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TOPICS IN RIEMANNIAN GEOMETRY *

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"Les services éminents qú a rendus et que rendra le Calcul Différentiel Absolu de Ricci et Levi-Civita ne doivent pas nous empêcher d'éviter les calculs trop exclusivement formels, oú les débauches d'indices masquent une réalité géométrique souvent trés simple"
E. Cartan (1925)

Avertissement

Nous avons rassemblé ici les notes de quatre cours que nous avons donnés au "5ᵉ Symposium Mathématique d'Abidjan: Géométrie différentielle et mécanique" sous le titre Eléments de Géométrie Riemannienne. Il ne s'agit pas d'une introduction à la Géométrie Riemannienne, mais bien d'éléments de cette géometrie choisis en fonction de l'auditoire et de ce que nous avons pensé être ses attentes. Ils sont répartis en quatre chapitres.

I: Riemannian metric on a differential manifold,

II: Curvature tensor fields on a Riemannian manifold,

III: Some classical functionals on a Riemannian manifold,

IV: Questions.

Ce dernier chapitre est bien spécial. Nous y avons regroupé une partie des "problèmes ouverts" exposés par différents conférenciers en liaison avec leurs propres cours ou communications au cours d'une séance consacrée à cet effet.

Remerciements

Nous remercions les organisateurs de nous avoir invité à donner ces cours et le Professeur Abdus Salam, Directeur des Centre International de Physique Theorique, où ces notes ont été rédigées ainsi que l'IAEA et l'UNESCO pour leur hospitalité.

I. RIEMANNIAN METRIC ON A DIFFERENTIAL MANIFOLD

I.1 Historical frame points

1827: Gauss "Theorema Egregium". By Euclidean Geometry one has known the notions of "arc length" and "measure of an angle" of curves on surfaces in an Euclidean space $\mathbb{R}^3$ a long time ago.

But the question whether there is for a surface (in $\mathbb{R}^3$) a notion of curvature depending only on the way the arclength is measured on the considered surface has been asked by C. Gauss. His investigations on this problem have been the very starting point of Riemannian Geometry by pointing the way towards the notion of abstract manifold even if only surfaces embedded into $\mathbb{R}^3$ are concerned in his own work.

His discoveries were announced in 1827 in his celebrated Theorema Egregium "the most Excellent Theorem" which asserts that there exists a measure of the curvature of a surface (embedded in $\mathbb{R}^3$) depending only on the notion of arclength of curves on the considered surface. In other words that curvature remains "unchanged by changes of shape which leave arc length unchanged". So there does exist an intrinsic geometry on a surface in $\mathbb{R}^3$, independent of the shape of this surface.

1830: Non–Euclidean Geometry. The discovery of a non–Euclidean Geometry by Boliai and Lobachevski (independently) was in the prolongation of the Gauss ideas.

1854: B. Riemann in his "Inaugural Address" at Göttingen pointed out the notion of "abstract manifold". He clarified what meant in this context arclength and generalized in an arbitrary dimension the ideas developed, sketched or only perceived by C. Gauss for surfaces in $\mathbb{R}^3$.

Later he elaborated an analytic function theory by using the properties of 2–dimensional "abstract manifolds" (called today Riemann Surfaces).

All those discoveries prove a great vitality of Geometry and its applications to other areas of mathematics in that period. The following famous mathematicians powerfully contributed by clarifying many–concepts which were being elaborated. H. Poincaré (Mechanics), E. Cartan, Killing (Differential Equations), S. Lie (Group Theory, Differential Equations).

The Italian School has illustrated itself by the formalisation of Differential Calculus ("Absolute Differential Calculus") on "abstract manifolds", initiated by G.C. Ricci at the end of the 19*th* century, and pursued by T. Levi–Civita until the beginning of the 20*th* century.

Nowadays Riemannian Geometry is oriented in various directions:

-A generalisation with the concept of Riemannian fibre bundles or Riemannian foliations with applications in Mathematical Physics and Global Analysis (Gauge theories, Differential Operator Theory).

-A best knowledge of the Geometry of manifolds, manifolds with prescribed curvatures (Einstein manifolds, manifolds with prescribed scalar curvature, manifolds with constant curvature, etc.).

-Deformations of the initial metric on manifolds (Riemannian Conformal Geometry and the famous Yamabe problem).

-Further applications in Mathematical Physics (Spin Geometry and Dirac Operators).

-Geometry of Riemannian submanifolds (minimal submanifolds).

-Complex Geometry and Analysis (Kähler manifolds, Riemann surfaces).

-Calculus of variations (Riemannian Functionals, minimal submanifolds, harmonic maps).

-Topology (characteristic classes, cohomology classes).

Note: We did not mention the generalisation to Semi–Riemannian Geometry (the considered metrics are assumed to be only non degenerate while Riemannian metrics are positive definite) although it has many important applications in Physics, Cosmology, Relativity, etc. Indeed, it will not be concerned here. The interested reader is referred to the excellent book [O'N] by B. O'Neill as well as the expositions by F. Allotey and F. Cagnac in this volume. Our lectures will be restricted to the somewhat narrow framework of Riemannian metrics, even if the stated results generally only require a slight modification to be adapted to Semi–Riemannian case.

I.2 Riemannian metrics on a manifold.

Let M be a smooth (C^∞) manifold of dimension n, n a positive integer. Let TM and T^*M denote its tangent and cotangent bundles, respectively.

2.1 Definition

A Riemannian metric (sometimes called metric tensor) is a smooth section g of the tensor algebra $T^*M \otimes T^*M$ of bilinear forms over M such that at each point x in M $g(x) \equiv g_x$ is symmetric and positive definite.

In other words a Riemannian metric on M is an assignment to each point x in M, an inner product on T_xM.

We denote with (M,g) a smooth manifold M equipped with a metric tensor g and we call it Riemannian manifold or *R.m.*

2.2 Do *R.m.* exist?

Before answering the question, let us prove the following

2.3 Proposition

Let $F: M_1 \to M_2$ be an immersion between two manifolds M_1 and M_2 and g_2 a Riemannian metric on M_2. Then $F^* g_2$ is a Riemannian metric on M_1.

Proof For every smooth map F one can easily check that $F^* g_2$ is a symmetric section of $T^* M_1 \otimes T^* M_1$ (that is a field of symmetric bilinear forms over M_1). Using the fact that F is an immersion one proves that $F^* g_2$ is positive definite (easy exercise left to the reader).

□

Thus, as soon as one can find a *R.m.* it would be easy to construct many others.

2.4 Examples

1) Let e be the Euclidean metric field over $\mathbb{R}^n$. Then $(\mathbb{R}^n, e)$ is a *R.m.*

2) Any submanifold M of $(\mathbb{R}^n, e)$ is a Riemannian manifold. (Use the immersion $i: M \hookrightarrow (\mathbb{R}^n, e)$).

For instance the unit sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$$

equipped with the metric $i^* e$ is a *R.m.*

The metric $i^* e$ is sometimes denoted with *can*. The Riemannian manifold (S^{n-1}, *can*) is called the canonical or standard $(n-1)$–sphere (in $\mathbb{R}^n$).

2.5 Proposition – Definition

Let $(M_1, g_1), (M_2, g_2)$ be two *R.m.*, and π_1 and π_2 be the projections of $M_1 \times M_2$ on M_1, M_2, respectively. Set

$$g = \pi_1^* g_1 + \pi_2^* g_2$$

Then g is a Riemannian metric on $M_1 \times M_2$.

The Riemannian manifold $(M_1 \times M_2, g)$ is called a Riemannian product.

Proof Exercise left to the reader.

2.6 The metric defined by Proposition (2.5) is sometimes denoted by $g = g_1 \times g_2$. More generally consider a positive function f on (M_2, g_2). One can easily prove that

$$g_f = \pi^* g_1 + (f \circ \pi_2)^2 \pi_2^* g_2$$

is a Riemannian metric on $M_1 \times M_2$. One denotes it with $g_f = g_1 \times_f g_2$ and calls it *warped product* of g_1 and g_2 and f is the *warping* function.

2.7 Theorem (Existence of Riemannian metrics on a manifold.)

On every differential manifold there exists a Riemannian metric.

Proof We outline two proofs of this theorem. Let M be a smooth manifold of dimenion n.

1st **proof:** By Whitney imbedding theorem [ST] there is N such that M may be embedded into $\mathbb{R}^N$ $(N \leq 2n+1)$. In particular this imbedding is an immersion $i: M \hookrightarrow (\mathbb{R}^N, e)$; hence, $i^* e$ is a Riemannian metric on M.

2nd **proof:** Let (U_i, φ_i) be an open covering of M. Then $\varphi_i : U_i \to (B^n, e)$ is a diffeomorphism, where (B^n, e) denotes an open ball in $\mathbb{R}^n$ equipped with the Euclidean metric; hence, $\varphi_i^* e$ is a Riemannian metric on U_i. Now, consider (f_i), a partition of unity subordinate to (U_i). Then $g = \sum f_i (\varphi_i^* e)$ is easily proven to be a metric tensor on M. For further details the reader is referred to [BO, p. 191 ff]

□

2.8 A manifold M may carry many Riemannian metrics; for instance, if g is a metric tensor on M and φ is a diffeomorphism from M onto M, then by Proposition (2.3) $\varphi^* g$ is a metric tensor on M. Whenever a metric g is specified on M, the tangent space at each point M is made into an Euclidean space by the inner product g_x. Hence, one is able to measure the angle of two intersecting curves on M at their common point M by measuring the angle of their tangents at M.

In the same fashion, we can define the length of a curve on M as follows. Let $t \mapsto \gamma(t)$, $a \leq t \leq b$, a smooth curve on (M, g) defined on an interval (a, b). Its length L is defined to be

$$L = \int_a^b g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)^{1/2} dt.$$

Notice that L is independent of the used parametrisation. Because of this invariance, one would like to use the simplest parametrisations to define L. One of them is obtained by means of *arclength parameter s* defined to be the length of curve γ from $\gamma(a)$ to $\gamma(t)$ and given by

$$s = L(t) = \int_a^t g\left(\frac{d\gamma}{dn}, \frac{d\gamma}{dn}\right)^{1/2} dn.$$

From these equalities, we deduce $\left(\frac{ds}{dt}\right)^2(t) = g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)(t) \equiv |\dot{\gamma}_t|^2$, for any parameter t, where $|\ |$ refers as to the norm on the tangent space at $\gamma(t)$ and $\dot{\gamma}$ is the *velocity vector field* along γ defined by

$$\dot{\gamma}_t(f) = \left(\frac{d\gamma}{dt}\right)_t (f) = \lim_{h \to 0} \frac{(f \circ \gamma)(h+t) - (f \circ \gamma)(t)}{h}$$

for any smooth function f on M such that $f \circ \gamma$ exists. In particular for $t = s$, one has $g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)(s) = |\dot\gamma_s|^2 = 1$. More generally, let $(x^1(t) \dots x^n(t))$ the local coordinates of $\gamma(t)$ in a local chart whose domain contains $\gamma(t)$. Since $\dot\gamma_t \in T_{\gamma(t)}M$, one has

$$\dot\gamma_t = \left(\frac{d\gamma}{dt}\right)_t = \sum_{\ell=1}^{n} \frac{dx^\ell}{dt}(t)\left(\frac{\partial}{\partial x^\ell}\right)_{\gamma(t)}$$

or more simply

$$\dot\gamma_t = \sum_\ell \dot x^\ell \partial_\ell = \dot x^\ell \partial_\ell$$

where we leave put $\frac{dx^\ell}{dt}(t) = \dot x^\ell$, $\left(\frac{\partial}{\partial x^\ell}\right)_{\gamma(t)} = \partial_\ell$ and from now on, summation for repeated indices is understood like in the last equality. Therefore

$$\left(\frac{ds}{dt}\right)^2(t) = g_{\gamma(t)}(\dot x^\ell \partial_\ell, \dot x^k \partial_k)$$
$$= g_{\gamma(t)}(\partial_\ell, \partial_k)\dot x^\ell \dot x^k$$

Setting

$$g_{\gamma(t)}(\partial_\ell, \partial_k) = g_{\ell k}$$

we have

$$ds^2 = g_{\ell k} dx^\ell dx^k.$$

2.9 Remark

The $g'_{\ell k}$'s are the coefficients of the matrix of the inner product induced by g on $T_{\gamma(t)}M$. Hence $[g_{\ell k}]$ is an invertible, symmetric, positive definite matrix. In particular, $\det[g_{\ell k}] > 0$. Let $[g_{\ell k}]^{-1}$ denote the inverse of $[g_{\ell k}]$. It is symmetric, positive definite. It defines an inner product on $T^*_{\gamma(t)}M$ for each t, and therefore, a smooth symmetric, positive definite section of the tensor algebra $TM \otimes TM$, which has sometimes been denoted with g^{-1}.

2.10 Let p, q two points in M. A piecewise C^1 curve from p to q is a curve γ defined on $[a, b]$ such that

i) $\gamma(a) = p$, $\gamma(b) = q$; ii) there is a subdivision $a < t_1 < t_2 < \dots < t_\ell < b$ so that $\gamma_{|[t_i, t_j]}$ is C^1.

Denote with $C(p, q)$ the set of piecewise C^1 curves from p to q. Set

$$d(p,q) = \inf_{\gamma \in C(p,q)} L(\gamma)$$

where

$$L(\gamma) = \int_a^b g(\dot\gamma_t, \dot\gamma_t)^{1/2} dt.$$

Then we have

2.11 Theorem

On a connected Riemannian manifold (M, g), the function $d : M \times M \to \mathbb{R}$, $(p, q) \mapsto d(p, q)$ is a distance, that is, for all p, q, r in M:

(i) $d(p, q) \geq 0$ and $d(p, q) = 0$ if and only if $p = q$;

(ii) $d(p, q) = d(q, p)$;

(iii) $d(p, q) + d(q, r) \geq d(p, r)$.

Furthermore the metric topology of M and its manifold topology agree.

Proof (Sketch): Since M is connected, it is arcwise connected; hence d is well-defined. Apart from "only if" part of (i), the statements (i), (ii) and (iii) are trivial to be checked. To prove that $d(p, q) > 0$ if $p \neq q$, one generally considers a local chart whose domain contains p and does not contain q (this is possible since M is Hausdorff space) and one then shows that $d(p, q) > 0$. This machinery is also used to prove the second part of the theorem. For details the reader is referred to [O'N, p. 134-136].

□

2.12 From a topological point of view, a Riemannian manifold (M,g) looks like a metric space. From the geometric point of view, it is then possible to introduce on (M,g) some familiar concepts from Euclidean Geometry, such as angles, distances, volumes and so on; furthermore, the existence of the metric tensor g itself yields the existence and the uniqueness of a very special connection as we will see in the next sections.

I.3 Canonical Volume Element on (M,g)

Consider a chart (U, φ) on (M,g). Let $(x^1, \dots, x^n)$ be the local coordinates of x in U. Let

$$[g_{U,\varphi}] = [g_{\ell k}(x^1, \dots x^n)]$$

denote the matrix of g in (U, φ). Since g is positive definite, det $[g_{U,\varphi}] > 0$. On $\varphi(U)$ $(\subset \mathbb{R}^n)$, one has the positive measure defined as follows

$$L_{U,\varphi} = \det[g_{U,\varphi}]^{1/2} dx^1 \dots dx^n.$$

We may then equip $U(\subset M)$ with the pullback (by φ) measure from $L_{U,\varphi}$, that is

$$N_{U,\varphi} = \varphi^*(L_{U,\varphi})$$

Now consider (U_i, φ_i), a covering of M by a neighbourhood system. Using a partition of unity subordinate to this covering, one defines a measure v_g on (M, g) by requiring

$$v_{g|U} = N_{U,\varphi}$$

where (U, φ) is any chart from the covering. The measure v_g is unique and it has sometimes been called canonical volume element of g. Its existence does not require the orientability of M.

I.4 Riemannian Metric and Connection

4.1 Theorem (fundamental lemma of Riemannian Geometry)

Let $\chi(M)$ be the algebra of vector fields on a Riemannian manifold (M,g). Then there is one and only one map

$$\nabla : \chi(M) \times \chi(M) \to \chi(M)$$

$$(X, Y) \mapsto \nabla_X Y$$

such that

LC1. ∇ is $\mathbb{R}$-bilinear

LC2. $\nabla_X fY = (X \cdot f)y + f\nabla_X Y$, for all $f \in C^\infty(M)$

LC3. $\nabla_{fX} Y = f\nabla_X Y$, that is, with L1C., ∇ is $C^\infty(M)$ linear in X.

LCT. $\nabla_X Y - \nabla_Y X = [X, Y]$

LCM. $X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

Here $C^\infty(M)$ denotes the algebra of smooth functions on M, and $[\ ,\ ]$ is the usual Poisson bracket in $\chi(M)$.

4.2 Remark

LC1., LC2., LC3. mean ∇ is a connection on M,

LCT means ∇ is torsion-free,

LCM means ∇ is metric.

4.3 Definition

The connection ∇ is called the Levi–Civita connection of (M,g)

Proof of Theorem 4.1 (Hint: for the existence, define $u : \chi(m) \times \chi(M) \to \chi(M)$, such that

$$2g(u(X,Y),V) = X \cdot g(Y,V) + Y \cdot g(V,X) - V \cdot g(X,Y) - g(X,[Y,V]) + g(Y,[V,X]) + g(V,[X,Y])$$

and prove that the properties are satisfied. For the uniqueness, proceed by contradiction). The above identity has sometimes been called Koszul formula; it is satisfied by ∇.

4.4 Examples

In a coordinate neighbourhood, setting $e_i = \left(\frac{\partial}{\partial x_i}\right)$, $i = 1,\ldots n$, one defines functions Γ^k_{ij} by

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$$

Since ∇ is $\mathbb{R}$-bilinear on $\chi(M) \times \chi(M)$, the Γ^k_{ij}'s define ∇ at each point. Conversely ∇ uniquely defines Γ^k_{ij}.

Exercise Show that $\Gamma^k_{ij} = \Gamma^k_{ji}$ and compute $\nabla_X Y$ in a local coordinate neighbourhood. Compute Γ^k_{ij} in terms of g_{ij}.

4.5 The vector field $\nabla_X Y$ has sometimes been called the *covariant derivative* of Y in the direction of X.

The Γ^k_{ij}'s are called the symbols of Christoffel.

II. CURVATURE TENSOR FIELDS ON A RIEMANNIAN MANIFOLD

II.1 General Definitions

1.1 Let (M,g) be an *R.m.* The Riemann curvature tensor field R is the curvature tensor field associated with the Levi–Civita connection ∇ on (M,g) and therefore defined by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z, \quad \text{for all } \; X,Y,Z \in \chi(M)$$

1.2 Remarks

i) The above formula defines a (3,1) tensor field (i.e. three times covariant and one contravariant).

ii) That formula associates every connection D on M with its curvature tensor field R^D. The Riemann curvature tensor field is the one associated with the canonical connection on (M,g).

iii) To study R, it is more convenient to start by some general algebraic considerations.

1.3 Algebraic preliminaries

Let $(V,<,>)$ be an Euclidean vector space of dimension $n \geq 2$. The vector space, $\Lambda^k V$, of its k–vectors is equipped with the following inner product:

$$< u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \wedge v_k >= \det|< u_i, v_j >|, u_i, v_j \in V$$

If $(e_i)_{i=1,\ldots,n}$ is an orthonormal basis of V, then $(e_{i_1} \wedge \ldots \wedge e_{i_k})$, is an orthonormal basis of $\Lambda^k V$.

Let End$(\Lambda^k V)$ denote the vector space of the endomorphisms of $\Lambda^k V$. One makes it into an Euclidean space by setting

$$(T,S) \equiv tr(T \circ {}^tS), \quad \text{for all } \; T,S \; \text{ in } \; End(\Lambda^k V)$$

where tr denotes the tracing operator on the elements of End$(\Lambda^k V)$ and ${}^tS$ is the adjoint of S.

1.4 When $k = 2$. We denote with $S^2(\Lambda^2 V)$ the subspace of End$(\Lambda^2 V)$ consisting of the symmetric endomorphisms of $\Lambda^2 V$. The elements of $S^2(\Lambda^2 V)$ will be called *curvature type operators*. One may identify such an operator T with a 2–form $\hat{T}$ (i.e. a skewsymmetric bilinear form) on V with values in the set of skewsymmetric endomorphisms of V by setting

$$< \hat{T}(x,y)u, v > = < T(x \wedge y), \; u \wedge v >$$

We will generally write T_{xy} instead of $\hat{T}(x,y)$. It is easy to check the following

1.5 Proposition

(a) $T_{xy} = -T_{yx}$

(b) $< T_{xy}u, v > = < T_{uv}x, y >$.

1.6 The trace of T, as an endomorphism of $\Lambda^2(V)$, is given by

$$trT(x) = \sum_{i=1}^{n} T_{e_i,x}e_i, \quad \text{for every } \; x \in V.$$

Thus $tr(T)$ is an endomorphism of V. It will be called *Ricci trace* of T and denoted with r, so that $r(x) = trT(x)$. The trace of r is called the *scalar curvature* of T and it will be denoted with s, so that

$$s = \sum_{i=1}^{n} < r(e_i), \; e_i >$$

Now consider the linear map

$$b : S^2\Lambda^2 \rightarrow S^2\Lambda^2$$

defined as

$$b(T)_{xy}u = T_{xy}u + T_{yu}x + T_{ux}y_1, \quad \text{for all vector } \; x,y,u \; \text{ in } \; V.$$

The endomorphism b is called the *Bianchi map*. Let $\mathcal{R}$ denote the kernel of b. Then for any symmetric endomorphism on Λ^2, say R, lying in $\mathcal{R}$, we have the following identity

(c) $b(R)_{xy}u = R_{xy}u + R_{yu}x + R_{ux}y = 0$

traditionally known as the *first Bianchi identity* or the algebraic Bianchi identity. An endomorphism in $S^2\Lambda^2$ which satisfies (c) is usually called *curvature operator*.

Exercise: Show that $\mathcal{R}^\perp$ *(the orthogonal of* $\mathcal{R}$ *in* $S^2\Lambda^2$*) is isomorphic to* $\Lambda^4(V)$ (*Hint:* prove that $\omega \mapsto S_\omega$, $\Lambda^4(V) \rightarrow S^2\Lambda^2$ given by $< S_\omega(x \wedge y), u \wedge v >= \omega(x,y,u,v)$, for all $x,y,u,v \in V$ is injective)

1.7 *For arbitrary two symmetric endomorphisms h and h' on V, one defines an element* $h \otimes h'$ in $S^2\Lambda^2(V)$ by setting

$$(h \otimes h')(x \wedge y, u \wedge v) = h(x,u)h'(y,v) + h(y,v)h'(x,u) - h(x,v)h'(y,u) - h(y,u)h'(x,v)$$

for any x,y,u,v in V.

From now on, we will set $< \; > = g$. One can then check that $g \otimes g = 2Id_{\Lambda_2}$; hence the space $\mathcal{R}$ of curvature operators splits into the direct sum

(1.8) $$\mathcal{R} = S_0^2\Lambda^2 \oplus \mathbb{R} g \otimes g$$

where we denote by $S_0^2\Lambda^2$ the space of trace-free curvature operators.

(1.9) Exercise: Show that

(i) $\textcircled{\wedge}$ is commutative

(ii) $h \textcircled{\wedge} h'$ is a curvature operator for all symmetric endomorphisms h, h'

(iii) the map $\textcircled{\wedge} g : S_0^2 V \to S_0^2\Lambda^2, h \mapsto h \textcircled{\wedge} g$ is injective and not surjective.

From (iii) of Excercise (1.9) one has the next very important theorem.

1.10 Theorem (decompisition of curvature operators).

Suppose $\dim V \geq 4$. As a module over the orthogonal group $O(V,g)$, the curvature operators space $\mathcal{R}$ admits the following orthogonal decomposition into a direct sum of unique irreducible subspaces

$$\mathcal{R} = \mathcal{U} \oplus \mathcal{Z} \oplus \mathcal{W}$$

where $\mathcal{U} = \mathbb{R} g \textcircled{\wedge} g$

$\mathcal{Z} = g \textcircled{\wedge} (S_0^2 V)$ (image of the injection (iii), Excercise 1.9)

$\mathcal{W}$ = complementary part of $\mathcal{Z}$ in $S_0^2\Lambda^2$.

Proof The existence of this decomposition is obvious from (1.8) and (1.9) above. The difficult part is the irreducibility of $\mathcal{W}$ under the action of the orthogonal group $O(V,g)$. For that the reader is referred to [BGM, p. 82–83].

1.11 The elements lying in $\mathcal{W}$ are called Weyl curvature operators. Thus a curvature operator R has the unique decomposition (as an element of the $O(V,g)$-module) into irreducible components:

$$R = U + Z + W \tag{1.12}$$

1.13 Exercise: Show that

$$U = \frac{s}{2n(n-1)} g \textcircled{\wedge} g; \quad Z = \frac{1}{n-2}\left(r - \frac{s}{n} g\right) \textcircled{\wedge} g$$

where s and r are scalar curvature and Ricci trace of R, respectively (Hint: take appropriate traces).

II.2 Back to geometry

2.1 Let x be a point of a Riemannian manifold (M,g) and let $T_x M$ denote the tangent space at x. The curvature tensor field R defined in (Sec. II.1.1) induces on $(T_x M, g_x)$ a curvature operator. Its trace will be denoted by Ric and the trace of Ric by S.

2.2 Definitions

Ric is called Ricci (curvature) tensor

S is called scalar curvature (function).

Using the foregoing section, one can easily prove the next propositions.

2.3 Proposition

For all vector fields X, Y, A, B

i) $\langle R_{XY} A, B \rangle = \langle R_{AB} X, Y \rangle$

ii) $\langle R_{YX} A, B \rangle = \langle R_{XY} A, B \rangle$

iii) $R_{XY} A + R_{YA} X + R_{AX} Y = 0$ (first Bianchi identity)

where for two vector fields A and B, we denote with $\langle A, B \rangle$ their inner product $g(A,B)$.

2.4 Proposition

The Ricci curvature tensor Ric is symmetric, and is given, relative to a frame field (E_i), by

$$Ric(X,Y) = \sum_{i=1}^{n} \langle R_{XE_i} Y, E_i \rangle$$

2.5 Definition

Denoting by ∇ the Levi-Civita connection on (M,g) one defines $\nabla_A R$ for a given vector field A by:

$$(\nabla_A R)(X,Y) = \nabla_A(R(X,Y)) - R(\nabla_A X, Y) - R(X, \nabla_A Y)$$

for all vector fields X, Y.

Using the definition of R, the Jacobi identity and a straighforward computation, one has the following

2.6 Proposition (second Bianchi identity)

For all vector fields A, X, Y on (M,g)

$$(\nabla_A R)(X,Y) + (\nabla_X R)(Y,A) + (\nabla_Y R)(A,X) = 0$$

2.7 Theorem (of fundamental decomposition of R)

Suppose dim $M \geq 4$. The Riemann curvature tensor R has the following decomposition into $O(g)$-irreducible components

$$(2.8) \qquad R = \frac{s}{2n(n-1)} g \bigotimes g + \frac{1}{n-2}(Ric - \frac{S}{n} g) \bigotimes g + W$$

where W is the "remainder" defined by (2.8).

W is called the Weyl curvature tensor $Z = Ric - \frac{s}{n} g$ is called trace-free (or traceless, or deviant) part of Ricci tensor.

Note: For details on (2.8) the reader is referred to [BE, p. 45-47].

2.9 Definition

(M,g) is called Einstein manifold if $Z = 0$. It is said to be conformally flat if $W = 0$ and to be flat if $R = 0$ and Ricci flat if $Ric = 0$.

2.10 Examples

1) $(\mathbb{R}^n, e)$ is a flat manifold.

2) If R_i is the Riemann curvature of (M_i, g_i) for $i = 1, 2$, then $R = R_1 + R_2$ is the Riemann curvature tensor of $(M_1 \times M_2, g_1 \times g_2)$

3) For dimension reasons, one has the following facts:

$$n = 2, \quad r = \frac{s}{4} g \bigotimes g; \quad Ric = \frac{s}{2} g; \quad W = 0$$

$$n = 3, \quad R = \frac{s}{12} g \bigotimes g + (Ric - \frac{s}{3} g) \bigotimes g; W = 0$$

2.11 For x in (M,g), denote with $G_x^2(M)$ the set of 2-planes contained in the tangent space $T_x M$. Then $G^2(M) = U_{x \in M} G_x^2(M)$ is a manifold. The sectional curvature is a map from $G^2(M)$ which assigns to each plane P the real number $\sigma(P)$ given by

$$(2.12) \qquad \sigma(P) \equiv \sigma(x, y) = < R_{xy} x, y > /(g(x, x) g(y, y) - g(x, y)^2))$$

where (x, y) is a basis of P.

2.13 Theorem

Suppose dim $M \geq 3$. If the sectional curvature is known on all sections of $T_x M$, then the Riemann curvature tensor is uniquely determined at x.

Proof (see [BO, p. 381]).

In other words, the sectional curvature determines the Riemann tensor. This the reason why one sometimes says curvature (without any precision) by referring to the sectional curvature or to the Riemann curvature tensor.

2.14 Exercise

1) Show that a Riemannian manifold (M,g) has constant sectional curvature k if and only if

$$R = \frac{k}{2} g \bigotimes g$$

Conclude that a manifold with constant sectional curvature is Einstein and conformally flat.

2) Express, for any $R.m.$ (M,g), Ric and S in terms of σ.

II.3 Curvature Forms, Structure Equations

3.1 Consider $(\partial_i)_{i=1,\dots,n}$ a basis of the tangent space $T_x M$ at a point x. One may then write R as

$$R(\partial_i, \partial_j)\partial_k = R^\ell_{kij} \partial_\ell$$

where ∂_i denotes $\frac{\partial}{\partial x_i}$ Using R^ℓ_{kij}'s, one defines n^2 2-forms by setting

$$\Omega^\ell_k = \sum_{i,j} R^\ell_{kij} dx^i \wedge dx^j$$

The 2-forms Ω^ℓ_k are given by

$$\Omega^\ell_k(\partial_i, \partial_j) = R^\ell_{kij}$$

By linearity, it then follows, for all vector fields X, Y

$$R(X, Y)\partial_k = \sum_\ell \Omega^\ell_k(X, Y)\partial_\ell$$

Hence $[\Omega^\ell_k(X, Y)]$ is the matrix of the curvature operator $R(X, Y)$ relative to the frame (∂_i). The Ω^ℓ_k's are called curvature 2-forms (in the neighbourhood of x). They depend both on the metric g and on (∂_i). (Exercise: Find some of their algebraic properties).

3.2 Assume that $(\partial_i)_{i=1,\dots,n}$ is an orthonormal frame field and let (ω^i) be the dual forms. The $(\omega^i)_{i=1,\dots,n}$'s uniquely determine n^2 1-forms ω^ℓ_k satisfying

i) $d\omega^\ell = \sum_{k=1}^n \omega^\ell_k \wedge \omega^k$

ii) $\omega^\ell_k + \omega^k_\ell = 0$

iii) $\Omega^\ell_k = d\omega^\ell_k - \sum_{l=1}^n \omega^i_k \wedge \omega^\ell_i, \quad s \leq \ell, \ k \leq n$

Exercise: Prove the previous statement.

3.3 Definition

The identities (i), (ii), (iii) have sometimes been called structure equations.

3.4 Example

When $\dim M = 2$, it is easy to show (show it) that

$$\Omega_1^2 = d\omega_1^2 = K\omega^1 \wedge \omega^2$$

where K is the Gauss curvature of (M,g) that is $2K = S$, S being the scalar curvature.

More generally one has the following

3.5 Proposition

Let (M,g) be a manifold with constant (sectional) curvature k. Then the curvature forms Ω_j^ℓ are given by

$$(**) \qquad \Omega_j^\ell = k\omega^\ell \lambda \omega^j, \quad 1 \leq \ell, j \leq n$$

Conversely, if locally on a manifold (M,g) there is an orthonormal frame field for which the curvature forms satisfy (**) for some constant k, then (M,g) has constant sectional curvature.

Proof (see [BO, p. 399]).

The Riemannian manifolds with constnat (sectional) curvature have been completely classified (modulo additional topological conditions). One has the following.

3.5 Theorem (Killing–Hopf)

Every complete, simply connected Riemannian manifold with constant curvature $k = +1, 0$ or -1 is isometric to (S^n, can) if $k = +1$ to $(\mathbb{R}^n, e)$ if $k = 0$, to (H^n, g_0) if $k = -1$, where (H^n, g_0) denotes the half plane

$$H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n, \quad x^n > 0\}$$

equipped with the metric

$$g_0 = \frac{(dx^1)^2 + \ldots + (dx^n)^2}{(x^n)^2}$$

3.6 Remark

Many people attempt to classify the manifolds with constant scalar curvature, but the answers are till partial.

III. SOME CLASSICAL FUNCTIONALS ON A RIEMANNIAN MANIFOLD

III.1 Geodesics. First variation formula

Consider an $R.m.$ (M,g) and an arclength parametrised C^∞ curve $\gamma : [a,b] \to M$. Let ε be a positive real number.

1.1 Definition

A variation of γ is a smooth map $\varphi : [a,b] \times]-\varepsilon, \varepsilon[\to M$ such that if γ_t denotes the map $[a,b] \to M$ given by $\gamma_t(s) = \varphi(s,t)$, then $\gamma_0 = \gamma$.

Suppose that φ is a variation of γ with fixed ends, that is, for all t, $-\varepsilon < t < \varepsilon$

$$\varphi(a,t) \equiv \gamma_t(a) = p$$

$$\varphi(b,t) = \gamma_t(b) = q$$

Then one has

$$\begin{aligned} L(\gamma_t) &= \int_a^b g(\dot\gamma_t, \dot\gamma_t)^{1/2} dt \\ &= \int_a^b g(\frac{\partial\varphi}{\partial s}(s,t), \quad \frac{\partial\varphi}{\partial s}(s,t))^{1/2} ds \\ &= \int_a^b E(s,t)^{1/2} ds \end{aligned}$$

where $E(s,t)$ denotes

$$g(\dot\gamma_t, \dot\gamma_t) = g(\frac{\partial\varphi}{\partial s}(s,t), \ \frac{\partial\varphi}{\partial s}(s,t)).$$

1.2 Question

What are the curves which minimise $L(\gamma_t)$ when φ is a variation with fixed ends of γ?

This a variational problem. Let us seek its Euler–Lagrange equation.

Since $\gamma = \gamma_0$ is a minimising curve we have

$$\left.\frac{dL(\gamma_t)}{dt}\right|_{t=0} = 0$$

But we formally have

$$\frac{dL(\gamma_t)}{dt} = \int_a^b \frac{\partial}{\partial t}\sqrt{E(s,t)}\, d = \frac{1}{2}\int_a^b \frac{1}{\sqrt{E}}\frac{\partial E}{\partial t}\, ds$$

so that we need to assume (and we will do so) that $\dot\gamma$ never vanishes. Furthermore, since γ is an arclength parametrized curve, $E(s,o) = g(\dot\gamma(s), \dot\gamma(s)) = 1$. Therefore one has

$$\frac{dL(\gamma_t)}{dt}\Big|_{t=0} = \frac{1}{2}\int_a^b \frac{\partial E}{\partial t}\Big|_{t=0}\, ds$$

The computation of the integrand gives

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t} g\left(\frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial s}\right) = 2g\left(\nabla_t \frac{\partial\varphi}{\partial s}, \frac{\partial\varphi}{\partial s}\right)$$

where ∇ is the Levi–Civita connection and ∇_t denotes $\nabla_{\partial/\partial t}$

1.3 Lemma (see BGM, p. 45)

Using this we obtain

$$\frac{\partial E}{\partial t} = 2\frac{\partial}{\partial s} g\left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial s}\right) - 2g\left(\frac{\partial\varphi}{\partial t}, \nabla_s \frac{\partial\varphi}{\partial s}, \right)$$

Therefore one has

$$\frac{dL(\gamma_t)}{dt}\Big|_{t=0} = \int_a^b [\frac{\partial}{\partial s} g(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial s})]_{t=0}\, ds - \int_a^b (\frac{\partial\varphi}{\partial t}\Big|_{t=0}, \nabla_{s_0}\dot\gamma)\, ds$$

Noticing that

$$(\frac{\partial\varphi}{\partial s})_{t=0} = (\frac{\partial}{\partial s})_{t=0} = (\dot\gamma_t)_{t=0} = \dot\gamma$$

one has

(1)
$$(\frac{dL(\gamma_t)}{dt})_{t=0} = [g((\frac{\partial\varphi}{\partial t})_{t=0}, \dot\gamma)]_a^b - \int_a^b g((\frac{\partial\varphi}{\partial t})_{t=0}, \nabla_{\dot\gamma}\dot\gamma)\, ds$$

Identity (1) is known as first variation formula. Since φ is a fixed ends variation, one has $\left(\frac{\partial\varphi}{\partial t}\right)_{t=0} = 0$ at a and b, which reduces (1) into

$$(\frac{dL\gamma(t)}{dt})_{t=0} = -\int_a^b g((\frac{\partial\varphi}{\partial t})_{t=0}, \nabla_{\dot\gamma}\dot\gamma)\, ds$$

Then with the fixed ends condition, Euler Lagrange equation is

(2)
$$\nabla_{\dot\gamma}\dot\gamma = 0$$

Hence if a smooth curve γ (c^2 class curve at least) realizes the distance between two fixed points p,q on a Riemannian manifold (M,g) then γ satisfies identity (2).

Let $(x^1(t), \ldots x^n(t))$ be the local coordinates of $\gamma(t)$ in a local chart, then (2) is equivalently the system

(2′)
$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \quad i = 1, \ldots, n$$

where Γ^i_{jk} are the coefficients of the Riemannian connection on (M,g).

1.4 Definition

A smooth curve $\gamma : [a,b] \to M$ which satisfies Eq.(2) or (2′) is called a geodesic of (M,g).

1.5 Theorem (existence of geodesics)

Let p be a point in (M,g). For every vector x in T_pM, there is one and only geodesic, say, γ_x, defined on an interval $]a,b[$ containing 0, nonextendible to a larger interval such that

$$\gamma_x(0) = p, \quad \dot\gamma_{x,o} = x$$

Proof This is an existence and uniqueness exercise for differential equation (2)′.

1.6 Definitions

The geodesic whose existence and uniqueness are stated by Theorem (1.4) is called maximal geodesic through p in (M,g).

A geodesic is said to be infinite if it is defined on all $\mathbb{R}$.

A Riemannian manifold (M,g) is said to be geodesically complete if every geodesic $\gamma : [a,b] \to M$ can be extended to an infinite geodesic.

1.7 Theorem (Hopf Rinow)

For a given Riemannian manidold (M,g) the following assertions are equivalent.

(i) (M,g) is a complete metric space

(ii) There is a point p in (M,g), such that for every vector x in T_pM, the maximal geodesic γ_x through p is infinite

(iii) (M,g) is geodesically complete

(iv) every closed bounded part of (M,g) is compact.

Proof (See for instance [KN1]).

1.8 Definition

A *R.m.* which satisfies one of the above equivalent conditions (i), (ii), (iii), (iv) is said to be complete.

1.9 Examples

1. ($\mathbb{R}^n e$) is complete by (i);

2) Every compact Riemannian manifold is complete, by (i.v).

1.10 Exponential map

Let p be a point in (M,g). Denote with D_p the set of tangent vector x lying in T_pM such that 1 belongs to $]a,b[$, the open interval from which γ_x, the maximal geodesic through p is defined. Set

$$D = \cup_{p \in M} D_p.$$

Notice that D_p (and so that D in TM) is an open set in T_pM.

1.11 Definition

The exponential map of the connection ∇ is the map $\exp : D \to M$ given by $\exp(x) = \gamma_x(1)$. One denotes with $\exp_p$ the restriction to $D_p = D \cap T_pM$ of exp.

Exercise Show that the tangent map to $\exp_p$ at the origin of T_pM is the identity map of T_pM.

1.12 Definition

The connection ∇ on (M,g) is said to be complete if the domain D of its exponential map is TM.

1.13 Jacobi Vector Field

Let γ be a geodesic of ∇ on (M,g). A Jacobi vector field along γ is a vector field J satifying the following

(i) J is a map which assigns to each t in $\mathbb{R}$ an tangent vector $J(t)$ in $T_{\gamma(t)}M$

(ii) the identity

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J + R(\dot\gamma, J)\dot\gamma = 0$$

holds. Here R denotes the Riemann curvature tensor.

Identity (ii) is Jacobi equation.

1.14 Remarks

1. Since γ is a geodesic and R is skewsymmetric with respect to the two first variables, the velocity vector field $\dot\gamma$ is a Jacobi vector field along γ.

2. Given a geodesic γ, there is one and only one Jacobi vector field along γ satisfying two prescribed initial conditions on γ and $\dot\gamma$, since Jacobi equation is a second order differential equation.

1.15 Let γ be a geodesic defined from an interval I containing 0. Two points $\gamma(0)$ and $\gamma(t)$ are said to be conjugate along γ if there is a Jacobi vector field J along γ such that $J(0) = 0 = J(t)$.

The conjugate points play an important role in Riemannian geometry. Unfortunately we are going to say nothing about them in our context here. The interested reader is referred to [CE].

III.2 Total Scalar Curvature. Einstein Equation in the Vacuum

Let us denote with $\mathcal{M}$ the space of all smooh Riemannian metrics on a manifold M. Suppose that M is compact and n–dimensional. For every g in $\mathcal{M}$, the total scalar curvature of (M,g) is defined to be

$$S(g) = \int_M s_g v_g$$

where s_g denotes the scalar curvature of g.

2.1 Proposition

The functional S is positively homogeneous of degree $\frac{n-2}{2}$.

Proof Let t be a positive real number. Then

$$s_{tg} = t^{-1} s_g \quad \text{and} \quad v_{tg} = t^{n/2} v_g$$

(use the definitions or see [BE, p. 59]). Then

$$S(tg) = t^{\frac{n-2}{2}} \int_M s_g v_g = t^{\frac{n-2}{2}} S(g)$$

□

2.3 The differential of S at a point g is defined on the symmetric (0,2) tensor fields by

$$S(g).h = \frac{d}{dt} S(g + th)|_{t=0}$$

Then one has the following

2.4 Theorem

The differential of S at g is given by

$$S'(g).h = \int_M g\left(\frac{Sg}{2} - Ric_g, h\right) v_g$$

where $g(k,k')$, for two symmetric (0.2) tensors fields is $tr(k \circ k')$, k_x, k'_x being viewed as symmetric endorphisms on T_xM, for each x in M.

Proof (Sketch, for details see [Br])

First observe that

(i) $V'_g \cdot h = \frac{1}{2}(trh)v_g \equiv \frac{1}{2}g(h,g)v_g$

(ii) $s'_g \cdot h = \Delta_g(trh) + \delta\delta(h) - g(h, Ric_g)$

where $\delta(T)$ the divergence of a symmetric (0,2)-tensor field is defined as $\delta(T)(X) = -(\nabla_{e_i}T)(e_i, X)$ with $(e_1, \ldots e_n)$ an orthonormal basis, and Δ_g is the Laplace operator associated to g.

Therefore,

$$\begin{aligned} S'(g) \cdot h &= \frac{d}{dt}\int_M (S_{g+th},\ v_{g+th})_{t=0} \\ &= \int_M s_g(v'g \cdot h) + (s'_g \cdot h)v_g \\ &= \int_M g(\frac{s_g}{2}\ g - Ric_g, h)v_g \end{aligned}$$

□

As an important consequence of this we have

2.5 Theorem

The critical points of S are the metrics which are solutions of equation

$$(E) \quad Ric_g - \frac{s_g}{2}\ g = 0$$

2.6 Definition

Equation (E) is known as Einstein Equations in the vacuum.

III.3 The functional of Yamabe. Metrics with Constant Scalar Curvature in a Conformal Class

3.1 Let M be a compact manifold. The Yamabe functional on M is defined to be

$$Y(g) = (\int_M s_g v_g)(\int v_g)^{-\frac{n-2}{n}}$$

3.2 Definition

Given a Riemannian metric g on M the conformal class of g is defined to be the set

$$conf(g) = \{\lambda^2 g | \lambda > 0,\ \lambda \in C^\infty(M)\}$$

Two metrics g and $\lambda^2 g$ on M are said to be homothetic if λ is constant. One then has the important following

3.3 Theorem

For a compact manifold (M,g) the following statements are equivalent

(i) the scalar curvature of g is constant

(ii) g is a critical point of Y(g) restricted to the metrics belonging to conf(g) and having the same volume.

Proof (see [Be, p. 121]).

3.4 Definition

Every metric within a conformal class conf(g) which minimises $Y(g)$ will be called Yamabe metric.

3.5 Suppose dim $M \geq 3$ and fix a metric g on M. Let N be defined by $N = \inf\{Y(\bar{g}), \bar{g} \in conf(g)\}$. By an analysis result N is a finite real number and furthermore $N \leq n(n-1)\omega_n^{2/n}$, where ω_n is the volume of the standard sphere $(S^2\ can)$[A]. This fact is extensively exploited in the proof of the following

3.6 Theorem

Let (M,g) be a compact Riemannian manifold of dimension ≥ 3. In the conformal class of g there is at least a metric with constant scalar curvature.

Proof (see [LP]).

In fact, this theorem has been being proved step by step depending on the sign of N since 1960. The last step ($N > 0$ and (M,g) conformally flat) has been announced by R. Schoen [Sc] for a forcoming joint paper with S.T. Yau.

Notice that Theorem 3.6 is an extension to higher dimension of the well-known uniformisation theorem of compact Riemannian 2-manifolds.

3.7 Question

Suppose g is a Riemannian metric with constant scalar curvature on a compact manifold M. Do there exist within conf(g) other metrics with constant scalar curvature?

This question has negatively been answered by Obata in the case (M,g) is an Einstein manifold unless (M,g) is the standard sphere. In the general case the answer is not known. In this direction we have the following obstruction.

3.8 Theorem [ER]

Let (M, g) be a compact manifold with constant scalar curvature which is not the standard sphere. Let Z_g denote the tracefree part of the Ricci curvature tensor of (M,g). If the conformal class of g contains a metric nonhomothetic to g, with constant scalar curvature, then there exists a function $\rho > 0$ such that the function $Z_g\,(\nabla\rho, \nabla\rho)$ is negative somewhere on M.

3.9 Corollary (Obata's theorem)

Let (M,g) be a compact Einstein manifold which is not the standard sphere. Then g is the only metric with constant scalar curvature lying in its conformal class.

Proof Using Theorem 3.8 one obtains the conclusion by contradiction since $Z_g \equiv 0$.

IV. QUESTIONS

We collect in this chapter a few open problems relative to the forgoing lectures as well as to the lectures by Temo Beko. Most of them already get partial answers in the literature (we do not like to give more details here).

A. From Riemannian Geometry

1. Describe the topology of the space of function on the sphere S^n which cannot be realised as scalar curvature functions of metrics lying in the conformal class of the standard metric.

2. Given a conformal class on a compact manifold, characterise the metrics of this class having constant scalar curvature. Uniqueness of such metric within their class.

3. It is known that the sphere (S^n, *can*) is the only compact, n–dimensional manifold with constant scalar curvature embedded into $(\mathbb{R}^{n+1}, e)$. Study the case where the ambient manifold is a complete Riemannian–manifold and the embedded (or immersed) one is assumed to be compact, $(n-1)$–dimensional and to have constant scalar curvature. Case of lower dimension.

4. Computation of the Yamabe number on compact manifolds, compact manifolds with boundary, complete manifolds.

5. Study Riemannian manifolds (M, g) on which vector fields X such that $g(X, X) = 1$ and $divX = 0$ live (By L. Barbosa in a private communication).

B. From Local Lie Algebras (by Temo Beko)

Let $M = \mathbb{R}^n$ be a smooth manifold and E a smooth real vector fibering over N with fibre $\mathbb{R}$. We denote by $\Gamma^\infty(E)$ the space of differentiable sections of E, in which we define a structure of a local Lie algebra such that

$$[f, f'] = \sum_i a^i (f D_{x^i} f' - f' D_{x^i} f) + \sum_{i,j} a^{ij} D_{x^i} f D_{x^j} f'$$

for all $f, f' \in C^\infty(M)$. (see the expositions by Temo Beko in this volume for notations). This shows that the structure of a local Lie algebra is defined in $C^\infty(M)$ by the pair (a,c) where $a = \sum_i a^i D_{x^i}$ is a vector field on $\mathbb{R}^n$, as $c = \sum_{i,j} a^{ij} D_{x^i} \wedge D_{x^j}$ is a field of bivectors on $\mathbb{R}^n$.

And a pair (a, c) defines the structure of a local Lie algebra iff $[a, c] = 0$ and $[c, c] = 2a \wedge c$ (*)

1. Suppose that the vector field a vanishes, then the conditions (*) on the bivector

field c take the form

$$\sum_{\ell} c^{i\ell}\partial_\ell c^{jk} = 0 \qquad (1)$$

where the summation is over the index ℓ and over cyclic permutations of i, j, k. We look for solutions of (1) in the form

$$c^{ij}(x) = \sum_k c_k^{ij} x^k$$

Then (1) are transformed into

$$\sum c_k^{i\ell} c_\ell^{jk} = 0$$

which means that the C_k^{ij} are the structure constants of some Lie algebra $\mathcal{G}$ and the commutator is defined by

$$[f, f'] = \sum_{i,j,k} c_k^{ij} x^k D_{x^i} f \; D_{x^j} f'$$

which was first considered in a paper by F.A. Berezin (Funct. Anal. Appl.1 (1967), pp. 91–102).

The algebra $\mathcal{G}$ is embedded into the local algebra $C^\infty(\mathbb{R}^n)$ as a space of linear functions. When the space $\mathbb{R}^n$ is identified with $\mathcal{G}^*$(= dual of $\mathcal{G}$) the fibres are orbits of the Lie group G and the local Lie algebra that arises on functions on a fibre corresponds to the canonical symplectic structure on an orbit. All these orbits are symplectic homogeneous G-spaces. This holds to be true for infinite dimensional Lie groups of the type $G = Diff_c(M_n)$ of diffeomorphisms with compact support of a smooth n-dimensional manifold M_n whose Lie algebra is the space $(\mathcal{X}_{M_n})$ of all smooth vector fields on M_n with compact support.

2. We consider any solution of (1) in homogeneous functions of degree λ on $\mathbb{R}^n$, then the corresponding commutator takes homogeneous functions of degree N and ν into homogeneous functions of degree $\lambda + N + \nu - 2$. In particular, homogeneous functions of degree $2 - \lambda$ form a subalgebra. The space of such functions is identified with the space of sections of some trivialisable fibering $E_{2-\lambda}$ over the projective space $\mathbb{P}^{n-1}(\mathbb{R})$. It would be interesting to study in more detail the local Lie algebras that arise here.

The case $\lambda = 0$ has been considered in the book by Ouzilou (Hamiltonian Actions on Poisson's Manifolds, Symplectic Geometry, Research Notes in Math. Pitman, 1983).

3. Finally, it is an interesting question to what extend local Lie algebras on a manifold M, which characterizes the manifold M itself.

In particular, the results by T. Nagano (J. Math. Soc. Japan 18 (1966), 398–404) and by M.E. Shoukro and L.E. Pursell (Proc. Amer. Math. Soc. 5 (1954), 468–472) give grounds for hoping that an isomorphism of transitive Lie algebras $\Gamma^\infty(E_1, M_1)$ and $\Gamma_2(E_2, M_2)$ implies an isomorphism of the fiberings E_1 and E_2.

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