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POSITION – DEPENDENT FRICTION IN QUANTUM MECHANICS

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Abstract: The quantum description of motion of a particle subjected to position-dependent frictional forces is presented. The two cases are taken into account: a motion without external forces and in the harmonic oscilator field. As an example, a frictional barrier penetration is considered.

1. INTRODUCTION

A study of frictional phenomena in quantum mechanics has had its roots in investigation of Langevin systems. A description of a motion of a Brownian particle requires the introduction of a frictional force, the term quite familiar in the field of the classical dynamics but somewhat embarrassing in quantum mechanics. Since the problem of a quantum description of a system subjected to dissipative as well as external and stochastic forces has no unambiguous solution, a number of methods developed in the last four decades is quite handsome /for review see Hasse /1975/, Messer /1979/, Dekker /1981/ /. In the Langevine equation describing a Brownian particle motion a friction force is proportional to the velocity, with a constant coefficient, and in most of these methods no other possibilities have been taken into account. As an exception, Remaud and Hernandez /1980/ consider a time-dependent friction coefficient.

In many physical problems, concepts concerning a constant friction coefficient do not suffice. In particular, friction is the term widely used in descriptions of heavy-ion collisions /mainly in the classical manner/. Frictional forces depend on position in this case and its range is comparable with the nuclear radius. Hahn and Hasse /1984/ investigated heavy-ion collisions quantum-mechanically using the realistic frictional forces

/with the position dependent strength/ but that work was purely numerical and required an immense computational effort.

The aim of this paper is to show what is to be expected if one allows a friction coefficient to vary with position. General formulae for the wave pocket time-evolution, without external forces, on the basis of Kanai's model are presented in §2. They are applied to the case of viscous barrier penetration in §3. §4 is devoted to the investigation of the wave pocket motion in the field of the damped harmonic oscilator from the Kanai's, as well as Kostin's model point of view.

2. FREE PARTICLE IN A VISCOUS MEDIUM

Let us assume that the classical dissipative force acting on a particle moving in an arbitrary external field $V(x)$ is proportional to the velocity of this particle: $F = -m\, f(t)\, v$, where m is the constant particle mass /for simplicity we restrict our considerations to one dimension/. $f(t)$ can be an arbitrary function of time. Following Kanai /1948/ /also Cardirola /1941/, Denman /1966/ /, quantisation of this system leads to the Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\, \varphi(t) + V(x)\, \varphi^{-1}(x) \right] \psi(x,t) \, , \qquad /1/$$

where $\varphi(t)$ is given by

$$\varphi(t) = \exp\left(-\int f(t)\,dt\right) . \qquad /2/$$

The wave pocket solution for the force-free motion /$V(x) = 0$/ with the initial condition

$$\psi(x,0) = A \exp\left(-\frac{x^2}{2a^2} + ik_0 x\right) , \qquad /3/$$

where a is the initial width of the pocket, $\hbar k_0$ - initial momentum and A - the normalisation constant, is easy to obtain by the method of integral transformation. It has the form

$$\psi(x,t) = \frac{A}{\sqrt{1+\frac{\hbar\tau(t)}{ma^2}i}} \exp\left[-\frac{x^2 - 2ia^2k_0x + i\frac{\hbar a^2k_0^2}{m}\tau(t)}{2a^2\left(1+i\frac{\hbar}{ma^2}\tau(t)\right)}\right] . \qquad /4/$$

The function $\tau(t)$ is given by

$$\dot{\tau}(t) = \varphi(t) . \qquad /5/$$

Since the density distribution is in the form

$$|\psi(x,t)|^2 = \frac{|A|^2}{\sqrt{1+\left(\frac{\hbar\tau(t)}{ma^2}\right)^2}} \exp\left[-\frac{\left(x-\frac{k_0\hbar}{m}\tau(t)\right)^2}{a^2\left[1+\left(\frac{\hbar\tau(t)}{ma^2}\right)^2\right]}\right] , \qquad /6/$$

the centroid of the pocket, moving like a classical particle, is given by

$$\langle x\rangle = \frac{k_0 t}{m}\tau(t) \equiv \eta\,\tau(t)\,. \qquad /7/$$

In this way, we have solved the problem of a free particle moving in a viscous medium, determined by a time-dependent friction coefficient $f(t)$. However, our task is to deal with position-dependent form-factors $f(x)$. For this purpose we make the approximation

$$f(x) \longrightarrow f(\langle x\rangle)\,. \qquad /8/$$

Now, the friction coefficient is merely a function of time. The approximation /8/ means that the whole pocket is subjected to the same value of frictional coefficient, determined by the classical motion of the centroid of the pocket.

Relations /2/, /5/, /7/ and /8/ imply that the function $\tau(t)$ fulfils the equation

$$\ddot{\tau} + f(\eta\tau(t))\,\dot{\tau} = 0\,. \qquad /9/$$

By these means, our problem resolves itself to the solution of eq. /9/. This equation is nonlinear, except the trivial case of the constant friction and the unphysical one when the friction form-factor depends lineary on position. In the next paragraph the exemplary solutions for typical physical problems are presented.

It is easy to realise that eq. /9/ is identical with the classical equation of motion of a damped free particle

It is the immediate consequence of the Ehrenfest theorem.

Up to now, we discussed the Kanai's method only. In §4 a few remarks will be made about the motion of a free particle from the Kostin's model point of view.

3. EXEMPLES AND DISCUSSION

In this paragraph we are dealing with the physically most interesting case of friction confined to a finite region. At first, we take $f(x)$ in the form

$$f(x) = \gamma\left[\theta(x) - \theta(x - x_0)\right], \quad x_0 \geqslant 0, \quad \gamma = \mathrm{const} \qquad /10/$$

where $\theta(x)$ is the Heaviside's function. In another words, friction is switched on at $x = 0$ and switched off at $x = x_0$. Our problem is to find such T that

$$f(t) = \gamma\left[\theta(t) - \theta\left(t - T(x_0)\right)\right]. \qquad /11/$$

From /2/ and /5/ we obtain

$$\tau(t) = \begin{cases} t & t \leqslant 0 \\ \frac{1}{\gamma}\left(1 - e^{-\gamma t}\right) & 0 < t < T \\ e^{-\gamma T}\left(t - T - \frac{1}{\gamma}\right) + \frac{1}{\gamma} & t \geqslant T \end{cases} \qquad /12/$$

where the integration constants have been chosen to satisfy the initial conditions

$$\tau(0)=0\ ,\ \dot{\tau}(0)=1\ . \qquad /13/$$

Since $x_0=\langle x\rangle(t=T)=\eta\tau(T)$, it is easy to calculate T:

$$T=\frac{x_0}{\eta}-\frac{1}{\gamma}\ln\left(1-\frac{\gamma x_0}{\eta}\right) \qquad /14/$$

and to obtain

$$\tau(t)=\left[1-\frac{\gamma x_0}{\eta}\right]t+\mathrm{const} \qquad /15/$$

for $t>T$.

The barrier penetration by the pocked results in slowing down of the centroid motion /in comparison to the friction-free case/ and slower spreading of the width of the pocket. However, if friction reaches the critical value / $\gamma x_0=\eta$ / the pocket never leaves the viscous region in finite time. The pocket eventually stops and ceases to spread /Buch and Denman /1974/ /. When friction approaches the critical value the uncertainty principle is violated. This difficulty has been already widelly discussed /Brittin /1950/, Senitzky /1960/, Messer /1978/ /.

Let us consider now a barrier in the form:

$$f(x)=\gamma e^{-\alpha|x|} \qquad /16/$$

Solution of the equation

$$\ddot{\tau}+\gamma e^{-\alpha\eta|\tau|}\dot{\tau}=0\ , \qquad /17/$$

with the initial conditions /13/, on the positive half-axis is

$$t=\frac{\alpha\eta}{\alpha\eta-\gamma}\left[\tau+\frac{1}{\alpha\eta}\ln\left(1+\frac{\gamma}{\alpha\eta}\left(e^{-\alpha\eta\tau}-1\right)\right)\right]. \qquad /18/$$

For $t \gg 0$:

$$\tau=\left(1-\frac{\gamma}{\alpha\eta}\right)t+\mathrm{const}\;. \qquad /19/$$

Thus, for a large time, the result is similar to /15/. The final energy is

$$\langle E\rangle=\frac{\hbar^2k_0^2}{2m}\,(\dot\tau)^2\xrightarrow[t\to\infty]{}\frac{\hbar^2k_0^2}{2m}\left(1-\frac{\gamma}{\alpha\eta}\right)^2, \qquad /20/$$

where $\hbar k_0$ stands for the average momentum at $t = 0$.

4. THE DAMPED HARMONIC OSCILLATOR

In the Kanai's method, the Hamiltonian for the damped harmonic oscillator has the form

$$H=-\frac{\hbar^2}{2m}\,\frac{\partial^2}{\partial x^2}\,\varphi(t)+\frac{m}{2}\,\Omega^2x^2\,\varphi^{-1}(t) \qquad /21/$$

where Ω is the classical frequency and $\varphi(t)$ is given by /2/. We will seek the centroid $\langle x\rangle$ and the square width χ of the distribution

$$|\psi|^2 = (2\pi\gamma)^{-1/2} \exp\left[-(x-\langle x\rangle)^2/2\gamma\right] . \qquad /22/$$

Due to the approximation /8/, from the Ehrenfest theorem we obtain:

$$\langle \ddot{x}\rangle + f(\langle x\rangle)\langle \dot{x}\rangle + \Omega^2\langle x\rangle = 0 , \qquad /23/$$

as an analogue to the eq. /9/. The moments of the pocket can be calculated /Hasse /1978/ / from the equation for the time-evolution of an arbitrary operator A:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar}\left[H, A\right] . \qquad /24/$$

The function $\gamma(t)$ is the second moment. Expression /24/ leads to the equation /obtained by Remaud and Hernandez /1980/ for the time-dependent friction/:

$$\ddot{u} + f(\langle x\rangle)\dot{u} + \Omega^2 u = \frac{1}{m^2} \exp\left[-2\int f(\langle x\rangle)\, dt\right] u^{-3} \qquad /25/$$

where

$$u(t) = \sqrt{\frac{2\gamma}{\hbar}} .$$

Equations /23/ and /25/ determine the density /22/ completely. However, in general, it is not easy to solve eq. /25/ analitically. We will do this for the case of rectangular barrier /10/.

Eq. /23/ for the barrier /10/, with the initial conditions

$$\langle x\rangle(t=0)=0\ .$$

$$\langle \dot{x}\rangle(t=0)=p_0/m\ , \qquad /26/$$

has the following solution

$$\langle x\rangle=\begin{cases}\frac{p_0}{m\Omega}\sin\Omega t & t\leq 0\\ \frac{p_0}{m\omega}e^{-\gamma t/2}\sin\omega t & 0<t<T\\ C_1\cos\Omega t+C_2\sin\Omega t & t\geqslant T\end{cases} \qquad /27/$$

where

$$\omega^2=\Omega^2-\gamma^2/4$$

and

$$C_1=\frac{p_0}{m\omega}e^{-\gamma T/2}\left(\sin\omega T\cos\Omega T-\frac{\omega}{\Omega}\sin\Omega T\cos\omega T+\frac{\gamma}{2\Omega}\sin\omega T\sin\Omega T\right)$$

$$C_2=\frac{p_0}{m\omega}e^{-\gamma T/2}\left(\sin\omega T\sin\Omega T-\frac{\gamma}{2\Omega}\sin\omega T\cos\Omega T+\frac{\omega}{\Omega}\cos\omega T\cos\Omega T\right)\ ;$$

T is given by

$$x_0=\frac{p_0}{m\omega}e^{-\gamma T/2}\sin\omega T\ .$$

The function $\langle x\rangle(t)$ and its first derivative are continuous for all t.

Eq. /25/ has been solved by Remaud and Hernandez /1980/ for the constant friction coefficient. In our case we have

$$\ddot{u}+\Omega^2u=\frac{1}{m^2}u^{-3} \qquad t<0\ ,\ t>T \qquad /28a/$$

$$\ddot{u} + \gamma\dot{u} + \Omega^2 u = \frac{1}{m^2} e^{-2\gamma t} u^{-3} \qquad 0 < t < T \qquad /28b/$$

Assuming that for $t < 0$ we can restrict ourselves to a constant solution, we have

$$u = \frac{1}{\sqrt{m\Omega}} \quad , \qquad t \leq 0 \qquad /29/$$

Defining the new function $W(t)$ by

$$u(t) = \frac{1}{\sqrt{m}} W(t) \exp\left(-\tfrac{1}{2}\gamma t\right) \qquad /30/$$

one can transform /28b/ to the form

$$\ddot{W} + \omega^2 W = W^{-3} \quad . \qquad /31/$$

The general solution is

$$W^2 = \omega^{-1}\left[\left(1 + A^2 + B^2\right)^{1/2} + A\cos 2\omega t + B\sin 2\omega t\right] \quad . \qquad /32/$$

Constants A and B can be determined from the continuity conditions at $t = 0$. Finally, we have

$$u(t) = \sqrt{\frac{\Omega}{m}}\, \omega^{-1} \sqrt{1 - \frac{\gamma}{2\Omega}\sin(2\omega t + \theta_1)}\, \exp\left(-\tfrac{1}{2}\gamma t\right) , \quad 0 < t < T \qquad /33/$$

where

$$\theta_1 = \operatorname{arctg}\left(-\frac{\gamma}{2\omega}\right) .$$

Thus, switching on the dissipative mechanism generates oscilations. They are damped and the width of the pocket tends to zero if friction continues to persist.

In the third region /t > T/ the solution is more complicated:

$$u(t)=\frac{1}{\sqrt{m\Omega}}\left[\frac{A_1^2+B_1^2+1}{2B_1}+\frac{B^2}{\sqrt{A^2+B^2}}\sin(2\Omega t+\theta_2)\right]^{1/2}, \quad t>T \qquad /34/$$

where

$$A=-A_1\sin 2\Omega T-\frac{\cos 2\Omega T}{2B_1}\left(A_1^2-B_1^2+1\right)$$

$$B=A_1\cos 2\Omega T-\frac{\sin 2\Omega T}{2B_1}\left(A_1^2-B_1^2+1\right)$$

$$A_1=-\frac{\gamma}{2\omega^2}\left(\omega\cos(2\omega T+\theta_1)-\frac{\gamma}{2}\sin(2\omega T+\theta_1)+1\right)$$

$$B_1=\frac{\Omega^2}{\omega^2}\left[1-\frac{\gamma}{2\Omega}\sin(2\omega T+\theta_1)\right]e^{-\gamma T}$$

$$\theta_2=\mathrm{arc\,tg}(A/B) .$$

It must be emphasized that after leaving the frictional region by the pocket, oscilations can never die down, regardless of the size of this region and the strenght of the frictional force. The larger the value of γT one assumes the larger the amplitude of the final oscilations becomes.

The question arises whether other models predict the similar phenomenon. In Hasse's method /eg. Albrecht /1975/ /, Hasse /1978/ /, with the constant friction coefficient, oscilations are present but they seem to behave unphysically, as it has been reported by Remaud and Hernandez /1980/. We will discuss now the Kostin's approach /Kostin /1972/, Kan and Griffin /1974/ /.

Kostin's method takes friction into account by inclusion an additional, nonlinear term into the Hamiltonian:

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\Omega^2 x^2 - \frac{i\hbar}{2} f(\langle x\rangle)\left(\ln\frac{\psi}{\psi^*} - \left\langle \ln\frac{\psi}{\psi^*}\right\rangle\right). \qquad /35/$$

The wave pocket solution for the constant friction has been found by Kan and Griffin /1974/. The Kostin's model has an important superiority over Kanai's one because it gives the correct uncertainty relation. Our problem of the barrier penetration appears to be very simple in this formalism.

As above, we seek the density distribution /22/. Eq. /23/ with the solution /27/ is still valid. The width of the pocket can be obtained from

$$\ddot{u} + \Omega^2 u = m^{-2}u^{-3} \qquad t<0\ ,\ t>T$$

$$\ddot{u} + \gamma\dot{u} + \Omega^2 u = m^{-2}u^{-3} \qquad 0<t<T \qquad /36/$$

The constant solution

$$u = \frac{1}{\sqrt{m\Omega}}$$

is stable and fulfils both of equations /36/. Thus, the presence of the frictional force does not affect the shape of the pocket at all, in contradiction to the Kanai's model predictions.

Now, return us for a moment to the case of free motion, discussed in §§2 and 3. The Kostin's method has the pocket solution in which the width is given by the equation:

$$\ddot{u} + f(\langle x\rangle)\dot{u} = m^{-2}u^{-3}. \qquad /37/$$

Unfortunately, even for a constant friction coefficient eq. /37/ has no analitical solution. Hasse /1975/ has shown numerically that $u(t)$ behaves similarly to the width in Kanai's method. For small times both results are almost identical. Thus, if friction is limited to a small region, Kanai's and Kostin's models are in full agreement.

5. SUMMARY AND CONCLUSIONS

We have discussed the behaviour of a particle subjected to a dissipative force, whose strenght depends on the position operator. The problem has been treated in an approximate way: a position dependence has been transfered on a time dependence by taking the average value of the position operator. The two cases have been considered: a free particle, without external field, and a particle in a quadratic field.

The problem of a free motion is especially simple in the Kanai's approach. The one second order differential equation /9/ determines the wave pocket evolution completely. By means of this equation the problem of frictional barrier penetration has been solved. As a result of such a penetration, the pocket travels and spreads similarly to the friction-free case but with the diminished speed. Qualitatively, this outcome does not depend on the particular shape and hight of the barrier.

It has been argued that results of the Kostin's method are very similar though more difficult to obtain.

A study of the case of the harmonic oscilator requires an investigation of the system /23/, /25/. For the rectangular barrier /10/ the Kostin's model gives a very simple picture. The pocket does not change its shape during the pass through the barrier. The width remains fixed everywhere. Only the classical motion of the centroid of the pocket is affected by dissipative forces. Results of Kanai's model are entirely different. It predicts that, as the final outcome, the width of the pocket will oscilate with a constant frequency and an amplitude depended on the friction barrier parameters. The fact that this amplitude /as well as an equilibrium position/ grows exponentially with γT, is especially strange. In Hasse's model, in turn, oscilations have a different character: they are not damped in the frictional region. This comparison reminds us that the unique quantum theory of dissipative phenomena is far from completness.

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