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THE COMPLEX STRUCTURES ON THE COADJOINT ORBIT SPACES OF Diff(S^1) AND ON BERS' UNIVERSAL TEICHMÜLLER SPACE ARE COMPATIBLE*

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ABSTRACT

Precisely two coadjoint orbit spaces of the group of string reparametrizations carry in a natural way the structure of infinite dimenisonal, holomorphically homogeneous complex manifolds. These are M_1 = Diff (S^1)/Rot (S^1) and M_2 = Diff (S^1)/Möb (S^1). M_2 can be naturally considered as (embedded in) the classical universal Teichmüller space $T(\Delta)$, simply by noting that a diffeomorphism of S^1 is a quasi-symmetric homeomorphism. $T(\Delta)$ is itself a homomorphically homogeneous complex Banach manifold. We prove that the inclusion of M_2 in $T(\Delta)$ is complex analytic. Every Teichmüller space of finite or infinite dimension is contained canonically and holomorphically in $T(\Delta)$. Our result thus appears to connect the loop space approach to bosonic string theory with the sum-over moduli (Polyakov path integral) approach.

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1. INTRODUCTION

The group Diff (S^1) occurs in string theory as the space of reparametrizations of a closed string. Two coadjoint orbit spaces of Diff (S^1), namely, M_1 = Diff $(S^1)/S^1$ and M_2 = Diff $(S^1)/SL(2,\mathbb{R})$, have occured in the physics literature as critically important because precisely these two carry the structure of infinite dimensional, holomorphically homogeneous, complex (Kähler) manifolds. The complex structures on the M_i, $i = 1,2$, are obtained by placing a natural physics-motivated almost complex structure on the appropriate spaces of real vector fields on S^1.

Now, considering diffeomorphisms of S^1 as quasi-symmetric homeomorphisms one can naturally identify M_2 as (embedded in) the classical universal Teichmüller space $T(\Delta)$. $T(\Delta)$ is a holomorphically homogeneous complex Banach domain from the famous Ahlfors-Bers theory of the Teichmüller spaces. Our chief result is that this inclusion of M_2 into $T(\Delta)$ is *complex analytic*. In fact, M_2 is one leaf of a holomorphic foliation of $T(\Delta)$. Also, since M_1 is a holomorphic disc bundle over M_2, it seems to us that this naturally and directly connects the string reparametrization complex manifolds with the complex analytic moduli of Riemann surfaces. It appears to have been an important question (see, for example, Bowick [2], Bowick and Rajeev [3]) to relate these reparametrization spaces with the spaces of moduli of Riemann surfaces because that would connect the loop space ("geometrical quantization") approach to string theory with the path integral ("sum over moduli") approach. Indeed, $T(\Delta)$ contains canonically within itself as complex submanifolds all the Teichmüller spaces of arbitrary Riemann surfaces or Fuchsian groups. If, therefore, strings are reparametrized using the more general quasi-symmetric homeomorphisms of the circle (rather than only by smooth diffeomorphisms), then the corresponding $SL(2,\mathbb{R})$ orbit space *is* the universal Teichmüller space of Riemann surfaces.

Our method of proof is to show that the almost complex structure obtained by the physicists (Bowick and Rajeev [3], [4]; Bowick and Lahiri [5]) on real vector fields on S^1 modulo the Möbius vector fields coincides with the almost complex structure of $T(\Delta)$ at the origin. The holomorphic homogeneity of both M_2 and $T(\Delta)$ under the action of (right-) translation then implies that the complex structures are compatible everywhere.

It is rather easy to consider the subsets of quasi-symmetric homeomorphisms (or, those infinitesimally real vector fields on S^1), which correspond to the Teichmüller space $T(G)$ embedded in $T(\Delta)$. The translation-invariant Kähler structure studied by the physicists should therefore give rise to a modular invariant Kähler metric on each $T(G)$, (for arbitrary Fuchsian group G). We will report on these matters in the future.

2. THE COMPLEX STRUCTURE OF Diff$(S^1)/S^1$ AND Diff$(S^1)/SL(2,\mathbb{R})$

Let M_1 and M_2 denote respectively these two orbit spaces. We will think of them as right coset spaces. Using $S^1 \cong$ Rot (S^1) and $SL(2,\mathbb{R}) = $ Möb (S^1) to normalize a given diffeomorphism (by following the given diffeomorphism by a normalizing one) we can identify M_1 (and M_2) as those diffeomorphisms of S^1 that fix one (respectively, three) points of S^1.

The Lie algebra of the Fréchet Lie group Diff (S^1) is the algebra of smooth real vector fields on S^1 (see Goodman [7]). The complexification of this Lie algebra is the Virasoro algebra generated by the $L_n = e^{in\theta}\frac{\partial}{\partial\theta} = iz^{n+1}\frac{\partial}{\partial z}$, $n \in \mathbb{Z}$. (Here $z = e^{i\theta}$.) A tangent vector to M_1 at its origin is a linear combination:

$$\vartheta = \sum_{m\neq 0} \vartheta_m L_m, \quad \overline{\vartheta}_m = \vartheta_{-m}, \tag{1}$$

where $\vartheta = u(\theta)\frac{\partial}{\partial\theta}$ is the corresponding smooth real vector field on the circle and the ϑ_m are the Fourier coefficients of $u(\theta)$ (The ϑ_k decay faster than any negative power of k since $u(\theta)$ is C^∞. See Katznelson [8], p.24.) For M_2, at its origin, a tangent vector will be of the form

$$\vartheta = \sum_{m\neq -1,0,1} \vartheta_m L_m, \quad \overline{\vartheta}_m = \vartheta_{-m} \tag{2}$$

Here one loses the coefficients $\vartheta_{-1}, \vartheta_0, \vartheta_1$ because an infinitesimal Möbius transformation of Δ allows one to normalise precisely these coefficients. One may also check that the Lie algebra generated by L_{-1}, L_0, L_1 is preciesly (the complexification of) $s\ell(2,\mathbb{R})$, as would be expected.

The *almost complex structure $\tilde{J}$ at the origin* of M_i is then defined (in both cases) by

$$\tilde{J}\vartheta = \sum_m -i\ sgn(m)\vartheta_m L_m. \tag{3}$$

See Bowick and Rajeev [4] and Bowick and Lahiri [5]. The formula (3) is, of course, the classic formula known in the theory of Fourier series as "conjugation". See, for example, Katznelson [8] Chapter III. One now follows [4], [5] to define the almost complex structure everywhere on these (right-) coset spaces M_i by right-translation invariance. As explained in [4] using the rather obvious involutivity of the (1,0) vector fields this $\tilde{J}$ is seen to be integrable and the right translations by elements of Diff (S^1) act as biholomorphic automorphisms on M_1 and M_2. (It is possible to get farily explicit holomorphic coordinates on the M_i as explained by Bruno Zumino in his July 1988 lectures at the ICTP.)

Remark We are purposely using right translations and right-invariant objects in order to finally coincide with the usual version of the theory of Teichmüller spaces. It is of course possible, as indicated in the last remark of the next section, to modify the definition of the Teichmüller spaces so that the left-invariant theory of the M_i works compatibly.

3. BERS' UNIVERSAL TEICHMÜLLER SPACE $T(\Delta)$:

Let Homeo $_{q.s.}(S^1)$ denote the group of quasi-symmetric homeomorphisms of the unit circle. These are the ones which allow some quasiconformal extension into the unit disc Δ bounded by S^1. By a well-known characterisation due to Ahlfors (see [1] or [11]) these are the homeomorphisms that alter cross ratios of points on S^1 by a bounded ratio. Now, Bers' universal Teichmüller space is

$$T(\Delta) = \text{Homeo}_{q.s.}(S^1)/SL(2,\mathbb{R}) \tag{4}$$

Again, $SL(2,\mathbb{R}) = $ Möb (Δ) can be thought of as normalizing a homeomorphism by following it by a Möbius transformation so that the composition fixes $+1, -1$ and i on S^1.

The complex analytic structure of $T(\Delta)$ comes by thinking of it as equivalence classes of proper Beltrami coefficients on Δ. These Beltrami coefficients comprise the unit ball $L^\infty(\Delta)_1$ of the complex Banach space $L^\infty(\Delta)$. Given any $\mu \in L^\infty(\Delta)_1$ one solves the Beltrami equation

$$w_{\bar{z}} = \mu w_z \tag{5}$$

to get a quasi conformal self-homeomorphism $w = w_\mu$ of Δ. The boundary values of w_μ on S^1 (which always exist) is the quasi-symmetric homeomorphism of S^1 representing the equivalence class $[\mu]$ in $T(\Delta)$. Thus,

$$T(\Delta) = L^\infty(\Delta)_1/\sim \tag{6}$$

where $\sim$ is the equivalence relation saying $\mu \sim \nu$ if and only if w_μ and w_ν (normalized as explained by post-composition with Möbius transformations) have identical boundary values on S^1.

Remark The way to get w_μ given μ in Δ is to first extend μ as a Beltrami coefficient to the exterior Δ^* of Δ by reflection (inversion in S^1). Apply then the Ahlfors-Bers theorem (see Nag [11], p.34) to this extended μ on the whole plane. The normalized solution of the Beltrami equation for this extended μ preserves each of Δ and Δ^*.

Bers proved that $T(\Delta)$ inherits the structure of a complex Banach manifold from the complex structure of the unit ball $L^\infty(\Delta)_1$. Namely there is a unique induced complex structure on $T(\Delta)$ such that the quotient projection $\Phi : L^\infty(\Delta)_1 \to T(\Delta)$ becomes a *holomorphic submersion*. For complete proofs see Nag [11]. More about $T(\Delta)$ is explained in Sec. 5.

Notice that $T(\Delta)$ is a group (though *not* a topological group). In fact, composition of quasi-symmetric homeomorphisms corresponds to the following group law on Beltrami coefficients (see [11], p. 54-55 and p. 227-228)

$$\begin{aligned} \lambda \cdot \mu &= \text{Beltrami coefficient of } (w_\lambda \circ w_\mu) \\ &= \frac{\mu + (\lambda \circ w_\mu)\gamma_\mu}{1 + \overline{\mu}(\lambda \circ w_\mu)\gamma_\mu} \quad \text{where} \quad \gamma_\mu = \frac{\overline{(w_\mu)_z}}{(w_\mu)_z} \end{aligned} \tag{7}$$

Since formula (7) depends holomorphically on λ we see that right translations act as biholomorphic automorphisms on $T(\Delta)$ (and on $L^\infty(\Delta)_1$).

Remark If we redefine universal Teichmüller space by associating to $\mu \in L^\infty(\Delta)_1$ the boundary values of w_μ^{-1}, then the *left* translations act biholomorphically. The usual conventions in the physics literature regarding the orbit spaces of Diff (S^1) can then be retained. We prefer to stick to the classical conventions in Teichmüller space theory.

4. $M_2 \hookrightarrow T(\Delta)$ IS A HOLOMORPHIC INCLUSION:

It is well-known that every diffeomorphism of S^1 extends to a diffeomorphism of the closed disk $\Delta \cup S^1$. So diffeomorphisms are certainly quasi-symmetric. Consequently, $M_2 =$ Diff $(S^1)/$Möb (S^1) sits canonically inside $T(\Delta) =$ Homeo $_{qs}(S^1)/$Möb (S^1).

Theorem The natural inclusion $M_2 \hookrightarrow T(\Delta)$ is holomorphic. M_2 can be thought of as one leaf of a holomorphic foliation of $T(\Delta)$ by injectively and holomorphically immersed leaves.

Using the holomorphic homogeneity of both M_2 and $T(\Delta)$ under right translations, one only needs to check the identity of the almost complex structures at the origin. The first problem is therefore to get a description of the almost complex structure, J, of $T(\Delta)$ at the origin, (so as to be able to compare it with the $\tilde{J}$ of Sec. 2).

Acknowledgment: The pretty description of J on $T(\Delta)$ given in the Proposition below is essentially an idea of S. Kerckhoff. The idea was explained to the first author in oral communication by C.J. Earle at Cornell University (1987–88).

A quasi-symmetric real vector field on S^1, to be thought of as an arbitrary tangent vector at the origin of $T(\Delta)$, is obtained from a one-parameter flow of quasi-symmetric homeomorphism $w_{t\mu}$, for any $\mu \in L^\infty(\Delta)$. The vector field on S^1 is then $\vartheta = \dot{w}[\mu]\frac{\partial}{\partial z}$ where $w_{t\mu}$ has the perturbation expansion:

$$w_{t\mu}(z) = z + t\dot{w}[\mu](z) + o(t), \quad t \to 0. \tag{8}$$

The problem is to obtain $J\vartheta$ on S^1, where

$$J\vartheta = \dot{w}[i\mu](z)\frac{\partial}{\partial z}, \quad \text{given} \quad \vartheta = \dot{w}[\mu](z)\frac{\partial}{\partial z}. \tag{9}$$

(Recall that the complex structure of $T(\Delta)$ is inherited from the complex structure of the space of μ's, as explained in Sec. 3. So J corresponds to sending μ to $i\mu$.)

Proposition Using θ as coordinate on S^1, $z = e^{i\theta}$, we can write $\vartheta = u(\theta)\frac{\partial}{\partial\theta}$ where $\dot{w}[\mu](z) = izu(z)$, $z \in S^1$. Then, $J\vartheta = u^*(\theta)\frac{\partial}{\partial\theta}$ where $\dot{w}[i\mu](z) = izu^*(z)$, $z \in S^1$. The formula for u^* is:

$$u^*(z) = Im(D(z)) + (cz + \bar{c}\,\bar{z} + b) \quad \text{on} \quad S^1 \tag{10}$$

for a certain $b \in \mathbb{R}$, $c \in \mathbb{C}$. Here $D(z)$ is a member of the disc algebra $A(\Delta)$ (namely, functions holomorphic in Δ and continuous on $\Delta \cup S^1$) such that $ReD = u$ on S^1.

Remark Notice that $u(z)$ is simply the *magnitude* of the vector field ϑ at the point $z \in S^1$.

Proof The first variation term $\dot{w}[\mu]$ can actually be explicitly written down (see [11], p. 38–40) in the form

$$\dot{w}[\mu](z) = \int\int_{\mathbb{C}} \tilde{\mu}(\varsigma)R(z,\varsigma)d\varsigma \wedge d\bar{\varsigma} \tag{11}$$

where $\tilde{\mu}$ is the extension of μ to the whole plane by reflection across S^1, as explained in a Remark in Sec. 3. (Explicitly, $\tilde{\mu}(\frac{1}{\bar{w}}) = \overline{\mu(w)}\frac{w^2}{\bar{w}^2}$ for w in Δ.) Here $R(z,\varsigma)$ is a certain rational function. The main feature of $\dot{w}[\mu]$ (from which actually formula (11) can be derived) is that

$$\bar{\partial}\dot{w}[\mu] = \mu \ \text{ a.e. on } \Delta, \quad (\text{here } \bar{\partial} = \partial/\partial\bar{z}). \tag{12}$$

See [11] p. 39–40 and p. 171 for a proof of this critical property. We also note for later use that formula (11) implies, since μ is L^∞, that $u(z) = \frac{\dot{w}[\mu](z)}{iz}$ (on S^1) satisfies a Hölder condition ($|u(z_1) - u(z_2)| \le c|z_1 - z_2|^\lambda$, $0 < \lambda < 1$) – in fact with λ arbitrarily close to 1.

Construct the function

$$F(z) = \dot{w}[i\mu](z) - i\dot{w}[\mu](z), \quad \text{on} \quad \Delta \cup S^1. \tag{13}$$

By formula (12) we see that $\bar{\partial}F = 0$ on Δ, so F is in the disc algebra $A(\Delta)$. Therefore the critical fact is:

$$izu^*(z) - izu(z) = F(z) \quad \text{on} \quad S^1, \quad \text{for} \quad F \in A(\Delta). \tag{14}$$

It is easy to derive the proposition from formula (14) as follows. Define $G \in A(\Delta)$ by $F(z) = F(0) + zG(z)$. Then (14) becomes

$$u(z) + iu^*(z) = G(z) + F(0)\bar{z} \quad \text{on} \quad S^1. \tag{15}$$

We therefore have

$$\begin{aligned} u(z) &= Re(G(z) + F(0)\bar{z}) \quad \text{on} \quad S^1 \\ &= Re(G(z) + \overline{F(0)}z) \quad \text{on} \quad S^1 \end{aligned} \tag{16}$$

Of course, $D(z) = G(z) + \overline{F(0)}z$ is also in $A(\Delta)$, and the above calculation shows that $D(z)$ solves the "holomorphic Dirichlet problem" for the real bounary values u on S^1.

Eq. (15) now allows us to relate u^* with u as desired:

$$\begin{aligned} u^*(z) &= Im(G(z) + F(0)\bar{z}), \quad \text{on} \quad S^1 \\ &= Im(D(z) - \overline{F(0)}z + F(0)\bar{z}), \quad \text{on} \quad S^1 \\ &= Im(D(z)) + kz + \bar{k}\,\bar{z}, \quad z = e^{i\theta}. \end{aligned} \tag{17}$$

Note that our $D(z)$ must be of the form

$$D(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big)d\theta + ib, \quad b \in \mathbb{R},$$

(the "Schwarz kernel" formula for the Dirichlet problem). The fact that u is Hölder is well known (see, for example, Gakhov [6]) to guarantee that D is in $A(\Delta)$, and therefore $Im(D(z))$ on S^1 is well-defined.

The Proposition is fully proved. □

Remark Notice that the normalisation of w_μ (to fix +1, -1 and i) implies that the vector fields $u(\theta)$, $u^*(\theta)$ must vanish at these three points. The constants b,c occuring in formula (10) can be related at least partly to the enforcement of this normalisation.

To prove our Theorem we need to show that J and $\tilde{J}$ act identically on the smooth real vector fields ϑ of S^1. Let, $\vartheta = u(\theta)\frac{\partial}{\partial\theta}$, with Fourier expansion

$$u(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \tag{18}$$

Then,

$$D(z) = a_0 + 2\sum_{n=1}^{\infty} a_n z^n \tag{19}$$

is in $A(\Delta)$ with $Re\,D = u$ on S^1 clearly.

By the Proposition we obtain therefore: $J\vartheta = u^*(\theta)\frac{\partial}{\partial\theta}$ where, for certain b,c

$$\begin{aligned} u^*(\theta) &= Im(D(e^{i\theta})) + (b + ce^{i\theta} + \bar{c}e^{-i\theta}) \\ &= \sum_{n=2}^{\infty}(-ia_n)e^{in\theta} + \sum_{n=2}^{\infty}\overline{(-ia_n)}e^{-in\theta} \\ &\quad + (\beta + \gamma e^{i\theta} + \bar{\gamma}e^{-i\theta}) \end{aligned} \tag{20}$$

The β and γ get normalised to zero via the $SL(2,\mathbb{R})$ normalisation. Thus, comparing (20) with formula (3) for $\tilde{J}$, we see that $J\vartheta \equiv \tilde{J}\vartheta$. The theorem is proved. □

5. REMARKS ON THE TEICHMÜLLER SPACES T(G) INSIDE T(Δ)

Given an *arbitrary* Fuchsian group G operating in the unit disc, the Teichmüller space $T(G)$ sits canonically in the universal Teichmüller space $T(\Delta) = T$ (trivial group). In fact, $T(G) = L^\infty(\Delta, G)_1/\sim$, for the same equivalence relation $\sim$ amongst Beltrami coefficients, where:

$$L^\infty(\Delta,G)_1 = \{\mu \in L^\infty(\Delta)_1 : \mu(g(z))\overline{g'(z)}/g'(z) = \mu(z) \ \text{ a.e. on } \Delta \text{ for every } g \in G\} \tag{21}$$

The Bers embedding (see [11]) embeds $T(\Delta)$ as a bounded domain in the complex Banach space

$$B_2(\Delta^*) = \{\varphi \in \text{Hol}\,(\Delta^*) : \ \|\varphi(z)(|z|^2-1)^2\|_\infty < \infty\}. \tag{22}$$

Then $T(G)$ is simply the complex submanifold $T(\Delta) \cap B_2(\Delta^*, G)$. Here $B_2(\Delta^*, G)$ is the closed subspace of $B_2(\Delta^*)$ consisting of those $\varphi \in B_2(\Delta^*)$ that are quadratic differentials for G (i.e. $\varphi(g(z))g'(z)^2 = \varphi(z)$ on Δ^* for each g in G). Δ^* denotes the exterior of Δ.

It is therefore easy to identify the quasi-symmetric homeomorphisms and the vector fields on S^1 that correspond to elements and tangent vectors of $T(G)$, for any G. Indeed, these are the ones corresponding to G-automorphic Beltrami coefficients as in formula (21). Now, in [5] (see also [3], [4], [9], [10], [12]) a translation invariant Kähler form has been studied on M_1 by the physicists. Restricted to the finite dimensional $T(G)$ this should produce a natural modular invariant Kähler metric on $T(G)$. We will report on these matters in the future. A specific question would be whether the volume form of this induced Kähler metric is the Polyakov volume form on $T(G)$, (when Δ/G is a compact Riemann surface).

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