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# Equivalence of Two Formalisms for Calculating Higher Order Synchrotron Sideband Spin Resonances*

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# Equivalence of Two Formalisms for Calculating Higher Order Synchrotron Sideband Spin Resonances*

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Synchrotron sideband resonances of a first order spin resonance are generally regarded as the most important higher order spin resonances in a high-energy storage ring. Yokoya's formula for these resonances is rederived, including some extra terms, which he neglected, but which turn out to be of comparable magnitude to the terms retained. Including these terms, Yokoya's formalism and the SMILE algorithm are shown to be equivalent to leading order in the resonance strengths. The theoretical calculations are shown to agree with certain measurements from SPEAR.

## 1 INTRODUCTION

Synchrotron sideband resonances of a first order spin resonance are generally regarded as the most important higher order spin resonances in a high-energy storage ring. The problem has previously been treated by Yokoya.[1] More recently, a formalism for arbitrary higher order spin resonances has been developed,[2] which includes synchrotron sideband resonances as a subset. Although both formalisms use perturbation theory, the terms are summed in different ways, and the results look quite different.

However, we shall show below that the two formalisms are equivalent, subject to certain caveats which will be described below. In particular, we shall rederive Yokoya's formula. We find that there are some extra terms, which he neglected, but which turn out to be of comparable magnitude to the terms retained. These terms are already present in the SMILE formalism. We also fit the results to some data from polarization measurements at SPEAR,[3] and obtain reasonable agreement for some resonances.

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The polarization is calculated using the Derbenev-Kondratenko formula[4]

$$P_{eq} = \frac{8}{5\sqrt{3}} \frac{\left\langle |\rho|^{-3}\ \hat{b}.\left[\hat{n} - \gamma\frac{\partial\hat{n}}{\partial\gamma}\right]\right\rangle}{\left\langle |\rho|^{-3}\left[1 - \frac{2}{9}(\hat{n}.\hat{v})^2 - \frac{11}{18}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right]\right\rangle} . \quad (1)$$

Here $\vec{v}$ is the particle velocity, $\hat{b} \equiv \vec{v} \times \dot{\vec{v}}/|\vec{v} \times \dot{\vec{v}}|$, $\rho$ is the local radius of curvature of the particle trajectory, $\hat{n}$ is the spin quantization axis on the particle trajectory, and the angular brackets denote an equilibrium average over the distribution of particle orbits and the ring azimuth. The direction of the equilibrium polarization vector is $\vec{P}_{eq} \parallel \langle\hat{n}\rangle$. Both Refs. [1] and [2] calculate $\hat{n}$ perturbatively, in ways to be described below, to evaluate $P_{eq}$.

## 2 YOKOYA'S FORMALISM

### 2.1 SOLUTION FOR $\hat{n}$

The notation and formalism below will mainly follow Ref. [1]. The ring azimuth is denoted by $\theta$. The horizontal betatron coordinate and energy offset of the particle are

$$\begin{aligned} x_\beta &= \sqrt{2I_x\beta_x}\,\cos(\psi_x - \tilde{\Psi}_x(\theta)) \\ \epsilon &= \sqrt{2I_z}\,\cos\psi_z\ . \end{aligned} \quad (2)$$

where $\{I_j, \psi_j, j = x, z\}$ are action-angle variables, and $\tilde{\Psi}_x(\theta)$ is the periodic part of the betatron phase advance,

$$\tilde{\Psi}_x(\theta) = \int_0^\theta \frac{R\,d\theta'}{\beta_x(\theta')} - Q_x\theta\ , \quad (3)$$

where $R$ is the average ring radius. The horizontal betatron and synchrotron tunes are called $Q_x$ and $Q_s$, respectively, in this note. The equation of motion for the Derbenev-Kondratenko $\hat{n}$ axis is

$$\frac{d\hat{n}}{d\theta} = (\vec{\Omega}_0 + \vec{\omega}\,) \times \hat{n}\ . \quad (4)$$

Here $\vec{\Omega}_0$ is the spin precession vector on the closed orbit and $\vec{\omega}$ is the contribution of the orbital oscillations ($\vec{\omega} = 0$ on the closed orbit). We write $\vec{\omega} = x_\beta\vec{\omega}_x + \epsilon\vec{\omega}_\epsilon$ to denote the couplings to the horizontal betatron and synchrotron oscillations, respectively. We express $\hat{n}$ in the form

$$\hat{n}(I_x, \psi_x, I_z, \psi_z, \theta) = \hat{n}_0\sqrt{1 - |\zeta|^2} + \mathrm{Re}(\vec{k}_0^*\zeta)\ , \quad (5)$$

where $\hat{n}_0$ and $\vec{k}_0$ are both solutions of Eq. (4) on the closed orbit, and

$$\begin{aligned} \hat{n}_0(\theta + 2\pi) &= \hat{n}_0(\theta) \\ \vec{k}_0(\theta + 2\pi) &= e^{i2\pi\nu}\vec{k}_0(\theta) \ . \end{aligned} \tag{6}$$

Here $\nu$ is the spin tune. The equation of motion for $\zeta$ is

$$\frac{d\zeta}{d\theta} = -i\vec{\omega}.\vec{k}_0\sqrt{1 - |\zeta|^2} - i\vec{\omega}.\hat{n}_0\zeta \ . \tag{7}$$

We approximate $\sqrt{1 - |\zeta|^2} \simeq 1$ and then solve for $\zeta$, which yields

$$\zeta = -ie^{-i\chi(\theta)}\int_{-\infty}^{\theta} d\theta' \, e^{i\chi(\theta')}\vec{\omega}.\vec{k}_0(\theta') \ , \tag{8}$$

where

$$\chi = -\int_0^{\theta} \vec{\omega}.\hat{n}_0 \, d\theta' \ . \tag{9}$$

The contributions of rapidly oscillating terms in $\chi$ are neglected, and only synchrotron oscillations are retained, i.e. $\omega.\hat{n}_0 \simeq \epsilon\vec{\omega}_\epsilon.\hat{n}_0$. In this context, we approximate $Q_s \ll 1$. We also define $u_\epsilon = \gamma a/Q_s$. Then

$$\chi \simeq \sqrt{2I_z}\, u_\epsilon \sin\psi_z \ . \tag{10}$$

Using the relation

$$e^{ir\sin\psi} = \sum_{m=-\infty}^{\infty} e^{im\psi}J_m(r) \ , \tag{11}$$

we obtain Yokoya's result

$$\zeta = e^{-i\chi}\sum_{m=-\infty}^{\infty}\Big[\frac{m}{u_\epsilon}A_m(\theta) + \sum_{\pm}\sqrt{2I_x\beta_x}e^{\pm i(\psi_x + \tilde{\Psi}_x)}B_{m,\pm x}(\theta)\Big]e^{im\psi_z}J_m(\sqrt{2I_z}u_\epsilon) \ , \tag{12}$$

where

$$\begin{aligned} A_m(\theta) &= \frac{-ie^{-imQ_s\theta}}{e^{i2\pi(\nu - mQ_s)} - 1}\int_{\theta}^{\theta+2\pi} e^{imQ_s\theta'}\vec{\omega}_\epsilon.\vec{k}_0 \, d\theta' \\ B_{m,\pm x}(\theta) &= \frac{-(i/2)e^{\mp i(\tilde{\Psi}_x - Q_x\theta) - imQ_s\theta}}{(e^{i2\pi(\nu \mp Q_x - mQ_s)} - 1)\sqrt{\beta_x}}\int_{\theta}^{\theta+2\pi} e^{\pm i(\tilde{\Psi}_x - Q_x\theta') - imQ_s\theta'}\sqrt{\beta_x}\vec{\omega}_x.\vec{k}_0 \, d\theta' \ . \end{aligned} \tag{13}$$

## 2.2 RESONANCE HARMONICS

The term in $A_m$ describes synchrotron sideband resonances centered on an integer, whereas the terms in $B_{m,\pm x}$ describe synchrotron sideband resonances around the betatron resonances

$\nu + n \pm Q_x = 0$, respectively. We shall focus on satellites of a betatron resonance, say $\nu + n - Q_x = 0$, so we neglect $A_m$ and $B_{m,x}$. We also keep only the harmonic closest to the resonance in $B_{m,-x}$, i.e.

$$
\begin{aligned}
B_{m,-x} \equiv B_m &= -\frac{1}{2}\sum_{n'} \frac{b_{n',-x}}{\nu + n' - Q_x + mQ_s} \frac{e^{-i\tilde{\Psi}_x + i(\nu+n')\theta}}{\sqrt{\beta_x}} \\
&\simeq -\frac{1}{2}\frac{b}{\nu + n - Q_x + mQ_s} \frac{e^{-i\tilde{\Psi}_x + i(\nu+n)\theta}}{\sqrt{\beta_x}} . \qquad (14)
\end{aligned}
$$

### 2.3 SOLUTION FOR $\gamma(\partial\hat{n}/\partial\gamma)$

Next, we want $\partial\zeta/\partial\epsilon$, to get $\gamma(\partial\hat{n}/\partial\gamma)$ in Eq. (1). Note that[1]

$$
\begin{aligned}
\frac{\partial}{\partial\epsilon}\left[\sqrt{2I_x\beta_x}e^{\pm i(\psi_x - \tilde{\Psi}_x)}\right] &= -\eta_x \pm i(\eta_x'\beta_x + \eta_x\alpha_x) \\
\frac{\partial}{\partial\epsilon}\left[J_m(\sqrt{2I_z}u_\epsilon)e^{im\psi_z}\right] &= \frac{u_\epsilon}{2}\left[J_{m-1}(\sqrt{2I_z}u_\epsilon)e^{i(m-1)\psi_z} - J_{m-1}(\sqrt{2I_z}u_\epsilon)e^{i(m-1)\psi_z}\right] , \qquad (15)
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{\partial\zeta}{\partial\epsilon} = e^{-i\chi}\sum_{m=-\infty}^{\infty} B_m(\theta)\Big[&\frac{\partial}{\partial\epsilon}\left(\sqrt{2I_x\beta_x}e^{-i(\psi_x + \tilde{\Psi}_x)}\right)J_m(\sqrt{2I_z}u_\epsilon)e^{im\psi_z} \\
&+\sqrt{2I_x\beta_x}e^{-i(\psi_x + \tilde{\Psi}_x)}\frac{\partial}{\partial\epsilon}\left(J_m(\sqrt{2I_z}u_\epsilon)e^{im\psi_z}\right)\Big] . \qquad (16)
\end{aligned}
$$

Yokoya neglected the second term, but we shall retain it. After Eq. (3.6) in Ref. [1], it is stated that "we have neglected the terms which are proportional to betatron oscillation amplitudes after differentiation." Using Eq. (15),

$$
\begin{aligned}
\frac{\partial\zeta}{\partial\epsilon} &= e^{-i\chi}\sum_{m=-\infty}^{\infty} B_m(\theta)\Big[\left(-\eta_x - i(\eta_x'\beta_x - \eta_x\alpha_x)\right)J_m(\sqrt{2I_z}u_\epsilon)e^{im\psi_z} \\
&\qquad + \sqrt{2I_x\beta_x}e^{-i(\psi_x - \tilde{\Psi}_x)}\frac{u_\epsilon}{2}\Big(J_{m-1}(\sqrt{2I_z}u_\epsilon)e^{i(m-1)\psi_z} \\
&\qquad - J_{m-1}(\sqrt{2I_z}u_\epsilon)e^{i(m+1)\psi_z}\Big)\Big] \\
&\equiv e^{-i\chi}\sum_{m=-\infty}^{\infty} C_m(\theta)J_m(\sqrt{2I_z}u_\epsilon)e^{im\psi_z} . \qquad (17)
\end{aligned}
$$

## 2.4 ENSEMBLE AVERAGES

Averaging over the synchrotron oscillations, assuming a Gaussian distribution,

$$
\begin{aligned}
\left\langle \left|\frac{\partial\zeta}{\partial\epsilon}\right|^2\right\rangle &= \sum_{m=-\infty}^{\infty} |C_m(\theta)|^2 \int_0^\infty \frac{dI_z}{\langle I_z\rangle}\, e^{-I_z/\langle I_z\rangle} J_m^2(\sqrt{2I_z}u_\epsilon) \\
&= \sum_{m=-\infty}^{\infty} |C_m(\theta)|^2 e^{-\alpha} I_m(\alpha)\,,
\end{aligned}
\tag{18}
$$

where $\alpha = \langle I_z\rangle u_\epsilon^2 = \langle I_z\rangle(\gamma a/Q_s)^2$. We also need to average over the betatron orbits in $C_m$. From Eq. (17)

$$
C_m(\theta) = \Big(-\eta_x - i(\eta_x'\beta_x - \eta_x\alpha_x)\Big)B_m + \sqrt{2I_x\beta_x}\,e^{-i(\psi_x-\tilde{\Psi}_x)}\frac{u_\epsilon}{2}(B_{m+1} - B_{m-1})\,. \tag{19}
$$

For brevity, put $\delta = \nu - n - Q_x$. We shall consider the terms proportional $\sqrt{2I_x\beta_x}$ later, but neglect them for now. Then

$$
\begin{aligned}
|C_m(\theta)|^2 &\simeq (\eta_x^2 + (\eta_x'\beta_x + \eta_x\alpha_x)^2)\,|B_m|^2 \\
&= \frac{1}{4}\frac{\eta_x^2 + (\eta_x'\beta_x + \eta_x\alpha_x)^2}{\beta_x}\frac{|b|^2}{(\delta - mQ_s)^2}\,,
\end{aligned}
\tag{20}
$$

and so

$$
\left\langle \left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle \simeq \frac{1}{4}\frac{\eta_x^2 + (\eta_x'\beta_x + \eta_x\alpha_x)^2}{\beta_x}\sum_{m=-\infty}^{\infty}\frac{|b|^2 e^{-\alpha} I_m(\alpha)}{(\delta - mQ_s)^2}\,, \tag{21}
$$

which agrees with the expression for $\langle|\gamma(\partial\hat{n}/\partial\gamma)|^2\rangle$ in Eq. (3.18) in Ref. [1].

## 2.5 ADDITIONAL SPIN INTEGRALS

Now let us include the extra terms, which are proportional to $\sqrt{2I_x\beta_x}$ in Eq. (19). To do so, note that[5]

$$
\begin{aligned}
\epsilon_\epsilon &\equiv \langle I_z\rangle = \frac{C_q\gamma_0^2\langle|\rho_0|^{-3}\rangle_\theta}{J_\epsilon\langle|\rho_0|^{-2}\rangle_\theta} \\
\epsilon_{x\beta} &\equiv \langle I_x\rangle = \frac{C_q\gamma_0^2}{J_x\langle|\rho_0|^{-2}\rangle_\theta}\left\langle\frac{1}{|\rho_0|^3}\frac{\eta_x^2 + (\eta_x'\beta_x + \eta_x\alpha_x)^2}{\beta_x}\right\rangle_\theta \equiv \frac{C_q\gamma_0^2}{J_x\langle|\rho_0|^{-2}\rangle_\theta}\left\langle\frac{H}{|\rho_0|^3}\right\rangle_\theta\,,
\end{aligned}
\tag{22}
$$

where $\langle\ldots\rangle_\theta$ denotes an average around the ring circumference, $C_q = 55/(32\sqrt{3})\hbar/(mc) = 3.84\times10^{-13}$ m, $\rho_0$ is the bending radius of the closed orbit, $\gamma_0$ is the average electron energy in units of $mc^2$, and $J_\epsilon$ and $J_x$ are the partition numbers of the synchrotron and betatron damping constants. For brevity, define $K = C_q\gamma_0^2/\langle|\rho_0|^{-2}\rangle_\theta$. Then, averaging over both the orbital action-angle variables and the ring circumference,

$$\left\langle \frac{1}{|\rho|^3} \left|\gamma\frac{\partial \hat{n}}{\partial \gamma}\right|^2 \right\rangle_{\theta,orbits} = \left\langle \frac{1}{|\rho|^3} \left|\frac{\partial \zeta}{\partial \epsilon}\right|^2 \right\rangle_{\theta,orbits}$$

$$= \sum_m e^{-\alpha} I_m(\alpha) \Big[ K^{-1} J_x \epsilon_{x\beta} |\beta_x^{\frac{1}{2}} B_m|^2 + \epsilon_{x\beta} K^{-1} J_\epsilon \epsilon_\epsilon \frac{u_\epsilon^2}{2}$$

$$\times \left( |\beta_x^{\frac{1}{2}} B_{m-1}|^2 + |\beta_x^{\frac{1}{2}} B_{m-1}|^2 - 2\mathrm{Re}(\beta_x B_{m-1} B_{m-1}^*) \right) \Big] . \quad (23)$$

We neglect the cross term $\mathrm{Re}(\beta_x B_{m-1} B_{m-1}^*)$ because it is not as singular as the other terms, and so

$$\left\langle \frac{1}{|\rho|^3} \left|\gamma\frac{\partial \hat{n}}{\partial \gamma}\right|^2 \right\rangle = \frac{\epsilon_{x\beta}}{K} \sum_{m=-\infty}^{\infty} e^{-\alpha} \Big[ J_x |\beta_x^{\frac{1}{2}} B_m|^2 I_m(\alpha)$$

$$+ J_\epsilon \frac{\alpha}{2} |\beta_x^{\frac{1}{2}} B_m|^2 (I_{m-1}(\alpha) + I_{m+1}(\alpha)) \Big]$$

$$= \frac{\epsilon_{x\beta}}{K} \sum_{m=-\infty}^{\infty} \frac{e^{-\alpha} |b|^2}{4(\delta - mQ_s)^2} \Big[ J_x I_m(\alpha) + J_\epsilon \frac{\alpha}{2} (I_{m-1}(\alpha) + I_{m+1}(\alpha)) \Big] . \quad (24)$$

One of the features of the above result is that because $I_{-m}(\alpha) = I_m(\alpha)$, the terms in $m$ and $-m$ have the same strength. This can be verified by substituting $m \to -m$ in Eq. (24). Thus Eq. (24) predicts the polarization should be symmetric about the first-order "parent" resonance. The inclusion of less singular terms, which have been neglected in the above derivation, will change this symmetry.

## 2.6 ENHANCEMENT FACTORS

### 2.6.1 Sidebands with $m > 0$

Let us now consider only the terms for which $m \geq 0$ in the above sum. These are the sidebands $\nu = n - Q_x - Q_s$, $\nu = n - Q_x - 2Q_s$, etc. We shall return to the $m < 0$ terms later. Using the relation

$$\alpha(I_{m-1}(\alpha) - I_{m+1}(\alpha)) = 2mI_m(\alpha) , \quad (25)$$

Eq. (24) can be written in the form

$$\left\langle \frac{1}{|\rho|^3} \left|\gamma\frac{\partial \hat{n}}{\partial \gamma}\right|^2 \right\rangle = K^{-1}\epsilon_{x\beta} \sum_m \frac{e^{-\alpha}}{4} \frac{|b|^2}{(\delta - mQ_s)^2} \Big[ J_x I_m(\alpha) + J_\epsilon (mI_m(\alpha) + \alpha I_{m+1}(\alpha)) \Big]$$

$$= K^{-1} J_x \epsilon_{x\beta} \sum_{m=0}^{\infty} \frac{e^{-\alpha} I_m(\alpha)}{4} \frac{|b|^2}{(\delta - mQ_s)^2} \left[1 + \frac{J_\epsilon}{J_x}\left(m - \alpha \frac{I_{m-1}(\alpha)}{I_m(\alpha)}\right)\right]. \quad (26)$$

We can write the above result as an "enhancement" of the parent resonance

$$\left\langle \frac{1}{|\rho|^3} \left|\gamma \frac{\partial \hat{n}}{\partial \gamma}\right|^2 \right\rangle = F \left\langle \frac{1}{|\rho_0|^3} \left|\gamma \frac{\partial \hat{n}}{\partial \gamma}\right|^2 \right\rangle_{1^{st}\ order} . \quad (27)$$

The first-order result is

$$\left\langle \frac{1}{|\rho|^3} \left|\gamma \frac{\partial \hat{n}}{\partial \gamma}\right|^2 \right\rangle_{1^{st}\ order} = \frac{1}{4} \left\langle \frac{H}{|\rho|^3} \right\rangle_\theta \frac{|b|^2}{\delta^2} . \quad (28)$$

Hence, for the $m > 0$ sidebands,

$$F_{m \geq 0} = \sum_{m=0}^{\infty} \left[1 + \frac{J_\epsilon}{J_x}\left(m + \alpha \frac{I_{m+1}(\alpha)}{I_m(\alpha)}\right)\right] e^{-\alpha} I_m(\alpha) \frac{\delta^2}{(\delta + mQ_s)^2} . \quad (29)$$

Using the asymptotic relations

$$\begin{aligned} I_m(\alpha) &\simeq \left(\frac{\alpha}{2}\right)^m \frac{1}{m!} \qquad (\alpha \ll m) , \\ &\simeq \frac{e^\alpha}{\sqrt{2\pi\alpha}} \qquad (\alpha \gg m) , \end{aligned} \quad (30)$$

we see that the term in $\alpha I_{m-1}/I_m$ is important only if $\alpha \gg m$.

### 2.6.2 Sidebands with $m < 0$

For the sideband resonances with $m < 0$ in Eq. (24), the enhancement factor is

$$\begin{aligned} F_{m \leq 0} &= \sum_{m=-\infty}^{0} \left[1 + \frac{J_\epsilon}{J_x}\left(-m + \alpha \frac{I_{m-1}(\alpha)}{I_m(\alpha)}\right)\right] e^{-\alpha} I_m(\alpha) \frac{\delta^2}{(\delta - mQ_s)^2} \\ &= \sum_{m=0}^{\infty} \left[1 + \frac{J_\epsilon}{J_x}\left(m + \alpha \frac{I_{m+1}(\alpha)}{I_m(\alpha)}\right)\right] e^{-\alpha} I_m(\alpha) \frac{\delta^2}{(\delta - mQ_s)^2} . \end{aligned} \quad (31)$$

The two enhancement factors are equal, as expected because the two sets of resonances have equal strength (see the statements below Eq. (24)).

### 2.6.3 Combined results

We can combine the above results into one factor

$$F = \sum_{m=-\infty}^{\infty} \left[1 + \frac{J_\epsilon}{J_x}\left( |m| + \alpha \frac{I_{|m|+1}(\alpha)}{I_{|m|}(\alpha)}\right)\right] e^{-\alpha} I_{|m|}(\alpha) \frac{\delta^2}{(\delta - mQ_s)^2} . \quad (32)$$

A similar, but slightly different, theoretical formula has been reported in Ref. [6], but without derivation, and only for the $m > 0$ sidebands. It is also convenient to write

$$F \equiv \sum_{m=-\infty}^{\infty} \frac{W_m^2}{(\delta + mQ_s)^2} \ . \qquad (33)$$

*If the resonances are well separated*, then it can be shown that the width of the resonance $\delta + mQ_s = 0$ is proportional to $W_m$. In particular, the ratio of the width of the sideband resonance $\delta - mQ_s = 0$ to that of the parent resonance $\delta = 0$ is equal, not merely proportional, to $W_m/W_0$. This result is not true, however, if several terms in the above sum contribute significantly to the width of a given resonance, i.e. if the resonances overlap.

### 2.7 NUMERICAL ESTIMATES FOR $\alpha$

At this point, let us estimate the value of $\alpha \equiv \epsilon_\epsilon(\gamma a/Q_s)^2$ for various rings. We obtain the value of $\epsilon_\epsilon$ using Eq. (22). For SPEAR at the horizontal betatron resonance $\nu = 3 + Q_x$, $E = 3.65$ GeV, and $\rho_0 \simeq 12$ m, and we put $J_\epsilon = 2$. From the data in Ref. [3], $Q_s \simeq 0.045$. For HERA, we use the values $E = 30$ GeV, $\rho = 600$ m, $J_\epsilon = 2$ and $Q_s = 0.06$, and for LEP we assume $E = 50$ GeV, $\rho = 3000$ m, $J_\epsilon = 2$ and $Q_s = 0.1$. The values of $\epsilon_\epsilon$ and $\alpha$ for these three models are given in Table 1. We see that $\alpha \ll 1$ for SPEAR, but is approximately unity for HERA and LEP.

Table 1: Numerical estimates for $\langle I_z \rangle$ and $\alpha$

| Ring | $\epsilon_\epsilon\,( = \langle I_z \rangle\,)$ | $\alpha$ |
|---|---|---|
| SPEAR | $8.2 \times 10^{-7}$ | $2.8 \times 10^{-2}$ |
| HERA | $1.1 \times 10^{-6}$ | 1.42 |
| LEP | $6.1 \times 10^{-7}$ | 0.79 |

## 3 SMILE FORMALISM

### 3.1 SOLUTION FOR $\hat{n}$

In this section the SMILE formalism[2] will be used to derive the above results. We need only consider sidebands with $m > 0$. In Ref. [2], the orbital motion is written as a sum of eigenvectors but here we shall write $\vec{\omega} = x_\beta\vec{\omega}_x + \epsilon\vec{\omega}_\epsilon$ instead, and now we put

$$x_\beta = a_{x\beta}\tilde{x}_\beta + \text{c.c.}\,, \qquad \epsilon = a_\epsilon\tilde{\epsilon} + \text{c.c.}\,, \tag{34}$$

where

$$\begin{aligned} a_{x\beta} &= \sqrt{I_x}\, e^{i(\psi_x - Q_x\theta)} & a_\epsilon &= \sqrt{I_s}\, e^{i(\psi_s - Q_s\theta)} \\ \tilde{x}_\beta &= \sqrt{\frac{\beta_x}{2}}\, e^{i(Q_x\theta + \tilde{\Psi}_x)} & \tilde{\epsilon} &= \frac{e^{iQ_s\theta}}{\sqrt{2}}\,. \end{aligned} \tag{35}$$

We decompose

$$\hat{n} = n_1\hat{l}_0 + n_2\hat{m}_0 + n_3\hat{n}_0 \tag{36}$$

in terms of a right-handed orthonormal basis $\{\hat{l}_0, \hat{m}_0, \hat{n}_0\}$ of solutions of the Thomas-BMT equation on the closed orbit, and we define

$$V_1 = -\frac{n_1 + in_2}{\sqrt{2}}\,, \qquad V_{-1} = \frac{n_1 - in_2}{\sqrt{2}}\,, \qquad V_0 = n_3\,. \tag{37}$$

Then $\vec{k}_0 = \hat{l}_0 + i\hat{m}_0$ and $\zeta = -\sqrt{2}V_1$. The solution for $\hat{n}$ is given by a time-ordered exponential[2]

$$\begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} = \mathrm{T}\left\{ \exp\left(i\int_{-\infty}^{\theta} d\theta'\vec{\omega}.\vec{J}^T\right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}\,, \tag{38}$$

where $\vec{J}$ is a vector of spin 1 angular momentum matrices. To obtain a practical solution, we expand the above exponential in a power series and sum the terms order by order.

### 3.2 SYNCHROTRON SIDEBAND RESONANCES

The above exponential contains all combinations of spin integrals, but to get the previous solution for $\zeta$ (Eqs. (6) - (9)), we consider only the terms with $x\vec{\omega}_x.\vec{k}_0$ at first order, followed by powers of $\epsilon\vec{\omega}_\epsilon.\hat{n}_0$, the coupling to the synchrotron oscillations, i.e.

$$\zeta = -\sqrt{2}V_1 \quad \simeq \quad -i\int_{-\infty}^{\theta} d\theta'\, x\vec{\omega}_x.\vec{k}_0$$

$$+\int_{-\infty}^{\theta} d\theta'\, \epsilon\vec{\omega}_\epsilon.\hat{n}_0 \int_{-\infty}^{\theta'} d\theta''\, x\vec{\omega}_x.\vec{k}_0$$

$$+i\int_{-\infty}^{\theta} d\theta'\, \epsilon\vec{\omega}_\epsilon.\hat{n}_0 \int_{-\infty}^{\theta'} d\theta''\, \epsilon\vec{\omega}_\epsilon.\hat{n}_0 \int_{-\infty}^{\theta''} d\theta'''\, x\vec{\omega}_x.\vec{k}_0 + \ldots \tag{39}$$

Eq. (38) also contains terms of the form

$$\zeta = \frac{i}{2}\int_{-\infty}^{\theta} d\theta'\, \epsilon\vec{\omega}_\epsilon.\vec{k}_0 \int_{-\infty}^{\theta'} d\theta''\, \epsilon\vec{\omega}_\epsilon.\vec{k}_0^* \int_{-\infty}^{\theta''} d\theta'''\, x\vec{\omega}_x.\vec{k}_0 - \ldots \tag{40}$$

i.e. terms involving $\vec{\omega}_\epsilon.\vec{k}_0$ rather than $\vec{\omega}_\epsilon.\hat{n}_0$, as well as other terms. Let us now evaluate Eq. (39) term by term. As before, we approximate

$$e^{-i(Q_x\theta+\tilde{\Psi}_x)}\sqrt{\beta_x}\vec{\omega}_x.\vec{k}_0 \simeq be^{i(\nu+n-Q_x)\theta} \equiv be^{i\delta\theta} \tag{41}$$

and, from above, we approximate $\epsilon\vec{\omega}_\epsilon.\hat{n}_0 \simeq -Q_s u_\epsilon\sqrt{2I_z}\cos\psi_z$ because $Q_s \ll 1$. Let us now write $\zeta = \zeta_1 + \zeta_2 + \zeta_3 + \ldots$, where $\zeta_n$ is the $n^{th}$ term in Eq. (39). Then

$$\begin{aligned}\zeta_1 &= -ia_{x\beta}^*\int_{-\infty}^{\theta}\tilde{x}_\beta^*\vec{\omega}_x.\vec{k}_0\, d\theta' \\ &= -a_{x\beta}^*\frac{b}{\sqrt{2}}\frac{e^{i\delta\theta}}{\delta}\ .\end{aligned} \tag{42}$$

We also need the results, which can be derived from Eq. (15),

$$\begin{aligned}\frac{\partial a'}{\partial\epsilon} &\equiv \frac{\partial}{\partial\epsilon}\left(a_{x\beta}^* e^{-i(\tilde{\Psi}_x-Q_x\theta)}\right) = \frac{-\eta_x - i(\eta'\beta_x + \eta_x\alpha_x)}{\sqrt{2\beta_x}} \\ \frac{\partial a_\epsilon}{\partial\epsilon} &= 1\ .\end{aligned} \tag{43}$$

### 3.3 FIRST ORDER

To first order,

$$\frac{\partial\zeta_1}{\partial\epsilon} = -\frac{\partial a'}{\partial\epsilon}\frac{b}{\sqrt{2}\delta}e^{i(\tilde{\Psi}_x+Q_x\theta)}e^{i\delta\theta} \tag{44}$$

and

$$\begin{aligned}\left\langle\frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle &= \left\langle\frac{1}{|\rho|^3}\left|\frac{\partial\zeta_1}{\partial\epsilon}\right|^2\right\rangle \\ &= \left\langle\frac{1}{|\rho|^3}\left|\frac{\partial a'}{\partial\epsilon}\right|^2\right\rangle\frac{b^2}{2\delta^2}\end{aligned}$$

$$= \frac{1}{4}\left\langle\frac{H}{|\rho|^3}\right\rangle\frac{b^2}{\delta^2}\ , \tag{45}$$

in agreement with the previous calculation.

## 3.4 SECOND ORDER

At second order,

$$\zeta_2 = -a^*_{x\beta}\frac{b}{\sqrt{2}\delta}a_\epsilon i\int_{-\infty}^{\theta}\frac{\epsilon^{iQ_s\theta'}}{\sqrt{2}}\bar{\omega}_\epsilon.\hat{n}_0 e^{i\delta\theta'}\,d\theta' + \ldots \tag{46}$$

We have neglected the term which gives the resonance $\delta - Q_s = 0$, because we are only considering sidebands $\delta + mQ_s$ with $m > 0$. Using the approximation for $\bar{\omega}_\epsilon.\hat{n}_0$ given above,

$$\begin{aligned}\zeta_2 &\simeq -a^*_{x\beta}\frac{b}{\sqrt{2}\delta}a_\epsilon i\int_{-\infty}^{\theta}\frac{-Q_s u_\epsilon}{\sqrt{2}}\epsilon^{i(\delta+Q_s)\theta'}\,d\theta' \\ &= a^*_{x\beta}\frac{b}{\sqrt{2}\delta}a_\epsilon\frac{Q_s u_\epsilon}{\sqrt{2}}\frac{\epsilon^{i(\delta-Q_s)\theta}}{\delta - Q_s}\ .\end{aligned} \tag{47}$$

Then

$$\frac{\partial\zeta_2}{\partial\epsilon} = \frac{1}{2}\left(\frac{\partial a'}{\partial\epsilon}a_\epsilon - a'\frac{\partial a_\epsilon}{\partial\epsilon}\right)\frac{b}{\delta}\frac{Q_s u_\epsilon}{(\delta - Q_s)}\epsilon^{i(\tilde{\Psi}_x - Q_x\theta)}\epsilon^{i(\delta+Q_s)\theta} \tag{48}$$

and

$$\begin{aligned}\left\langle\frac{1}{|\rho|^3}\left|\frac{\partial\zeta}{\partial\epsilon}\right|^2\right\rangle &\simeq 2\left\langle\frac{1}{|\rho|^3}\left|\frac{\partial a'}{\partial\epsilon}\right|^2\right\rangle\frac{b^2}{4\delta^2} \\ &\quad -4\left\langle\frac{1}{|\rho|^3}\left[\left|\frac{\partial a'}{\partial\epsilon}\right|^2\epsilon_\epsilon - \epsilon_{x\beta}\left|\frac{\partial a_\epsilon}{\partial\epsilon}\right|^2\right]\right\rangle\frac{b^2}{4\delta^2}\frac{(Q_s u_\epsilon)^2}{4(\delta - Q_s)^2} \\ &= K^{-1}\epsilon_{x\beta}J_x\frac{b^2}{4\delta^2} - 2K^{-1}\epsilon_{x\beta}\epsilon_\epsilon(J_x - J_\epsilon)\frac{b^2}{4\delta^2}\frac{(Q_s u_\epsilon)^2}{4(\delta - Q_s)^2}\ .\end{aligned} \tag{49}$$

Recall the expression for the enhancement factor $F$ in the previous section could be written in the form

$$F = \sum_{m=-\infty}^{\infty}\frac{W_m^2}{(\delta + mQ_s)^2}\ . \tag{50}$$

Unlike Eq. (50), the sum in Eq. (49) cannot be separated into terms with distinct resonance denominators $\delta - mQ_s$. The higher order terms also contain lower order resonance denominators. Thus the second term has both $\delta^2$ and $(\delta - Q_s)^2$ in the denominator. This gives a correction to the first order resonance strength of $O(\epsilon_\epsilon u_\epsilon^2) = O(\alpha)$. In the previous calculation, this was given by the $\epsilon^{-\alpha}I_m(\alpha)$ and $\alpha I_{m-1}/I_m$ factors.

Let us concentrate on the leading order contributions (in powers of the beam emittances) to the resonances, i.e. the leading power of $\alpha$ in $W_m$. From Table 1, this is a good approximation for SPEAR. It is not such a good approximation at higher energies. To obtain the strength of the first sideband resonance, i.e. $W_1$, we can approximate $\delta - Q_s \simeq 0$ near the resonance. Hence we can put $\delta - Q_s = 0$ in the coefficient of $(\delta - Q_s)^{-2}$ in the r.h.s of Eq. (49)

$$W_1^2 \simeq 2\Big(1 - \frac{J_\epsilon}{J_x}\Big)\frac{\epsilon_\epsilon u_\epsilon^2 \delta^2}{4} = \Big(1 - \frac{J_\epsilon}{J_x}\Big)\frac{\alpha\delta^2}{2} \ . \tag{51}$$

Recall that in squaring and averaging $\gamma(\partial\hat{n}/\partial\gamma)$ in Eq. (24), the interference terms between distinct resonances were neglected. This was permissible as long as we considered only the most singular terms in each resonance. We shall also neglect interferences between resonances here. From Eq. (29),

$$\begin{aligned} W_1^2 &= \Big[1 - \frac{J_\epsilon}{J_x}\Big(1 - \alpha\frac{I_1(\alpha)}{I_0(\alpha)}\Big)\Big]\epsilon^{-\alpha} I_1(\alpha)\delta^2 \\ &\simeq \Big[1 - \frac{J_\epsilon}{J_x}\Big(1 - \frac{\alpha^2}{2}\Big)\Big]\frac{\alpha}{2}\delta^2 \\ &\simeq \Big(1 - \frac{J_\epsilon}{J_x}\Big)\frac{\alpha\delta^2}{2} \ , \end{aligned} \tag{52}$$

and so the two formalisms agree.

### 3.5 GENERAL CASE

Let us now consider the $m^{th}$ sideband $\delta - mQ_s$. It can easily be verified that

$$\zeta_{m-1} \simeq -a^*_{x\beta}\frac{b}{\sqrt{2}\delta}\frac{a_\epsilon^m(-Q_s u_\epsilon)^m}{2^{m/2}(\delta - Q_s)\dots(\delta - mQ_s)}\epsilon^{i(\delta - mQ_s)\theta} ; \tag{53}$$

and so

$$\begin{aligned} \Big\langle \frac{1}{|\rho|^3}\Big|\frac{\partial\zeta_{m-1}}{\partial\epsilon}\Big|^2\Big\rangle &\simeq \Big\langle\frac{1}{|\rho|^3}\Big[2\Big|\frac{\partial a'}{\partial\epsilon}\Big|^2 |a_\epsilon|^{2m} - 2|a^*_{x\beta}|^2\Big|m\frac{\partial a_\epsilon}{\partial\epsilon}a_\epsilon^{m-1}\Big|^2\Big]\Big\rangle \\ &\qquad\times\frac{b^2}{4\delta^2}\frac{(Q_s u_\epsilon)^{2m}}{2^m(\delta - Q_s)^2\dots(\delta - mQ_s)^2} \\ &= K^{-1}\epsilon_{x\beta}J_x\frac{b^2}{4\delta^2}\Big[m!\epsilon_\epsilon^m - m^2(m-1)!\epsilon_\epsilon^m\frac{J_\epsilon}{J_x}\Big] \\ &\qquad\times\frac{(Q_s u_\epsilon)^{2m}}{2^m(\delta - Q_s)^2\dots(\delta - mQ_s)^2} \end{aligned}$$

$$= K^{-1}\epsilon_{x\beta}J_x\frac{b^2}{4\delta^2}\left[1+m\frac{J_\epsilon}{J_x}\right]\frac{m!\alpha^m Q_s^{2m}}{2^m(\delta-Q_s)^2\dots(\delta-mQ_s)^2}\ . \tag{54}$$

Putting $\delta + mQ_s = 0$ in the coefficient of $(\delta - mQ_s)^{-2}$,

$$\begin{aligned}
W_m^2 &\simeq \frac{m!\alpha^m Q_s^{2m}}{2^m(\delta+Q_s)^2\dots(\delta-(m-1)Q_s)^2}\left(1+m\frac{J_\epsilon}{J_x}\right)\\
&= \frac{m!\alpha^m Q_s^2}{2^m(m-1)^2(m-2)^2\dots1^2}\left(1+m\frac{J_\epsilon}{J_x}\right)\\
&= \frac{m!\alpha^m\delta^2}{2^m(m!)^2}\left(1+m\frac{J_\epsilon}{J_x}\right)\\
&= \left(1-m\frac{J_\epsilon}{J_x}\right)\frac{\alpha^m\delta^2}{2^m m!}\ .
\end{aligned} \tag{55}$$

Once again, using Eq. (29) yields the same result

$$\begin{aligned}
W_m^2 &= \left[1-\frac{J_\epsilon}{J_x}\left(m-\alpha\frac{I_{m-1}(\alpha)}{I_m(\alpha)}\right)\right]e^{-\alpha}I_m(\alpha)\delta^2\\
&\simeq \left[1-\frac{J_\epsilon}{J_x}\left(m-\frac{\alpha^2}{2(m-1)}\right)\right]\frac{\alpha^m}{2^m m!}\delta^2\\
&\simeq \left(1-m\frac{J_\epsilon}{J_x}\right)\frac{\alpha^m\delta^2}{2^m m!}\ .
\end{aligned} \tag{56}$$

# 4 NUMERICAL RESULTS

## 4.1 SIDEBANDS WITH $m > 0$

### 4.1.1 Enhancement formula

Fig. 1 shows a graph of polarization vs. energy measured at SPEAR.[3] In this section we shall compare the above results with some data from this graph, specifically, the horizontal betatron resonance $\nu = 3 - Q_x$ at 3.65 GeV and its sideband $\nu = 3 + Q_x - Q_s$. The resonance widths are obtained from Fig. 2 (which is a subset of Fig. 1 containing the relevant resonances) by measuring the interval in which $P/P_0 < 50\%$, where $P_0 = 8/(5\sqrt{3}) \simeq 92.4\%$. We find that the widths are approximately 4.1 – 6.4 MeV for the resonance $\nu = 3 - Q_x$ and 1.5 MeV for the sideband $\nu = 3 + Q_x - Q_s$. The values of the relevant parameters used in

Table 2: Resonance widths

| A. Resonance widths in MeV | |
|---|---|
| $\nu = 3 + Q_x$ | 4.1 - 6.4 |
| $\nu = 3 - Q_x - Q_s$ | 1.5 |
| **B. Values for $W_0/W_1$ in Fig. 2.** | |
| Expt. | 2.7 - 4.3 |
| Theory (Eq. (29)) | 4.9 |
| Theory (Eq. (21)) | 8.5 |

the theoretical fit (Eq. (29)) are the same as the ones used to calculate $\alpha$ in Table 1. so $\nu = \gamma a = 8.28$, $Q_s = 0.045$, $\alpha = 2.8 \times 10^{-2}$, $J_\epsilon = 2$ and $J_x = 1$. The results are given in Table 2. The result obtained by using only Eq. (21) is also given. The second sideband resonance $\nu = 3 - Q_x - 2Q_s$ could not be fitted by this method because it is so narrow that $P/P_0$ does not drop below 50% in the experimental graph.

#### 4.1.2 SMILE program

A simplified version of the SMILE program. called SMILE2. which calculates only the leading order contributions to the spin resonances (in powers of the emittances. as defined in the previous section). was used to fit the above resonances in the data. Using SMILE2. it is possible to fit both the $\nu = 3 - Q_x - Q_s$ and $\nu = 3 + Q_x - 2Q_s$ sidebands. The result is shown in Fig. 3. and it also agrees with the data. A model storage ring with the approximate properties of SPEAR was used to fit the width of the first order betatron resonance. The widths of the sideband resonances then followed without further adjustment of the model.

In this context, one should note that both SMILE and SMILE2 require some imperfections in the storage ring model in order to yield *any* resonances — they give absolute resonance widths. from which the ratios of the widths of distinct resonances can be calculated. The enhancement formula Eq. (29) gives the ratios of resonance widths, but not their absolute values. These are obtained by multiplying the ratio $W_m/W_0$ by the strength of the first order resonance. and this again requires a model of lattice imperfections.

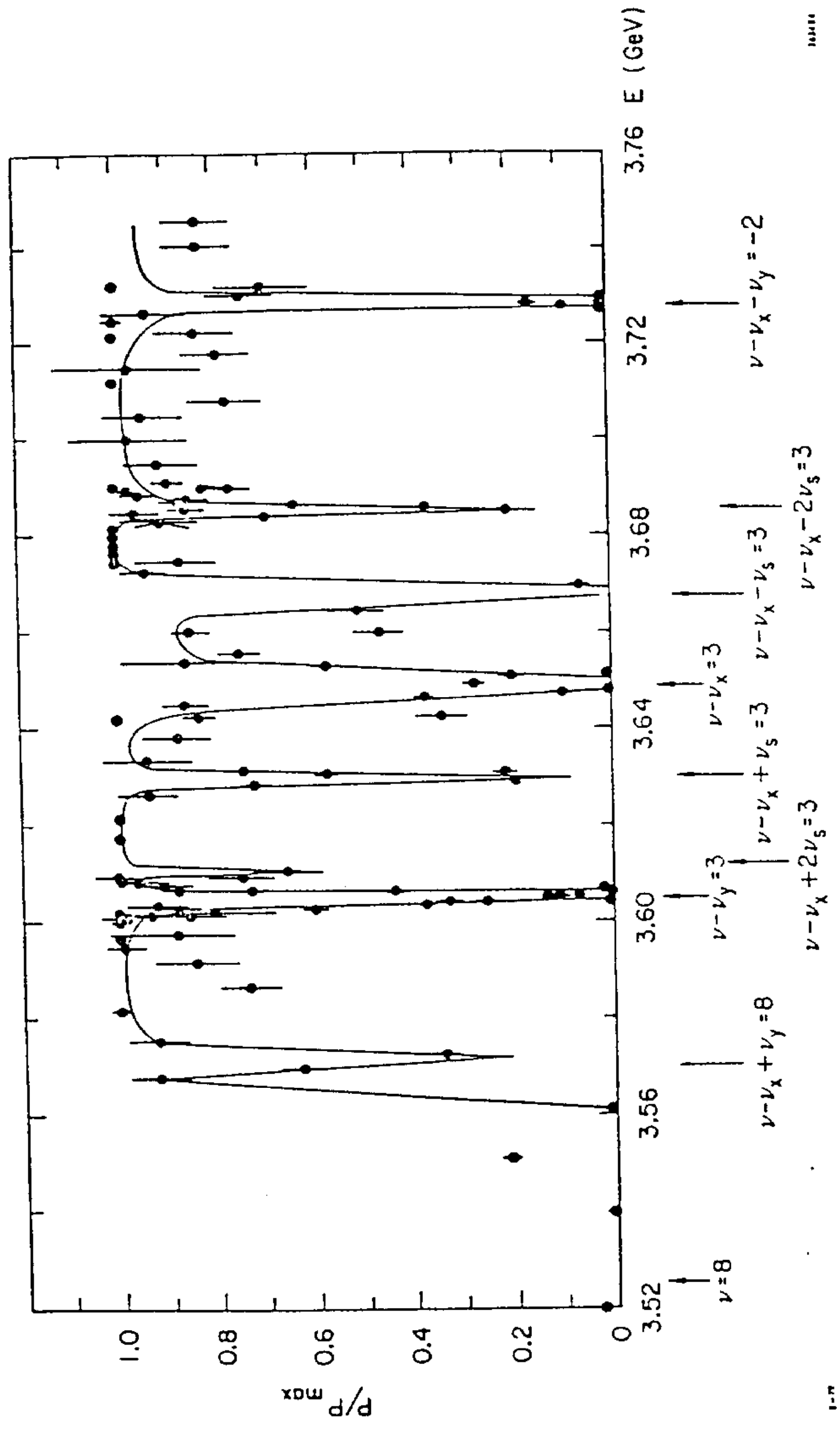


Figure 1: Polarization measurements at SPEAR (from Ref. [3])

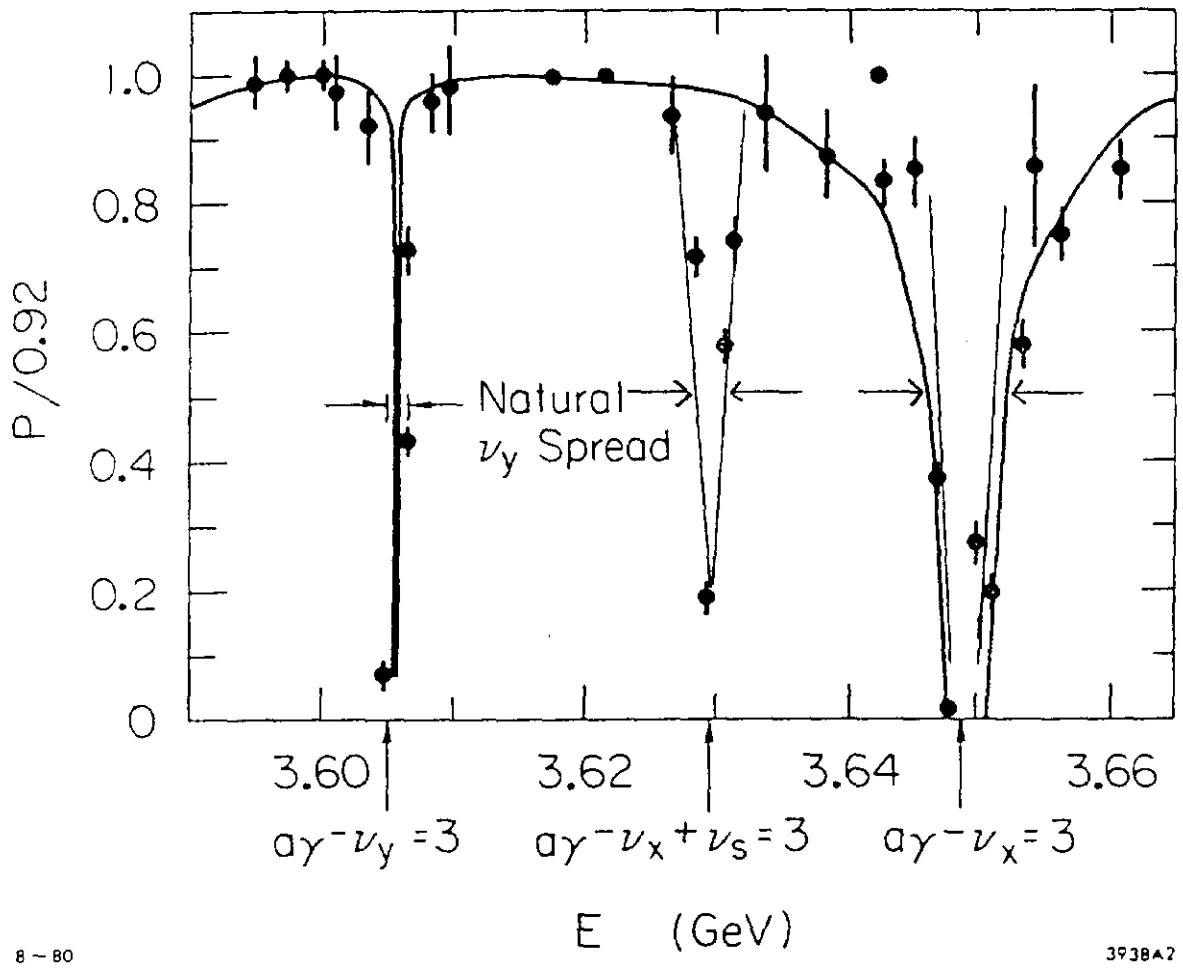


Figure 2: Polarization measurements at SPEAR (from Ref. [3])

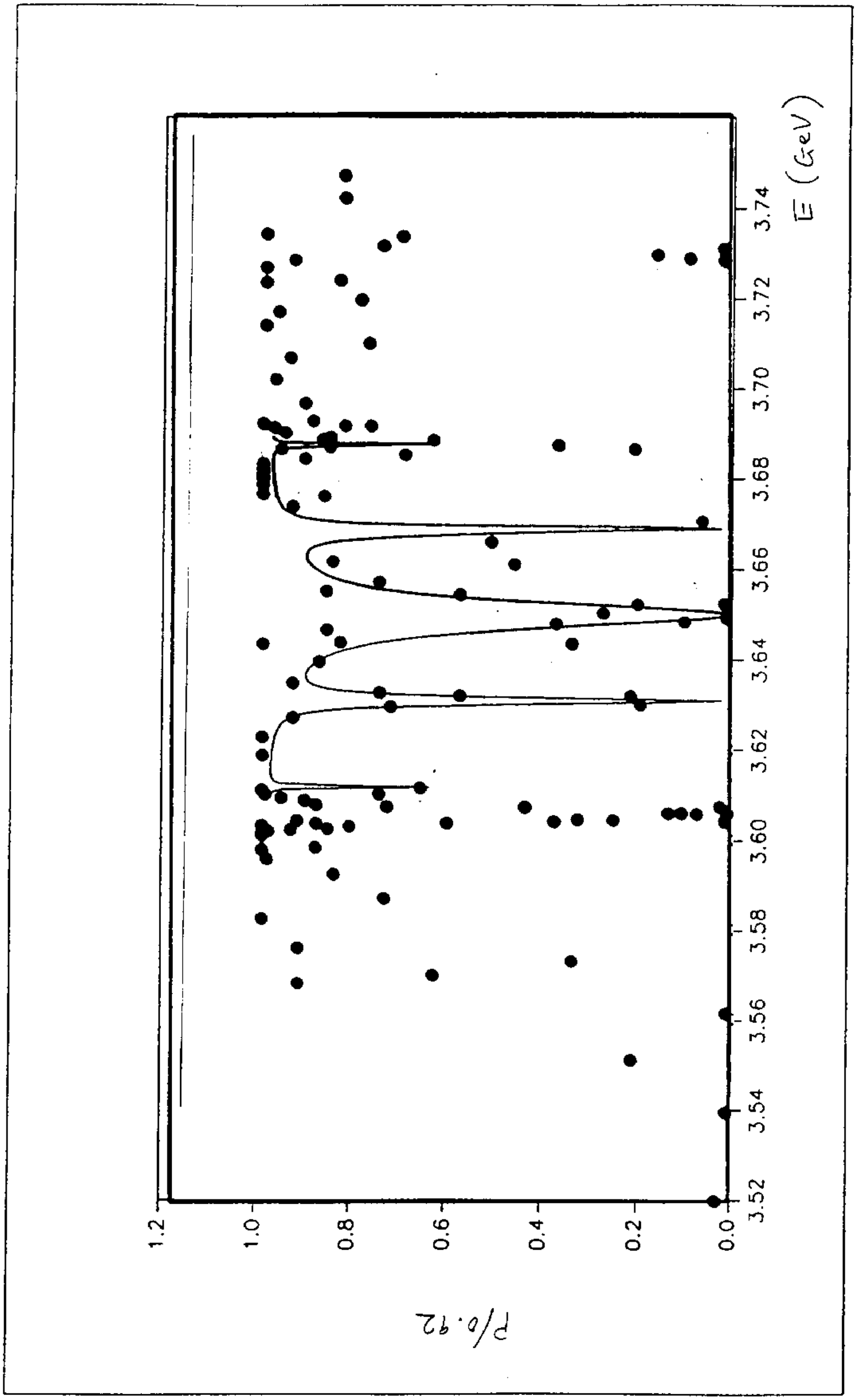


Figure 3: Theoretical fit to SPEAR polarization measurements using SMILE2.

### 4.2 SIDEBANDS WITH $m < 0$

Eq. (32) predicts that the sidebands with $m < 0$ should have the same widths as their counterparts with $m > 0$. This point is not mentioned in Ref. [6], and a formula is not presented for the $m < 0$ sidebands. There is not enough experimental data to test this prediction quantitatively, and in addition there are other complications, both experimental and theoretical.

First, it is reported in Ref. [3] that the polarization measurements were not all made in the same run, and the machine parameters, such as the tunes and the closed orbit, varied between runs. Thus the resonance widths cannot all be obtained from one model. Second, the data around the region of the sideband resonance $\nu = 3 - Q_x - Q_s$ is sparse, and so the resonance width is difficult to estimate accurately. Even the width of the first order resonance $\nu = 3 - Q_x$ is difficult to estimate accurately. With regard to the second sidebands $\nu = 3 - Q_x \mp 2Q_s$, the resonance $\nu = 2Q_y - 2$ is close to the resonance $\nu = 3 - Q_x - 2Q_s$ at 3.685 GeV, so there are really two resonances there, and the resonance $\nu = 2Q_y - 2$ is not treated in the above calculation. Further, many terms have been neglected in the above calculations such as the r.h.s. of Eq. (40), and they are not always symmetric about the parent resonance. A few such integrals will be calculated below, to demonstrate the asymmetry of their effect on the resonances $\delta \mp mQ_s = 0$.

The version of the SMILE program used above, called SMILE2, is a simplified version which calculates only the leading order terms considered above for each resonance, and also neglects interferences between resonances. It also predicts that the sidebands should be symmetric about the first order resonance. The full SMILE program, called SMILE1, includes non-leading terms as well, but requires much more time and storage space for its computations. An example of such non-leading terms is given in the appendix below. In addition, the inclusion of non-leading integrals introduces more adjustable parameters into the calculation. Hence a better numerical representation of SPEAR is required. Work on the above matters is in progress.

## 5 CONCLUSIONS

It has been shown that Yokoya's formula[1] for synchrotron sidebands of a first order betatron spin resonance together with some terms he neglected (Eq. (29)), and the SMILE spin integrals[2] (Eqs. (45), (52) and (56)), are equivalent. There are numerous caveats to the previous statement; in particular it was assumed that $Q_s \ll 1$, and only the leading contribution (in powers of the beam emittances) to each resonance was retained. The latter approximation is not very good for HERA and LEP, and so one expects that higher order terms are required in the power series in Ref. [2], and the terms in $\sqrt{1-|\zeta|^2}$ in Eq. (5) are required in the formalism of Ref. [1]. Both SMILE2 and Ref. [1] also neglect interferences between resonances.

One can also show that the two formalisms yield the same result for synchrotron spin resonances centered on an integer. The proof requires the same approximations made for the case of sidebands of a betatron spin resonance, and involves similar calculations, hence it will not be shown explicitly here.

The formula Eq. (29) was used to fit certain data from SPEAR polarization measurements, specifically the ratio of the widths of the resonances $\nu = 3 + Q_x$ and $\nu = 3 + Q_x - Q_s$. It was also shown that the SMILE2 program could fit the above resonances, including in addition the $\nu = 3 + Q_x - 2Q_s$ resonance. However, the above theoretical formulas also predict equal width for the resonances $\nu = n + Q_x \pm mQ_s$, but the experimental data do not necessarily support this conclusion. The full SMILE1 program can handle this case in principle (i.e. a more comprehensive range of spin integrals, and including interference terms between integrals), but a more detailed knowledge of SPEAR is required, in particular a more detailed model of the lattice, and a more sophisticated model of lattice imperfections.

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## A ADDITIONAL SPIN INTEGRALS

Here we shall calculate some additional spin integrals which were neglected in the main text. The goal there was to prove the equivalence of the formalisms in Refs. [1] and [2]. The approximations in the main text (the neglect of various spin integrals) led to the prediction that the resonances $\delta \pm mQ_s = 0$ should have equal width. Here we shall show that some of the terms which were neglected above can cause the resonances $\delta - mQ_s = 0$ and $\delta - mQ_s = 0$ to have unequal width. The terms considered in the example chosen below come from the $\sqrt{1-|\zeta|^2}$ factor in Eq. (5).

Let us calculate the contribution of a term such as the r.h.s. of Eq. (40). It contributes to $\zeta_3$. We shall call the previous solution $\zeta_3^{(1)}$ and the new term $\zeta_3^{(2)}$. Then

$$
\begin{aligned}
\zeta_3^{(2)} &= \frac{i}{2}\int_{-\infty}^{\theta} d\theta'\, \epsilon\vec{\omega}_\epsilon.\vec{k}_0 \Big[\int_{-\infty}^{\theta'} d\theta''\, \epsilon\vec{\omega}_\epsilon.\vec{k}_0^* \int_{-\infty}^{\theta''} d\theta'''\, x\vec{\omega}_x.\vec{k}_0 \\
&\qquad\qquad + \int_{-\infty}^{\theta'} d\theta''\, x\vec{\omega}_x.\vec{k}_0 \int_{-\infty}^{\theta''} d\theta'''\, \epsilon\vec{\omega}_\epsilon.\vec{k}_0^* \Big] \\
&= \frac{i}{2}\int_{-\infty}^{\theta} d\theta'\, \epsilon\vec{\omega}_\epsilon.\vec{k}_0 \Big(\int_{-\infty}^{\theta'} d\theta''\, \epsilon\vec{\omega}_\epsilon.\vec{k}_0^*\Big)\Big(\int_{-\infty}^{\theta'} d\theta'''\, x\vec{\omega}_x.\vec{k}_0\Big) \; .
\end{aligned}
\tag{57}
$$

We have included another integral because it is very similar, and the sum of the two integrals is actually easier to calculate than the individual terms. Note that we now need expressions not only for $\vec{\omega}_x.\vec{k}_0$ and $\vec{\omega}_\epsilon.\hat{n}_0$, but also $\vec{\omega}_\epsilon.\vec{k}_0$ and $\vec{\omega}_\epsilon.\vec{k}_0^*$. We approximate

$$
\vec{\omega}_\epsilon.\vec{k}_0 \simeq a_+ e^{i\delta'\theta} \; , \qquad \vec{\omega}_\epsilon.\vec{k}_0^* \simeq a_- e^{-i\delta'\theta} \; , \tag{58}
$$

where $\delta' = \nu - n'$ and $n'$ is the most singular harmonic in the above functions. Then, making the previous approximations for the other functions, and keeping only terms which contribute to the resonance $\delta + 2Q_s = 0$,

$$
\begin{aligned}
\zeta_3^{(2)} &\propto -\frac{i}{2}\int_{-\infty}^{\theta} \frac{a_+}{\sqrt{2}} e^{i(Q_s-\delta')\theta'} \Big(\frac{a_-}{\sqrt{2}(Q_s-\delta')} e^{i(Q_s-\delta')\theta'}\Big)\Big(\frac{b}{\sqrt{2}\delta} e^{i\delta\theta'}\Big) \\
&= \frac{1}{2}\frac{a_- a_- b}{2^{3/2}} \frac{e^{i(\delta+2Q_s)\theta}}{\delta(\delta+2Q_s)(\delta'-Q_s)} \; .
\end{aligned}
\tag{59}
$$

We have omitted constants of proportionality which are exactly the same as in $\zeta_3^{(1)}$. The above solution must be added to $\zeta_3^{(1)}$ coherently, i.e. *before* differentiating and squaring to obtain $\gamma(\partial\hat{n}/\partial\gamma)^2$. The result is

$$
\zeta_3^{(1)} + \zeta_3^{(2)} \propto \frac{b}{\sqrt{2}\delta} \frac{e^{i(\delta+2Q_s)\theta}}{2(\delta+2Q_s)} \left[\frac{(Q_s u_\epsilon)^2}{\delta+Q_s} + \frac{1}{2}\frac{a_- a_-}{\delta'-Q_s}\right] \; . \tag{60}
$$

The contribution to $\langle \rho^{-3}\gamma(\partial\hat{n}/\partial\gamma)^2\rangle$ therefore changes to

$$
\left\langle \frac{1}{\rho^3}\, \frac{\partial\zeta_3}{\partial\epsilon}^2 \right\rangle \longrightarrow \left\langle \frac{1}{\rho^3}\, \frac{\partial\zeta_3^{(1)}}{\partial\epsilon}^2 \right\rangle \left[1 + \frac{1}{2}\frac{a_- a_-}{(Q_s u_\epsilon)^2}\frac{\delta+Q_s}{\delta'-Q_s}\right]^2 \; . \tag{61}
$$

The contribution to the resonance $\delta - 2Q_s = 0$, on the other hand, is

$$
\zeta_3^{(1)} + \zeta_3^{(2)} \propto \frac{b}{\sqrt{2}\delta} \frac{e^{i(\delta-2Q_s)\theta}}{2(\delta-2Q_s)} \left[\frac{(Q_s u_\epsilon)^2}{\delta-Q_s} + \frac{1}{2}\frac{a_- a_-}{\delta'-Q_s}\right] \; . \tag{62}
$$

The proportionality factor is the same as before. In the main text, the strengths $W_{\pm m}$ of the resonances $\delta \pm mQ_s = 0$ were unchanged by the transformation $Q_s \to -Q_s$ (only the resonance denominators changed), and so the resonance widths were equal. Here, the terms in $a_- a_-$ have opposite signs in Eqs. (60) and (62), and so the resonances $\delta \pm 2Q_s = 0$ no longer have equal width.

We therefore see that the additional terms in the solution for $\zeta$ can cause the resonances $\delta \pm mQ_s = 0$ to have unequal width. The example chosen involved the $\sqrt{1 - |\zeta|^2}$ term in Eq. (5), and applied only to the resonance $\delta \pm 2Q_s$, but the conclusion obviously holds more generally. Unfortunately, it is also necessary to introduce additional lattice dependent parameters such as $a_-$ and $a_-$.

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