A} \tag{3.13}$$

for some constants Q and Q'. Computing the derivative of the first expression, we obtain

$$(K + A'^2)' = 2 A' A'' = - \frac{Q}{A^2} A' \quad ; \qquad (3.14)$$

so the second equation is a consequence of the first and in particular we have that $Q' = -Q/2$.

The first equation is nothing other than the Friedmann equation. It is then somewhat surprising that by playing around with the idea of R-W spacetimes which look future asymptotically flat, we do not find a trivial or extremely complicated new model, but instead the dust Friedmann model.

It can easily be seen that in the Friedmann model the curvature components are given by

$$\Phi_{22} = \frac{K + A'^2 - A A''}{4} = \frac{3Q}{8 A} \qquad (3.15)$$

$$\Phi_{11} = \frac{K + A'^2 - A A''}{4 A^2} = \frac{3Q}{8 A^3} \qquad (3.16)$$

$$\Phi_{00} = \frac{K + A'^2 - A A''}{A^4} = \frac{3Q}{2 A^5} \qquad (3.17)$$

$$\Lambda = \frac{K + A'^2 + A A''}{4 A^2} = \frac{Q}{8 A^3} \quad . \qquad (3.18)$$

We will now relate these expressions with the family of conformal factors Ω which are proportional to the inverse of the affine null distance, that is

$$\Omega \propto \frac{1}{r} \quad . \qquad (3.19)$$

It will be useful to note that

$$dA = \frac{\partial A}{\partial u} du + \frac{\partial A}{\partial r} dr = A' \frac{\dot{r}_o}{A} du + \frac{A'}{A} dr = \frac{A'}{A} d(r_o + r) \quad ; \qquad (3.20)$$

since then we can express

$$r + r_o = \int \frac{A\, dA}{A'} = \int \frac{A \quad dA}{\sqrt{\frac{Q}{A} - K}} \qquad (3.21)$$

$$= \begin{cases} \sqrt{\frac{Q}{A} + 1}\; \left(\frac{A^2}{2} - \frac{3QA}{4}\right) + \frac{3Q^2}{8}\, ln\left(\frac{\sqrt{\frac{Q}{A} + 1} + 1}{\sqrt{\frac{Q}{A} + 1} - 1} \right) & \text{for } K = -1 \\ \frac{2 A^{5/2}}{5 \sqrt{Q}} & \text{for } K = 0 \end{cases} .$$

It is observed from the last expression that in the limit $r \longrightarrow \infty$ one also has $A \longrightarrow \infty$; and the leading order of this equation gives:

$$r \approx \frac{A^2}{2} \qquad \text{for } K = -1 \qquad (3.22)$$

and

$$r \approx \frac{2 A^{5/2}}{5 \sqrt{Q}} \qquad \text{for } K = 0 \quad . \qquad (3.23)$$

Let us next study the two cases corresponding to the two possible values of the constant K.

CASE K = -1

In this case we can take the conformal factor Ω to be

$$\Omega = \frac{1}{A^2} \quad ; \qquad (3.24)$$

that is Ω is asymptotically given in terms of the affine distance r by

$$\Omega \approx \frac{1}{2 r} \quad . \qquad (3.25)$$

Let us see what the conformal metric looks like at future null infinity. From equation (1.16) we obtain

$$d\tilde{s}^2 = \Omega^2 ds^2 = A^{-4} (2 \dot{r}_o - A^2) du^2 + 2 A^{-4} du\, dr - A^{-2} f_k^2 d\Sigma^2 \quad . \qquad (3.26)$$

The first two terms of this equation present no problem; however we should study in detail the asymptotic behavior of the third term. Using equation (1.14) asymptotically one obtains

$$\frac{\partial r}{\partial \chi} = \left.\frac{\partial r}{\partial \chi}\right|_u = A^2 \approx 2 r \quad ; \qquad (3.27)$$

from which it is deduced that for very large r

$$r \approx e^{2\chi} \quad . \tag{3.28}$$

Then since for very large χ one has that $f_k^2 \approx e^{2\chi}/4$, it is deduced that asymptotically

$$A^{-2} f_k^2 \approx \frac{r}{4\, A^2} \approx \frac{1}{8} \quad . \tag{3.29}$$

So the conformal metric at future null infinity is given by

$$d\tilde{s}^2 \Big|_{\Omega=0} = - du\, d\Omega \; - \frac{1}{8}\, d\Sigma^2 \quad , \tag{3.30}$$

which is a non-degenerate regular metric. This means that the present choice of Ω does the job of bringing infinity to a finite distance and provides us with a well behaved conformal metric at future null infinity in this case.

It is then reasonable to ask under what conditions on A(t) will the choice of Ω as the inverse of the affine distance along null geodesics have this property. We will study this in a later section.

The Riemann tensor can now be expressed in terms of Ω, obtaining

$$\Phi_{22} = -\frac{3Q}{8}\ \Omega^{1/2} \quad , \tag{3.31}$$

$$\Phi_{11} = \frac{3Q}{8}\ \Omega^{3/2} \quad , \qquad \Lambda = \frac{Q}{8}\ \Omega^{3/2} \tag{3.32a),(3.32b}$$

$$\Phi_{00} = \frac{3Q}{2}\ \Omega^{5/2} \quad . \tag{3.33}$$

Equation (3.2) implies asymptotic behavior of the form

$$\Phi_{22} = h(\Omega)\ \hat{\Phi}_{22} + \delta\Phi_{22} \quad , \tag{3.34}$$

$$\Phi_{11} = h(\Omega)\ \Omega^2\ \hat{\Phi}_{11} + \delta\Phi_{11} \quad , \qquad \Lambda = h(\Omega)\ \Omega^2\ \hat{\Lambda} + \delta\Lambda \tag{3.35a),(3.35b}$$

$$\Phi_{00} = h(\Omega)\ \Omega^4\ \hat{\Phi}_{00} + \delta\Phi_{00} \quad ; \tag{3.36}$$

where it can be seen from the last equation that

$$h(\Omega) = \Omega^{5/2-8/2} = \Omega^{-3/2} \quad . \tag{3.37}$$

This means that the Riemann tensor diverges as $\Omega^{-3/2}$ as one approaches null infinity. And to avoid any confusion it is convenient to remark here that the Friedmann K = -1 model is not asymptotically flat, and the scalar ω

used above does not coincide with the conformal factor Ω which is used in the construction of the conformal metric.

Let us now consider the other expanding Friedmann model.

CASE K = 0

We try now to repeat the same construction as before by taking the conformal factor Ω proportional to the inverse of the affine distance; more precisely we take

$$\Omega = \frac{1}{A^{5/2}} \quad . \tag{3.38}$$

The conformal metric is given in this case by

$$d\tilde{s}^2 = \Omega^2\, ds^2 = A^{-5}\,(2\,\dot{r}_o - A^2)\, du^2 + 2\, A^{-5}\, du\, dr \; - A^{-3}\, f_k^2\, d\Sigma^2 \, . \tag{3.39}$$

Studying the asymptotic behavior of $A^{-3}\, f_k^2$ in a similar fashion, we obtain

$$\frac{\partial\, r}{\partial\, \chi} = \frac{\partial\, r}{\partial\, \chi}\Big|_u = A^2 \approx \left(\frac{5\, Q^{1/2}\, r}{2} \right)^{4/5} \quad ; \tag{3.40}$$

which means that asymptotically we have

$$\chi \approx \left(\frac{2}{5\, Q^{1/2}} \right)^{4/5} 5\, r^{1/5} \quad . \tag{3.41}$$

Then for large r one obtains

$$A^{-3}\, f_k^2 = A^{-3}\, \chi^2 \approx \left(\frac{5\, Q^{1/2}\, r}{2} \right)^{-6/5} \left(\frac{2}{5\, Q^{1/2}} \right)^{8/5} 5^2\, r^{2/5}$$

$$\approx \left(\frac{5\, Q^{1/2}}{2} \right)^{-14/5} 5^2\, r^{-4/5} \quad ; \tag{3.42}$$

so the conformal metric obtained in this way turns out to be degenerate at future null infinity.

Note that traditionally the Friedmann models have been studied through their conformal representation in the Einstein universe, where this bad behavior of the conformal metric is absent. This reinforces the initiative of relating every R-W model conformally to the Einstein universe. However we should also note that in this last example the bad behavior of the metric at future null infinity is associated with the choice of the conformal factor as the inverse of the affine distance, since it produces complications with the asymptotic behavior of the angular part of the metric; and so it is clear that these complications will disappear if we choose the conformal factor as the inverse of the luminosity distance, although probably at the expense of introducing other complications.

It is therefore clear that we need a systematic study of future null infinity in the R-W models. We do this in the next section.

4 THE GENERAL CASE

We are therefore confronted with the question of how to choose an appropriate conformal factor that will make the conformal metric regular (meaning at least continuous) at future null infinity. Of course this is associated with the behavior of the function A(t) which is the only nontrivial input in the metric.

Let us be more precise in our discussion. We will define the manifold $\mathcal{I}^+$ to be future null infinity of a non-contracting C^∞ Robertson-Walker spacetime (M, g_{ab}) if there exists a manifold $\tilde{M}$ with boundary $\mathcal{I}^+$, metric $\tilde{g}_{ab}$ and a function Ω on $\tilde{M}$ such that a neighborhood of $\mathcal{I}^+$ in $\tilde{M}$ is diffeomorphic to a neighborhood of $\mathcal{I}^+$ in the manifold $M \cup \mathcal{I}^+$ and

a) on M: Ω is C^∞, $\Omega > 0$ and $\tilde{g}_{ab} = \Omega^2 g_{ab}$;

b) at $\mathcal{I}^+$: $\Omega = 0$, Ω is C^0; at every point of $\mathcal{I}^+$ there end future directed null geodesics of $\tilde{M}$, and $\tilde{g}_{ab}$ is non-degenerate.

Note that we are implicitly requiring the conformal metric to be C^1 at scri. Also it should be observed that nothing is said about the differentiability properties of Ω at scri, that is we only require it to be continuous.

At this point it is important to recall that if v is an affine parameter along the null geodesics contained in the null hypersurfaces u = constant, but with respect to the conformal metric $\tilde{g}_{ab}$, then it can be related to r by the equation

$$\frac{\partial v}{\partial r} = -\Omega^2 \quad . \qquad (4.1)$$

Since v is a natural coordinate of $\tilde{M}$, it is then clear how the choice of Ω determines the differentiable structure of the conformal manifold.

Due to the present symmetries, the scalar Ω is taken to be a function depending only on u and r; so one can define v to be

$$v = -\int_{r_1}^{r} \Omega(u,r')^2\, dr' + v_0(u) \quad , \qquad (4.2)$$

with $v = 0$ at scri. In this way we will have

$$dv = -\Omega^2\, dr + \left(\dot{v}_0 - \int_{r_1}^{r} 2\,\Omega\,\dot{\Omega}\, dr' \right) du \quad ; \qquad (4.3)$$

where as before a dot means partial derivative with respect to the coordinate u; for example

$$\dot{\Omega} = \left.\frac{\partial \Omega}{\partial u}\right|_{r=\text{const.}} \quad . \qquad (4.4)$$

Let us note that the existence of the coordinate v in the conformal manifold $\tilde{M}$ requires that

$$\lim_{r \longrightarrow \infty} \int_{r_1}^{r} \Omega(u,r')^2\, dr' \qquad \text{must exist.} \qquad (4.5)$$

The conformal metric can then be expressed by

$$d\tilde{s}^2 = \Omega^2\, ds^2 = \Omega^2\,(2\,\dot{r}_0 - A^2)\, du^2 + 2\,\Omega^2\, du\, dr - \Omega^2 A^2 f_k^2\, d\Sigma^2 \qquad (4.6)$$

$$= \left(\Omega^2\,(2\,\dot{r}_0 - A^2) - 2 \int_{r_1}^{r} 2\,\Omega\,\dot{\Omega}\, dr' + 2\,\dot{v}_0 \right) du^2 - 2\, du\, dv - \Omega^2 A^2 f_k^2\, d\Sigma^2 \; .$$

It is crucial to notice that this expression for the conformal metric is an invariant one, that is, it has been geometrically defined; and that du, dv and $d\Sigma^2$ have clear invariant meaning. Therefore the three terms appearing in the above expression are geometrically well defined and it makes sense to refer to the asymptotic behavior of each of them. From the last term, in particular, we observe that the algebraic condition that will make this term regular at $\mathcal{I}^+$ is $\Omega^2 A^2 f_K^2 \propto$ constant.

One may ask: is it possible then that by taking Ω to be the inverse of the luminosity distance r_L the conformal metric is regular at $\mathcal{I}^+$?

The luminosity distance is a scalar that satisfies

$$\frac{\partial r_L}{\partial r} = -\rho\, r_L \quad . \qquad (4.7)$$

The natural choice for r_L is

$$r_L = A\, f_K \quad . \qquad (4.8)$$

So by taking the conformal factor Ω to be the inverse of the luminosity distance we obtain

$$\Omega = \frac{1}{r_L} = \frac{1}{A\, f_K} \quad , \qquad (4.9)$$

and

$$d\tilde{s}^2 = \left[\frac{1}{f_K^2} \left(\frac{2\,\dot{r}_o}{A^2} - 1\right) - 2 \int_{r_1}^{r} 2\,\Omega\,\dot{\Omega}\,dr' + 2\,\dot{v}_o \right] du^2 - 2\,du\,dv - d\Sigma^2. \quad (4.10)$$

It is then clear that for this metric to be regular at scri we should have the expression

$$W = \int_{r_1}^{r} \Omega\,\dot{\Omega}\,dr' \quad (4.11)$$

regular at scri.

The last expression can be further transformed by noting that

$$\dot{\Omega} = \frac{1}{A\,f_K^2}\sqrt{1 - K f_K^2} - \frac{\dot{r}_o}{A\,f_K}\left[\frac{\sqrt{1 - K f_K^2} + A'\,f_K}{A^2\,f_K}\right]$$

$$= \frac{1}{A\,f_K^2}\sqrt{1 - K f_K^2} + \frac{\dot{r}_o}{A\,f_K}\,\rho = \Omega\,\frac{\sqrt{1 - K f_K^2}}{f_K} + \Omega\,\dot{r}_o\,\rho$$

$$= \Omega\left[\frac{\sqrt{1 - K f_K^2}}{f_K} + \dot{r}_o\,\rho\right] \quad ; \quad (4.12)$$

which implies

$$W = \int_{r_1}^{r} \Omega^2 \left[\frac{\sqrt{1 - K f_K^2}}{f_K} + \dot{r}_o\,\rho\right] dr' \quad . \quad (4.13)$$

Then using the relation

$$\frac{\partial\Omega}{\partial r} = -\frac{1}{r_L^2}\frac{\partial r_L}{\partial r} = \frac{1}{r_L}\,\rho = \Omega\,\rho \quad , \quad (4.14)$$

we can express W by

$$W = \int_{r_1}^{r} \left[\Omega^2\,\frac{\sqrt{1 - K f_K^2}}{f_K} + \dot{r}_o\,\Omega\,\frac{\partial\Omega}{\partial r}\right] dr'$$

$$= \int_{r_1}^{r} \left[\Omega^2\,\frac{\sqrt{1 - K f_K^2}}{f_K}\right] dr' + \dot{r}_o\,\frac{\Omega^2}{2}\Big|_{r_1}^{r} \quad . \quad (4.15)$$

This means that we only need to consider the first term in the asymptotic region in order to determine the regularity of the conformal metric at $\mathcal{I}^+$.

At this stage it is clear that we need a detailed study of the asymptotic behavior of the functions A and χ as r goes to infinity. We have explicitly mentioned that we consider $A' \geq 0$. Next let us observe that the optical scalar ρ is given by

$$\rho = -\frac{\sqrt{1 - K f_K^2} + A'\,f_K}{A^2\,f_K} \quad , \quad (4.16)$$

and therefore satisfies $\rho < 0$; which implies that

$$\frac{\partial r_L}{\partial r} > 0 \quad . \quad (4.17)$$

But furthermore, by taking the conformal factor $\Omega = 1/r_L$, we are implicitly assuming that in the limit $r \longrightarrow \infty$, we should have $\Omega \longrightarrow 0$, or equivalently $r_L \longrightarrow \infty$. Can we have $\lim_{r \to \infty} r_L < \infty$? We will see later that under the present conditions $r_L \to \infty$ as one approaches $\mathcal{I}^+$.

In order to obtain information on the asymptotic behavior of the coordinate χ along the null geodesics contained in the null hypersurfaces u = const., it is convenient to study the equation

$$du = \frac{dt}{A(t)} - d\chi \quad , \quad (1.10)$$

since when u = const. we have

$$\chi = \int_{t_o}^{t} \frac{dt'}{A(t')} \quad . \quad (4.18)$$

Defining χ_∞ by

$$\chi_\infty = \int_{t_0}^{\infty} \frac{dt'}{A(t')} \quad , \tag{4.19}$$

we want to know whether χ_∞ is bounded (finite) or unbounded (infinite). We can then classify the R-W spacetimes in two classes such that:

Class B corresponds to χ_∞ bounded and

Class U corresponds to χ_∞ unbounded.

Case B coincides with the appearance for each observer of an event horizon; since for example an observer traveling along the geodesic $\chi = \text{const.} > \chi_\infty$, $\theta = \text{const.}$, and $\varphi = \text{const.}$ will never be able to get information from the events with $t > {}_0t$ and $\chi < \chi_\infty$. The case U refers to those models where there is no event horizon.

CASE B:

Since in this case $f_K(\chi_\infty)$ is finite we deduce that a sufficient condition for W to be regular at $\mathcal{J}^+$ is that

$$\lim_{r \to \infty} \int_{r_1}^{r} \Omega^2 \, dr' \quad \textbf{exists} \; ; \tag{4.20}$$

which agrees with eq. (4.5), and it is also equivalent in this case to the condition that

$$\int_{r_1}^{r} \frac{1}{A^2} \, dr' \quad \textbf{exists for } r \longrightarrow \infty \quad . \tag{4.21}$$

CASE U:

Since χ_∞ is unbounded, we disregard the possibility K = 1, and we consider the cases K = -1 and K = 0 separately.

CASE U, K = -1:

We now also require that the expression

$$\int_{r_1}^{r} \Omega^2 \, dr' = \int_{r_1}^{r} \frac{1}{A^2 \, f_K^2} \, dr' \quad \textbf{exists for } r \longrightarrow \infty \quad ; \tag{4.22}$$

which also coincides with equation (4.5).

CASE U, K = 0:

In this situation it is required that

$$\int_{r_1}^{r} \frac{\Omega^2}{f_K} \, dr' = \int_{r_1}^{r} \frac{1}{A^2 \, f_K^3} \, dr' \quad \textbf{exists for } r \longrightarrow \infty \quad ; \tag{4.23}$$

which is weaker than the condition for the existence of the function v.

It is clear in all cases considered that $r_L \longrightarrow \infty$ as $r \longrightarrow \infty$, since at least A or f_K is unbounded in this limit, and both satisfy $A > 0$ and $f_K > 0$.

We have that the condition of regularity of W in the cases considered above is satisfied if the function v is well defined. Let us now observe that in all cases v is well defined.

In case B we have that the integral (4.19) exists; while for v to be well defined we should have that

$$\int_{r_1}^{r} \frac{1}{A^2 \, f_K^2} \, dr' \quad \textbf{exists for } r \longrightarrow \infty; \tag{4.24}$$

which in this case is equivalent to the condition that

$$\int_{r_1}^{r} \frac{1}{A^2} \, dr' \quad \textbf{exists for } r \longrightarrow \infty \quad . \tag{4.25}$$

This integral is easily shown to be equivalent to (4,19) since

$$\int_{r_1}^{\infty} \frac{1}{A^2} \, dr' = \int_{t_1}^{\infty} \frac{1}{A^2} \frac{\partial r'}{\partial t}\Big|_u \, dt = \int_{t_1}^{\infty} \frac{1}{A} \, dt \quad . \tag{4.26}$$

Therefore v is well defined in case B.

In case U one has

$$\int_{t_1}^{\infty} \frac{1}{A} \, dt = \infty \quad , \tag{4.27}$$

and one would like to have

$$\int_{r_1}^{\infty} \frac{1}{A^2 \, f_K^2} \, dr' < \infty \quad . \tag{4.28}$$

But let us note that since K can only have the values -1 and 0, one has that

$$f(\chi) \geq \chi \quad ; \tag{4.29}$$

from which one concludes that

$$\lim_{r \to \infty} \int_{r_1}^{r} \frac{1}{A^2 f_K^2}\, dr' = \lim_{t \to \infty} \int_{t_1}^{t} \frac{1}{A^2 f_K^{\,2}} \left.\frac{\partial r}{\partial t'}\right|_u dt'$$

$$= \lim_{t \to \infty} \int_{t_1}^{t} \frac{1}{A\, f_K^{\,2}}\, dt' = \lim_{\chi \to \infty} \int_{\chi_1}^{\chi} \frac{1}{A\, f_K^{\,2}} \left.\frac{\partial t'}{\partial \chi'}\right|_u d\chi'$$

$$= \lim_{\chi \to \infty} \int_{\chi_1}^{\chi} \frac{1}{f_K^{\,2}}\, d\chi' \le \lim_{\chi \to \infty} \int_{\chi_1}^{\chi} \frac{1}{\chi'^{\,2}}\, d\chi' < \infty \quad . \tag{4.30}$$

Therefore we see that also in case U the function v is well defined.

Then since the regularity conditions of W are satisfied if v exists, we conclude that the choice

$$\Omega = \frac{1}{A\, f_K} \tag{4.31}$$

provides us with a construction of $\mathcal{J}^+$ for all non-contracting R-W models.

It now remains to be seen when $d\Omega$ is regular at $\mathcal{J}^+$, and what type of hypersurface is $\mathcal{J}^+$, namely when it is timelike, null or spacelike.

In terms of the coordinates $(u, v, \zeta, \bar{\zeta})$ of the conformal manifold $\tilde{M}$ the differential of Ω is given by

$$d\Omega = \left[\frac{\sqrt{1 - K f_K^2}}{A\, f_K^2} + \Omega_v \left(2\,W - \dot{v}_0 - \frac{\dot{r}_0}{A^2\, f_K^2} \right) \right] du + \Omega_v\, dv \quad , \tag{4.32}$$

where

$$\Omega_v = \frac{A'\, f_K + \sqrt{1 - K f_K^2}}{A} \quad . \tag{4.33}$$

Then $d\Omega$ will be regular at $\mathcal{J}^+$ if and only if Ω_v is regular at $\mathcal{J}^+$; furthermore at $\mathcal{J}^+$ we have

$$d\Omega\Big|_{\mathcal{J}^+} = \Omega_v\Big|_{\mathcal{J}^+} dv \quad . \tag{4.34}$$

If $d\Omega$ is well behaved at $\mathcal{J}^+$ the character of the hypersurface $\mathcal{J}^+$ can be studied from the expression

$$\tilde{g}(d\Omega, d\Omega) = \left(\frac{A'}{A} \right)^2 - \left(\frac{\sqrt{1 - K f_K^2}}{A\, f_K} \right)^2 \quad ; \tag{4.35}$$

where one can see that since it is the difference of two positive terms, in principle it could be positive, negative or zero.

In order to have a concrete picture of the different possible behavior let us consider the following examples:

a) constant $\quad A = 1 \quad , \quad \frac{A'}{A} = 0$ (4.36)

b) very slow $\quad A = \frac{t}{t+1} \quad , \quad \frac{A'}{A} = \frac{1}{t\,(t+1)}$ (4.37)

c) slow $\quad A = \ln t \quad , \quad \frac{A'}{A} = \frac{1}{t\, \ln(t)}$ (4.38)

d) power law $\quad A = t^n \, , \; n>0 \quad , \quad \frac{A'}{A} = \frac{n}{t}$ (4.39)

e) fast $\quad A = \cosh t \quad , \quad \frac{A'}{A} = \tanh t$ (4.40)

The first three cases belong to class U. The examples d) with $n \le 1$ are also of class U, while when $n > 1$ they belong to class B. Finally the last example is of class B. So we see that $A = t$ is a boundary case among our examples which divide them into classes U and B.

Let us now state the value of Ω_v, and $\tilde{g}(d\Omega, d\Omega)$ at $\mathcal{J}^+$:

Example	Class	K	$\Omega_v\big\rvert_{\mathcal{J}^+}$	$\tilde{g}(d\Omega,d\Omega)\big\rvert_{\mathcal{J}^+}$	remarks
a)	U	-1	∞	- - -	
		0	1	0	Minkowski space
b)	U	-1	∞	- - -	
		0	1	0	
c)	U	-1	∞	- - -	
		0	1	0	
d, n<1)	U	-1	∞	- - -	$A = t^{2/3}$ is a Friedmann model
		0	0	- - -	
d, n=1)	U	-1	2	0	Minkowski space
		0	1	0	
d, n>1)	B	-1	0	- - -	
		0	0	- - -	
		1	0	- - -	
e)	B	-1	$\sinh \chi_\infty$	1	
		0	χ_∞	1	
		1	$\sin \chi_\infty$	1	de Sitter space

TABLE 1: Values of Ω_v and $\tilde{g}(d\Omega, d\Omega)$ at scri.

It is observed that although we are forced to take the conformal factor as the inverse of the luminosity distance, in order to obtain a regular metric at scri, this conformal factor is not suitable for the study of the nature of scri, since we have seen that $d\Omega$ is sometimes zero or not defined at scri.

However we can choose the coordinate v such that $v = 0$ at $\mathcal{J}^+$ by appropriately defining $v_o(u)$; and by doing so we can use dv for the study of the nature of $\mathcal{J}^+$, since obviously v is a regular function at scri. In fact one finds that

$$\tilde{g}(dv,dv) = \frac{1}{f_K^2} \quad ; \tag{4.41}$$

so scri is non-timelike, and furthermore in case U it is null and in case B spacelike.

We have just proved the following theorem,

THEOREM:

The future null infinity of non-contracting Robertson-Walker models is null or spacelike according to the absence or presence of cosmic event horizons respectively.

This completes the statements appearing in the literature[3] which claimed, based on intuitive arguments, that when $\mathcal{J}^+$ is null one expects no event horizons, and when $\mathcal{J}^+$ is spacelike each observer will be assigned an event horizon. It is important to emphasize that although the same intuitive arguments were used for the nature of past null infinity, which normally coincides with the initial cosmic singularity, the analog theorem is not true. For example the K = -1 de Sitter model would violate it, since past null infinity of de Sitter space (which is spacelike) does not coincide with the initial null cone cosmic singularity of the K = -1 model (which does not possess particle horizons).

It was asked previously under what circumstances would the choice of the conformal factor as the inverse of the affine distance provide us with a construction of $\mathcal{J}^+$. To see this let us note that

$$\frac{\partial r_L}{\partial r} = \Omega_v \quad . \tag{4.42}$$

So, since we know that the choice of the inverse of the luminosity distance as the conformal factor provides us with a construction of $\mathcal{J}^+$, we are sure that the affine distance will do the job if

$$0 < \Omega_v \Big|_{\mathcal{J}^+} < \infty \quad . \tag{4.43}$$

It is clear then from the above table why in the case K = 0 of the Friedmann model it is inappropriate to take $\Omega = 1/r$.

5 FINAL COMMENTS

We will proceed here with a quick recapitulation of the topics we have covered, and take the opportunity to insert some comments.

In order to put our work into perspective we have reviewed in sections 2 and 3 what could be considered as the most significant Robertson-Walker models, namely: Minkowski space, de Sitter space, and the open dust Friedmann models. We have approached the first two spaces through their standard conformal representations in the static Einstein universe.

In section 3 we have observed that a naive comparison of the asymptotic behavior of a regular asymptotically flat spacetime with the Riemann tensor of a R-W model leads us to the dust Friedmann models. Actually, as we have seen, these models are not asymptotically flat. This might seem a little curious, since for example in the K = -1 dust Friedmann model the scalar A(t) asymptotically approaches the functional form $A \approx t$ for large t. And since Minkowski space can be represented as a R-W model with K = -1 and A = t, one would be tempted to conclude that the K = -1 dust Friedmann model is asymptotically flat at future null infinity. Instead we have seen that this is not the case; in fact the curvature tensor of this model diverges at future null infinity. One might however introduce the notion of timelike infinity and argue that the K = -1 dust Friedmann model is asymptotically flat in that region; we instead for the moment concentrate our attention on future null infinity.

We have also studied the use of the conformal factor as the inverse of the affine distance and found the following: while in the case K = -1 for the dust Friedmann model it provides us with a well behaved metric at future null infinity, in the case K = 0 it produces a degenerate metric at scri.

From the contents of section 4 we have proved that the choice of the conformal factor as the inverse of the luminosity distance permits the construction of $\mathcal{J}^+$ for all non-contracting R-W models. We have also seen that although this choice is necessary if one wants to obtain a regular metric at scri, one can not always use the gradient of this conformal factor for the study of the nature of $\mathcal{J}^+$, since in some cases $d\Omega$ is zero or not defined at scri. In any case it was proved that in the non-contracting R-W models $\mathcal{J}^+$ is not timelike, and more precisely the theorem of the last section relates its nature to the appearance or absence of cosmic event horizons.

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