IC/88/275

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

REVERSED SENSE OF THE "OUTWARD" DIRECTION FOR DYNAMICAL EFFECTS OF ROTATION CLOSE TO A SCHWARZSCHILD BLACK HOLE

Marek A. Abramowicz

and

A.R. Prasanna

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

IC/88/275

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

REVERSED SENSE OF THE "OUTWARD" DIRECTION FOR DYNAMICAL EFFECTS OF ROTATION CLOSE TO A SCHWARZSCHILD BLACK HOLE *

Marek A. Abramowicz
International Centre for Theoretical Physics, Trieste, Italy,
and
Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy,

and

A.R. Prasanna **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Anderson and Lemos (1988) noticed that the direction in which viscous torque transports angular momentum changes, close to a black hole, from outwards to inwards. We find here that close to a black hole the centrifugal force attracts particles towards the hole. We argue that these are particular examples of a general reversal in sense of the inward- and outward directions for all dynamical effects of rotation close to the hole. Using results from the recent paper by Abramowicz, Carter and Lasota (1988) we explain that the reversal is not connected with dragging of inertial frames or with the difference between the angular velocities of the hole and of the surrounding matter but rather, it is an effect of curvature. For a Schwarzschild black hole the reversal takes place at the circular photon orbit ($r = 3\tilde{M}$) because the geodesic curvature, $\kappa = r(1 - 3\tilde{M}/r)$, of the circles r = const. changes it sign there.

MIRAMARE - TRIESTE
October 1988

* To be submitted for publication.

** Permanent address: Physical Research Laboratory, Navrangpura, Ahmedabad-380009, India.

I. INTRODUCTION

Many astronomical objects are explained today as consisting of rapidly rotating accretion disks around black holes. Accretion of matter with non-zero angular momentum occurs because of the balance between gravity and several rotational effects in which the physical sense of the inward and outward directions plays an important role. Gravity always acts inwards by attracting matter towards the hole. In Newtonian dynamics the centrifugal force acts outwards, repelling rotating matter away from the axis of rotation and hence also away from the black hole. In connection with this, the angular momentum distribution in an accretion disk is stable against local axially symmetric perturbations if the angular momentum increases outwards (the Rayleigh criterion). Within a stable accretion disk, viscous torque transports angular momentum outwards. Both the inward direction of gravity and the outward direction of rotation may be locally defined in physical terms: the inward direction of gravity is defined with respect to a plumb line and the outward direction of rotation is defined with respect to photon trajectories: the "out" vector is on the same side of a circular orbit as the photon path tangent to it (Fig.1). A strong Newtonian intuition tells us that the inward direction of gravity and the outward direction of rotation should be opposite but this need not necessarily be the case in the curved, non-Euclidean geometry of Einstein's theory of gravitation.

In the gravitational field of a Schwarzschild black hole (with gravitational mass $\tilde{M} = GM/c^2$) the circle $r = 3\tilde{M}$ is a closed, circular photon path. For a circle $r = r_a$ (with $r_a > 3\tilde{M}$) all of the points on the photon path tangent to it have $r > r_a$, while for a circle $r = r_b$ (with $r_b < 3\tilde{M}$) they have $r < r_b$. This illustrates how the sense of the "outward" direction reverses at the circular photon orbit (Fig.2).

Using the optical reference geometry introduced recently by Abramowicz, Carter and Lasota (1988) (hereafter referred to as ACL) we will discuss the physical implications of this reversal. We will show that inside the photon orbit ($r < 3\tilde{M}$):

a) The centrifugal force is reversed. The centrifugal force attracts particles towards the axis of rotation: the faster a particle orbits around, the more strongly it is pushed towards the black hole.

b) The Rayleigh stability criterion is reversed. For a stable angular momentum distribution the specific angular momentum decreases with increasing distance from the axis of rotation.

c) The direction of the viscous torque is reversed. For a fluid with constant angular momemtum distribution, angular momentum is transported inwards by viscous torque. This was recently noticed by Anderson and Lemos (1988).

In this paper we will stress physical meaning rather than formal aspects. However, we do not make any approximations in our calculations which are fully consistent with general relativity.

II. OPTICAL REFERENCE GEOMETRY

Following ACL, consider a static spacetime with metric $g_{\alpha\beta}$ and covariant derivative operator ∇_α (Greek indices refer to the spacetime and run through 0,1,2,3). The static spacetime admits a timelike Killing vector η_α which obeys the Killing equation $\nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha = 0$ and has the norm

$$\Phi = \eta^\alpha \eta^\beta g_{\alpha\beta} \equiv (\eta\eta) \quad . \tag{1}$$

(The scalar product of vectors x_α and y^α is here denoted by (xy).)

Using the Killing vector η_α, ACL invariantly defined the time coordinate t and the three-dimensional, spacelike <u>optical reference geometry</u> $\tilde{g}_{ik}$:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = \Phi\, (dt^2 + \tilde{g}_{ik}\, dx^i\, dx^k) \quad . \tag{2}$$

The Latin indices run through 1,2,3. The covariant derivative in the metric $\tilde{g}_{ik}$ will be denoted by $\tilde{\nabla}_i$.

ACL referred the motion of the test particles and photons in the original spacetime $g_{\alpha\beta}$ to their traces ("trajectories") in the three-dimensional optical reference geometry. Let P^α be the four-momentum of a test particle $(m > 0)$ or of a photon $(m = 0)$. The corresponding three-momentum $\tilde{P}^i$ in the optical reference geometry is

$$\tilde{P}^i = \Phi\, P^i \quad . \tag{3}$$

The three-dimensional force $\tilde{f}_i$ in the optical reference geometry is defined by

$$m\, \tilde{f}_i = \Phi^{-1}\, (P^\alpha\, \nabla_\alpha\, P_i) \quad . \tag{4}$$

This was then expressed in terms of quantities defined in $\tilde{g}_{ik}$:

$$m\, \tilde{f}_i = \tilde{P}^j\, \tilde{\nabla}_j\, \tilde{P}_i + \frac{1}{2} m^2\, \tilde{\nabla}_i\, \Phi \tag{5}$$

and they pointed out that the geodesic lines in the optical reference geometry coincide with the photon trajectories $(m = 0)$. They stressed that the term

$$m\, \tilde{F}_i = \tilde{P}^j\, \tilde{\nabla}_j\, \tilde{P}_i \tag{6}$$

is proportional to the square of the velocity in the 3-space, and is connected with the force which acts on particles which move on trajectories other than photon paths. Thus, $\tilde{F}_i$ should be identified with the <u>centrifugal force</u>. The term

$$m\, \tilde{G}_i = \frac{m^2}{2}\, \tilde{\nabla}_i\, \Phi \tag{7}$$

does not depend on the motion and is proportional to the gradient of a scalar. Thus G_i should be identified with the <u>gravitational force.</u>

Let us now discuss circular motion. Circles in the optical reference geometry $\tilde{g}_{ik}$ are defined as trajectories of the Killing vector $\tilde{\xi}_i$ connected with the axial symmetry of $\tilde{g}_{ik}$, The norm of the vector $\tilde{\xi}^i$ defines the proper circumferential radius of the circle $\tilde{r}$

$$\tilde{r}^2 = -\, \tilde{\xi}^i \tilde{\xi}^k\, \tilde{g}_{ik} = -\, (\tilde{\xi}\tilde{\xi}) \quad . \tag{8}$$

The vector $\tilde{\xi}^i$ is the trace of the Killing vector ξ^α connected with the axial symmetry of the original spacetime $g_{\alpha\beta}$

$$\tilde{\xi}^i = \Phi^{-1/2}\, \xi_i \tag{9}$$

and obeys the Killing equation in the optical reference geometry, $(\tilde{\nabla}_i \tilde{\xi}_k + \tilde{\nabla}_k \tilde{\xi}_i = 0)$ from which one can easily derive

$$\tilde{\xi}^j\, \tilde{\nabla}_j\, \tilde{\xi}_i = -\frac{1}{2}\, \tilde{\nabla}_i\, (\tilde{\xi}\tilde{\xi}) = \tilde{r} \tilde{\nabla}_i \tilde{r} \quad . \tag{10}$$

for circular motion, the momentum $\tilde{P}^i$ can be written s

$$\tilde{P}_i = m\, \tilde{v}\, \tilde{\tau}_i \tag{11}$$

where $\tilde{v}$ is the speed along the circle and

$$\tilde{\tau}_i = \tilde{r}^{-1}\, \tilde{\xi}_i \tag{12}$$

is a unit tangent to the circle.

The speed $\tilde{v}$ is defined in the following (standard) way. The four velocity u_α for circular motion is a linear combination of the Killing vectors η_α and ξ_α

$$u_\alpha = A(\eta_\alpha + \Omega \xi_\alpha) \quad . \tag{13}$$

Here Ω is the angular velocity (measured by stationary observers) and A is the redshift factor

$$A^2 = (\eta\eta) + \Omega^2(\xi\xi) = \Phi\,(1 - \Omega^2\tilde{r}^2) \qquad (14)$$

The specific angular momentum ℓ is defined by

$$\ell = -\frac{(u\xi)}{(u\eta)} = \Omega\tilde{r}^2 \qquad (15)$$

The relationship between Ω, ℓ, and the speed $\tilde{v}$ is given by

$$\tilde{v}^2 = \frac{\Omega\ell}{1-\Omega\ell} = \frac{\Omega^2\tilde{r}^2}{1-\Omega^2\tilde{r}^2} \qquad (16)$$

From the definitions of $\tilde{v}$ it follows that for particle motion (timelike lines in $g_{\alpha\beta}$), $0 \leq \tilde{v} < \infty$.

The geodesic curvature $\tilde{\mathcal{R}}$ of a curve with unit tangent vector $\tilde{\tau}_i$ is defined by (cf. Synge and Schild, 1959):

$$\tilde{\tau}^j\tilde{\nabla}_j\tilde{\tau}_i = \tilde{\mathcal{R}}^{-1}\lambda_i \ , \qquad (17)$$

where the unit vector λ_i is the first normal to the curve. Directly from this definition it follows that the norm of the centrifugal force $\tilde{F}^2 = -\tilde{F}^i\tilde{F}^k\tilde{g}_{ik}$ is given by

$$\tilde{F} = \frac{m\tilde{v}^2}{\tilde{\mathcal{R}}} \qquad (18)$$

which is identical to its Newtonian equivalent. However, in the Newtonian theory (based on Euclidean geometry) the geodesic curvature of a circle is always equal to its proper circumferential radius $\mathcal{R} = \tilde{r}$. In the general case, as follows from (8), (12) and (17),

$$\tilde{\mathcal{R}}^2 = -\frac{1}{\tilde{r}^2}\,(\tilde{\nabla}_i r)\,(\tilde{\nabla}_k\tilde{r})\,\tilde{g}^{ik} \ . \qquad (19)$$

Let us now consider a Schwarzschild black hole with mass M (and corresponding geometrical mass $\tilde{M} = GM/c^2$). In Schwarzschild co-ordinates one has

$$ds^2 = (1 - \frac{2\tilde{M}}{r})\left[dt^2 - (1 - \frac{2\tilde{M}}{r})\,dr^2 - r^2\,(1 - \frac{2\tilde{M}}{r})^{-1}\,(d\theta^2 + \sin^2\theta d\varphi^2)\right] . \quad (20)$$

From this we see that

$$\Phi = (1 - \frac{2\tilde{M}}{r}) \ , \quad \tilde{r} = r\,\Phi^{1/2} \ , \quad \tilde{g}^{rr} = -\Phi^2 \qquad (21)$$

Thus, the geodesic curvature radius of the circle r = const. is given by

$$\tilde{\mathcal{R}} = r\,(1 - \frac{3\tilde{M}}{r})^{-1} \qquad (22)$$

This is infinite at the circular photon orbit $r = 3\tilde{M}$ because, as we have already seen, a photon trajectory is a geodesic in the optical reference geometry $\tilde{g}_{ik}$. Obviously, a geodesic line must have infinite geodesic curvature radius. Note that the proper circumferential radius $\tilde{r}$ of a circle r = const. never goes to zero in the optical reference geometry $\tilde{g}_{ik}$ for a Schwarzschild black hole. Instead,

$$\tilde{\nabla}_i\tilde{r} = 0 \quad \underline{\text{at the circular photon orbit}} \ . \qquad (23)$$

This corresponds to the minimum value of the proper circumferential radius $\tilde{r}_{min} = 3\sqrt{3}\,\tilde{M}$.

III. CENTRIFUGAL FORCE REVERSED

We start by repeating an argument of Abramowicz and Lasota (1974; 1986) which explains why no centrifugal forces acts on particles which move along the circular photon orbit around a Schwarzschild black hole. The argument points out that the relative acceleration $\Delta\tilde{a}$ between two observers moving along the circular photon orbit with arbitrary (but constant) speeds, must be exactly zero. Because the gravitational acceleration $\tilde{a}_G = \tilde{G}/m = -1/9\tilde{M}$ is equal for all observers, a non-zero relative acceleration would imply a non-zero difference in centrifugal accelerations and therefore a non-zero centrifugal force. On the other hand, if the relative acceleration is zero independent of speed $\tilde{v}$ (i.e. including the case $\tilde{v} = 0$) then by reversing the above reasoning one concludes that the centrifugal force must be zero:

$$\{\Delta\tilde{a} = 0\} \equiv \{\tilde{F} = 0\} \ . \qquad (24)$$

Let us consider two observers A and B moving with different but constant speeds $\tilde{v}_A$ and $\tilde{v}_B$. The observers use radar tracking in order to detect their relative motion. The radar signals follow exactly the same paths as the observers since the signals must move along the circular photon orbit. Thus, an observer will detect no change in the direction from which he receives signals which come from the other observer. In other words, he observes his colleague as moving along a straight line. Since $\tilde{v}_A$ and $\tilde{v}_B$ are fixed in time, the relative speed $\tilde{v}_A - \tilde{v}_B$ is also constant in time. Each observer therefore sees the other to be moving along a straight line with constant speed. This clearly means that the relative acceleration between the observers is equal to zero and hence that the centrifugal force acting on particles moving along the circular photon orbit around a Schwarzschild black hole is exactly zero.

This physical argument fully agrees with the formally derived expression for the centrifugal force [cf. Eqs.(18) and (22)]:

$$\tilde{F} = \frac{m\tilde{v}^2}{r}\left(1 - \frac{3\tilde{M}}{r}\right) \quad . \tag{25}$$

Because for particle motion ($m > 0$, timelike curves in $g_{\alpha\beta}$) $\tilde{v}^2 \geq 0$, the centrifugal force is positive (i.e. it repells particles away from the axis of rotation) for circular motion with $r > 3\tilde{M}$. However it is negative (i.e. it attracts particles towards the axis) for the circular motion with $r < 3\tilde{M}$. The reversal in direction of the centrifugal force is connected with a change in sign of the geodesic curvature radius $\tilde{\mathcal{R}}$ of circles r = const. [cf. Eqs.(18), (22) and (25)]. It is the most important example of the reversed sense of the "outward" direction for dynamical effects of rotation close to a Schwarzschild black hole.

The gravitational and centrifugal forces acting on a particle with angular momentum ℓ are [cf. Eqs.(6), (7), (15), (16), (18) and (22)]:

$$\tilde{G} = \left[-\frac{\tilde{M}m}{r^2}\right] \tag{26}$$

$$\tilde{F} = \left[\frac{\ell^2 m}{r^3}\right] \frac{\left(1 - \frac{2\tilde{M}}{r}\right)\left(1 - \frac{3\tilde{M}}{r}\right)}{1 - \ell^2 r^{-2}\left(1 + \frac{2\tilde{M}}{r}\right)} \quad . \tag{27}$$

In terms of the angular velocity Ω, the centrifugal force is $\tilde{F} = [m\Omega^2 r]$ $(1 - 3\tilde{M}/r)A^2$, Te terms in the square brackets are identical to the corresponding Newtonian expressions.

The condition for free circular motion (Keplerian orbits) is

$$\tilde{F}(R, \ell^2) + \tilde{G}(r) = 0 \tag{28}$$

Fig.1 shows the curves $\tilde{F}(r, \ell^2)$ for several values of ℓ^2 and also the curve $\tilde{G}(r)$. The Keplerian orbits correspond to intersections of these curves.

The properties of Keplerian orbits around Schwarzschild black holes are, of course, well known (e.g. Shapiro and Teukolsky, 1983). Our discussion based on the optical reference geometry and summarized in Fig.3 gives an opportunity of seeing these properties from a different viewpoint. For example, the marginally stable orbit located at $r = 6\tilde{M}$ is the most curved of all the Keplerian orbits as it has the minimal geodesic curvature radius $\tilde{\mathcal{R}}_{min} = 12\tilde{M}$:

$$\left(\frac{d\tilde{\mathcal{R}}}{dr}\right)_{r = 6\tilde{M}} = 0 \quad . \tag{29}$$

This is the purely geometrical reason (explained in Fig.4) why the marginally stable orbit is located at $r = 6\tilde{M}$.

IV. RAYLEIGH CRITERION REVERSED

The relativistic concept of centrifugal force, introduced by ACL and discussed in the previous section, is quite useful for deriving the generalized Rayleigh criterion for local stability with respect to infinitesimal, axially symmetric perturbations. Such perturbations conserve angular momentum ℓ .

Let us consider a collection of particles orbiting a Schwarzschild black hole on circular orbits. The angular momentum distribution for this collection is given by $\ell = \ell(r)$. The equilibrium condition is given by the balance of forces

$$\tilde{F}(r, \ell^2) + \tilde{X}(r) = 0 \quad . \tag{30}$$

Here $\tilde{X}(r)$ is a force which depends only on location and not on any of the conserved quantities connected with the matter and its motion.

When a particle with angular momentum $\ell_0 = \ell(r_0)$ is displaced from its original position r_0 to a new one $r_1 = r_0 + \delta r$, the perturbation of the total force will be

$$\delta T = \left[\tilde{F}(\ell_1^2, r_1) + \tilde{X}(r_1)\right] - \left[\tilde{F}(\ell_0^2, r_1) + \tilde{X}(r_1)\right] = -\frac{\partial\tilde{F}}{\partial\ell^2}\frac{d\ell^2}{dr}\,\delta r \quad . \tag{31}$$

In a stable situation the perturbed force must bring the displaced particle back towards equilibrium: $\delta T\,\delta r < 0$. This is equivalent to the generalized Rayleigh criterion for stability

$$\frac{\partial\tilde{F}}{\partial\ell^2}\frac{d\ell^2}{dr} > 0 \quad . \tag{32}$$

In the familiar Newtonian situation $\partial F/\partial\ell^2 = r^{-3}$ [cf. the term in square brackets in Eq.(27)] and therefore we recover the standard form of the Rayleigh criterion: $d\ell^2/dr > 0$, which is equivalent to the following statement: for a stable configuration, specific angular momentum increases in The direction of decreasing gravity (i.e. increasing r).

In the general case, when all of the terms in Eq.(27) are taken into account for computing the derivative $\partial F/\partial\ell^2$, the stability criterion can be written as

$$\check{\mathcal{R}}^{-1} \frac{d\ell^2}{dr} > 0 . \qquad (33)$$

We see that for circular motion inside the circular photon orbit (where $\check{\mathcal{R}} < 0$) the Rayleigh criterion is reversed: for a stable configuration, specific angular momentum decreases in the direction of decreasing gravity (i.e. increasing r).

This is another example of the reversal at the circular photon orbit of the sense of the inward and outward directions for dynamical effects of rotation. It is again due to the change in sign of the geodesic curvature of the circles r = const. at that radius.

The criterion (33) may also be written in the form $(\partial \ell^2/\partial \tilde{r}) > 0$, where $\tilde{r} = [-(\xi\xi)(\eta\eta)^{-1}]^{1/2}$ is the proper circumferential radius of the circles r = const. in the optical reference geometry $\tilde{g}_{ik}$. In this form the criterion is almost identical to the one derived formally by Seguin (1975). His criterion for the case of an isentropic matter distribution in a static spacetime, demands for stability

$$[-g^{\alpha\beta}]\ \nabla_\alpha [-(\xi\xi)(\eta\eta)]\ \nabla_\beta \ell^2 > 0 \qquad (34)$$

which is in full agreement with our criterion (33).

V. ACTION OF VISCOSITY REVERSED

Anderson and Lemos (1988) noted in a recent paper that close to a black hole the viscous torque transports angular momentum inwards rather than outwards. This very interesting and important remark was the main motivation for our paper. As we have already said in the Introduction, we shall argue that the reversed sense of direction of the action of viscosity is due to the general reversal in the sense of outward and inward directions for all dynamical effects of rotation close to a black hole.

We start by recalling a general formula which describes the angular momentum flux through a spacelike surface S:

$$\dot{J} = \dot{M} j - Q . \qquad (35)$$

This formula is strictly valid in both the Newtonian theory and in general relativity. It consists of two terms: the advective flux $\dot{M}j$ and the viscous torque Q. The advective term describes the angular momentum transport connected with the macorscopic flow of matter: $\dot{M}$ is the net rest mass flux through the surface and j is the (average) specific angular momentum of matter crossing the surface S. The viscous torque W is

connected with the microscopic transport of angular momentum through the surface S in which there is no net mass flow through the surface. The details of this microscopic transport are totally ignored in the hydrodynamical macroscopic treatment. Instead, a purely phenomenological expression for the torque is assumed which is the same in both Newtonian and relativistic hydrodynamics. The torque is non-zero only if non-zero shear $\sigma_{\alpha\beta}$ is present in the fluid

$$Q = -2 \int \eta\, \sigma_{\alpha\beta}\, \xi^\beta\, dS^\alpha . \qquad (36)$$

Here dS^α is the oriented element of the surface S. In the present problem the surface S is a part of a spherical surface r = const. and therefore the vector dS^α does not change sign anywhere in $2\tilde{M} < r < \infty$. Since the viscosity coefficient η is non-negative it then follows that the viscous torque Q reverses sign if and only if the vector $\sigma_{\alpha\beta}\xi^\beta$ changes sign. For the circular motion described by formula (13)-(16) this vector can be written in a convenient form derived by Kozlowski, Jaroszynski and Abramowicz (1978) (hereafter referred to as KJA):

$$-\sigma_{\alpha\beta}\, \xi^\beta = \frac{1}{\Omega}\, \sigma_{\alpha\beta}\, \eta^\beta = \frac{1}{2} A^3 R\, \nabla_\alpha \Omega . \qquad (37)$$

For a static spacetime $R^2 = -(\eta\eta)/(\xi\xi)$. This formula is valid also for the much more general case of a stationary spacetime (e.g. Kerr) with $R^2 = (\eta\xi)^2 - (\eta\eta)/(\xi\xi)$ and $A^{-2} = (\eta\eta) + 2\Omega(\eta\xi) + \Omega^2(\xi\xi)$. It follows from this that the magnitude of the shear $\sigma^2 = -\frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ is equal to (Cf. KJA):

$$\sigma^2 = -\frac{1}{4} \frac{R^2 A^2}{(u\eta)^2 (1-\Omega\ell)^2} (\nabla_\alpha \Omega)(\nabla_\beta \Omega)\, g^{\alpha\beta} . \qquad (38)$$

The function $\sigma^2 = \sigma^2(r)$ is shown in Fig.5, taken from KJA.

The case ℓ = const. is physically important and especially easy to treat and so we shall now assume this particular angular momentum distribution. From formula (37) we get

$$\sigma_{\alpha\beta}\, \xi^\beta = -\frac{1}{2} A^3 R^2 \ell\, \nabla_\alpha \tilde{r}^2 . \qquad (39)$$

We have already noted that the proper circumferential radius $\tilde{r}$ has its minimum at the circular photon orbit, which means that the vector $\nabla_\alpha \tilde{r}^2$ changes its sign there.

This demonstrates that $\sigma_{\alpha\beta}\xi^\beta$, and therefore also the viscous torque Q, changes its sign at the circulr photon orbit. For $r < 3\tilde{M}$ the viscous torque transports the angular momentum inwards. This is yet another example of the general reveral in the sense of the inward and outward directions for the dynamical effects of rotation.

Fig.5 shows that the magnitude of the shear is non-zero at both the inner edge of the accretion disk and at the horizon of the black hole. We shall now comment on this.

(a) Viscous torque at the inner edge of the accretion disk

The standard Keplerian accretion disk model assumes that the viscous torque is equal to zero at the inner edge which is located at $r = 6M$. There, in addition, the Keplerian angular momentum has a minimum and so is locally constant. Thus, all of our formulae derived for ℓ = const. are valid. As both the viscosity coefficient and the shear tensor are non-zero there, the torque can only be zero if the density (or the thickness) of the disk is zero at the inner edge, i.e. if the properties of the flow at the inner edge are singular. This implies that the no-torque inner boundary condition is wrong. The issue of the "correct" boundary condition for the viscous torque was discussed intensely ten years ago. Today we know that the real problem was not connected with the inner boundary conditions but rather with the totally inadequate treatment of physical processes at the inner edge of the standard disk model (see, e.g. Abramowicz 1988, for a recent review). The standard model ignores the horizontal pressure gradient, horizontal heat transport, the possibility of a non-Keplerian angular momentum distribution and the dynamical importance of the accretion velocity. One consequence of this is that the standard model cannot describe the transonic part of the flow which is located very close to the inner edge.

In the correct treatment which includes all physical processes relevant for the innermost part of the flow (Muchotrzeb and Paczynski 1982; Abramowicz et al. 1988) the regularity conditions at the sonic radius give a constraint which must be obeyed by the integration constants describing the flow in order for a regular transonic stationary solution to be possible. This means, in particular, that when one assumes a value for the integration constant connected with the continuity equation (i.e. the accretion rate $\dot{M}$), the value of the integration constant connected with the angular momentum conservation cannot be assumed a priori: it is an eigenvalue of the problem and must be computed.

In the stationary case both $\dot{J}$ and $\dot{M}$ in Eq.(35) are independent of location in the accretion disk. Therefore, Eq.(35) can be written as

$$\dot{M}j(r) - Q(r) = \dot{M}j(r_0) - Q(r_0) \quad , \qquad (40)$$

where r_0 is a fixed radius (e.g. the inner radius of the disk, $r_0 = r_{in}$, or the radius of the black hole horizon, $r_0 = r_G$). The above equation can also be written in a slightly different form (which is the standard one used in accretion disk theory).

$$\dot{M}[j(r) - C_0] = Q(r) \quad . \qquad (41)$$

Here C_0 is the integration constant for angular momentum conservation and also the eigenvalue of the problem. In general

$$C_0 = j(r_0) - Q(r_0)/\dot{M} \quad , \qquad (42)$$

but when the torque vanishes at r_0, the eigenvalue $C_0 = j(r_0)$. In the Newtonian theory of Keplerian disks $j = (GMr)^{1/2}$ and therefore, for the zero torque inner boundary condition, one gets $\dot{M}[1 - (r_0/r)^{1/2}] = Q(r)/(GMr)^{1/2}$.

The question of whether or not the viscous torque vanishes at some particular radius cannot be, and more importantly does not need to be decided a priori. It can always be answered uniquely when the eigenvalue C_0 hs been found and the model constructed. In the above sense the whole problem of the "correct" inner boundary condition for the viscous torque is only of academic interest.

(b) Viscous torque at the horizon of the black hole

From Fig.5 it is clear that the shear at the horizon is not zero. One could think that because the other quantities in the integral (36) are non-zero, the viscous torque across the horizon should be non-zero as well. However, this is not necessarily true.

Let us consider a possible microphysical process of momentum transport (the situation for angular momentum transport is similar). Momentum is not a massless fluid a la flogiston. On the microphysical level it is always carried by matter. *) Macrophysically it may appear that some momentum was transported across a surface S without a net transport of mass but this is just because the microphysical mass fluxes inwards and outwards exactly cancel. This point has been nicely discussed in the Feynman Lectures on Physics.

To see this better, let us consider a simple situation (Fig.6) in which the inward and outward matter fluxes are $\dot{M}_-$ and $\dot{M}_+$, respectively, and the average momenta on either side of the surface S are denoted by j_- and j_+. The inward and outward fluxes of momentum are $\dot{J}_- = \dot{M}_- j_-$ and $\dot{J}_+ = \dot{M}_+ j_+$, and the net fluxes of mass and momentum through the surface S are $\dot{M} = \dot{M}_+ - \dot{M}_-$ and $\dot{J} = \dot{J}_+ - \dot{J}_-$. The net advective flux is $\dot{J}_0 = (\dot{M}_+ - \dot{M}_-)\, j_0$, where j_0 is the mass averaged momentum. (If the outward and inward directions are physically equivalent, j_0 is just the momentum at the surface S.) The part of the total momentum flux which is not connected witht he net mass flux is

$$Q_0 = \dot{J} - \dot{J}_0 = \dot{M}_+(j_+ - j_0) - \dot{M}_-(j_- - j_0) \qquad (43)$$

*) Matter, that is, including photons. Inclusion of radiation in our arguments is straightforward and obvious.

We shall now consider two cases: $\dot{M}_+ = \dot{M}_-$ and $Q = 0$.

i) No net mass flux ($\dot{M}_+ = \dot{M}_- = \dot{M}_0$). In the case

$$Q_0 = \dot{M}_0 \, (j_+ - j_-) \quad . \tag{44}$$

Let us consider the case when the mass flux $\dot{M}_0$ is microphysical and due to random molecular motion with mean velocity $\bar{v}$. Denoting the density by ρ and the area of the surace S by S_0, we have $\dot{M}_0 = \rho \, \bar{v} \, S_0$. The difference between the specific momenta is $j_+ - j_- = 2(dv/dr)\, \delta r$ where δr is the mean free path and $v = v(r)$ is the distribution of the velocity component v tangent to the surface S in the direction of co-ordinate r (which is orthogonal to this surface). The derivative (dv/dr) is just the shear at the surface S. We can write

$$Q_0 = 2(\rho \, \bar{v} \, \delta r) \, (dv/dr) \, S_0 \quad . \tag{45}$$

Comparing this with the well-known expression for the molecular viscosity coefficient (Lifshitz and Pitaevskii, 1981)

$$\eta = \rho \, \bar{v} \, \delta r \tag{46}$$

and with formula (36), one sees that the quantity Q_0 should be identified with the viscous torque Q. Thus, the viscous torque is equal to that part of the momentum flux which is not connected with the net mass flux through the surface.

ii) No viscous torque ($Q = 0$). It follows from Eq.(43) that a sufficient condition for this is

$$(\dot{M}_+ = 0 \text{ and } j_0 = j_-) \text{ or } (\dot{M}_- = 0 \text{ and } j_0 = j_+) \tag{47}$$

which is equivalent to saying that for vanishing viscous torque across the surface S it is sufficient that either only inward or only outward flux of matter occurs through S.

Since there is no outward flux of matter across the event horizon of a black hole, we conclude that there is no viscous torque across the horizon of a black hole. This can also be seen in another way: gravity will influence the microphysical properties of molecular motion when the gravitational acceleration g (as measured by a stationary observer) is strong enough

$$g > \frac{(\bar{v})^2}{\delta r} \tag{48}$$

When this happens, the formula (46) should include a redshift factor and so the "properly redshifted" viscosity coefficient η is related to the "standard" one by

$$\tilde{\eta} = \Phi^{1/2} \, \eta \quad . \tag{49}$$

Note the similarity between this formula and the formulae for red-shifted temperature ($T = \Phi^{1/2} T$) and chemical potential ($\mu = \Phi^{1/2} \mu$), constancy of which is necessary for thermodynamical equilibrium in a strong gravitational field.

Because $\Phi = 0$ at the horizon, the viscosity coefficient vanishes there. Consequently, there is no viscous torque across the horizon despite the fact that the shear itself is non-zero.

VI. THE CASE OF A KERR BLACK HOLE

When a black hole rotates, the spacetime $g_{\alpha\beta}$ is not static but only stationary and is described by the Kerr geometry. This introduces a few complications (fully discussed by ACL) which are connected with the fact that the Killing vectors η^α and ξ^α are not orthogonal: $(\eta\xi \neq 0)$. Because of this the formula (12) must be changed to

$$ds^2 = \Phi \, [(dt + 2\alpha_i \, dx^i)^2 + \tilde{g}_{ik} \, dx^i \, dx^k] \tag{50}$$

The off-diagonal term $\alpha_i = g_{ti}$ results from the non-zero angular momentum of the source of the gravitational field (i.e. from the rotation of the black hole), and is connected with the Coriolis force C_i which appears, together with the centrifugal force F and gravitational force G, in the generalized version of Eq.(5), derived by ACL:

$$m\tilde{f}_i = \tilde{P}^j \, \tilde{\nabla}_j \, \check{P}_i + \frac{1}{2} m^2 \, \tilde{\nabla}_i \, \Phi + 2\mathcal{E} \, \tilde{P}^j \, \omega_{ij}$$

$$\omega_{ij} \equiv (\tilde{\nabla}_i \alpha_j - \tilde{\nabla}_j \alpha_i) \quad . \tag{51}$$

This formula is valid for motion in which the total energy $= (P\eta)$ is conserved (see ACL for a more general case). The Coriolis force is defined by

$$\tilde{C}_i = 2\mathcal{E} \, \tilde{P}^j \, \omega_{ij} \quad . \tag{52}$$

We will now discuss the circular orbits. (These are described by Eq.(13); the other equations should be changed to include $(\eta\xi) \neq 0$.) Unlike the centrifugal force, the radial component of the Coriolis force, $\tilde{C} \equiv \tilde{C}_r$, changes its sign together with the velocity: $\tilde{C}(-v) = -\tilde{C}(v)$. The Coriolis force splits the unique circular photon orbit of the Schwarzschild spacetime (photons move in both positive and negative senses of the co-ordinate φ along the $r = 6\tilde{M}$ circle) into two separate orbits on which photons move in either retrograde or prograde directions (in the sense of the angular momentum of the Kerr black hole).

These orbits are located at $r = r^{+}_{ph}$ (retrograde) and $r = r^{-}_{ph}$ (prograde) with $r^{+}_{ph} > r^{-}_{ph}$. As Eq.(51) indicates, the positions of these orbits are given by the condition that the total rotational force vanishes:

$$\text{circular photon orbits:} \quad \tilde{R}(r,\ell) \equiv \tilde{F}(r,\ell^2) + \tilde{C}(r,\ell) = 0 \quad . \tag{53}$$

However, the photon orbits do not correspond to geodesic lines in the optical reference geometry g_{ik} . Therefore, the geodesic curvature radius $\tilde{\mathcal{R}}$ is not infinite at the circular photon orbits and $\tilde{\mathcal{R}}^{-1}$ does not change its sign there. Instead, this happens at $r = r_\infty$, between the retrograde and prograde circular photon orbits.

The explicit expression for $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(r,a,\tilde{M})$ in the Kerr geometry is:

$$\tilde{\mathcal{R}} = r \frac{\left(1 - \frac{4\tilde{M}}{r} + \frac{4\tilde{M}^2 + a^2}{r^2} - \frac{2\tilde{M}a^2}{r^3}\right)^{1/2}}{1 - \frac{5\tilde{M}}{r} + \frac{6\tilde{M}^2}{r} - \frac{2\tilde{M}a^2}{r^3}} \quad . \tag{54}$$

Here, a is the specific angular momentum of the Kerr black hole and r is the radial (Boyer-Lindquist) co-ordinate. Note, that $\tilde{\mathcal{R}}$ depends only on the square of the specific angular momentum of the hole, a . This means that r_∞ is the same for retrograde and prograde particle orbits. The circle $\tilde{\mathcal{R}}^{-1} = 0$ lies between the retrograde and prograde circular photon orbits:

$$r^{-}_{ph}(a) \leq r_\infty(a) < r^{+}_{ph}(a) \quad . \tag{55}$$

These three functions are shown in Fig.6.

As in the case of a Schwarzschild black hole, the total force connected with rotation, $R = F + C$, changes the sense of the inward and outward directions in very close relation with the circular photon orbits

$$\tilde{R} > 0 \quad \text{for} \quad r > r^{+}_{ph} \tag{56}$$

$$\tilde{R} < 0 \quad \text{for} \quad r < r^{-}_{ph} \tag{57}$$

independent of velocity. In between the circular photon orbits, the sense of the total rotational force is different for prograde orbits (indicated by -) and retrograde orbits (indicated by +) as shown in Table I. Other examples of the change in sense of rotational effects in the Kerr geometry will be discussed in a follow-up paper.

VII. CONCLUSIONS

We have formally demonstrated (and physically explained) that the dynamical effects of rotation change the sense of the inward and outward directions at the circular photon orbit around a Schwarzschild black hole. The reason for this is purely geometrical: the photon orbit gives a physical model for a straight line despite the fact that it is circular. The particles which move along its circular trace in the three-dimensional dspce feel no centrifugal force, because they move (in a sense formally explaiend in the paper) along a straight line which has infinite geodesic curvature radius. The circles inside the circular photon orbit have negative geodesic curvature radius. This is the reason why in this region the centrifugal force attracts the particles towards the axis of rotation, a stable angular momentum distribution has specific angular momentum outwards decreasing outwards, and viscous torque transports angular momentum inwards.

The effects described here may have some direct astrophysical consequences. When reading the first version of this paper, Dr. John Miller suggested that the reversed sense of relationship between angular velocity and eccentricity for very compact spheroids with constant density found in Chandrasekhar and Miller (1974) may be explained by reversed sense of centrifugal force in the deep interiors of these spheroids. He also pointed out that ideas presented in our paper may be relevant to discussion of the latest stages of dynamical collapse of rotating matter.

ACKNOWLEDGMENTS

One of the authors (A.R.P.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

Abramowicz M.A. 1988 in General Relativity and Gravitation, Ed. M.A.H. MacCallum (Cambridge University Press, Cambridge).

Abramowicz M.A., Carter B. and Lasota J.P., 1988 General Relativity and Gravitation, in press.

Abramowicz M.A., Czerny B., Lasota J.P. and Szuszkiewicz E. 1988 Astrophys. J. 332 646.

Abramowicz M.A. and Lasota J.P. 1974 Acta Phys. Pol. B5, 327.

Abramowicz M.A. and Lasota J.P. 1986 Am. J. Phys. 54, 936.

Anderson M.R. and Lemos J.P.S. 1988 Mon. Not. Roy. astr. Soc. 233, 489.

Chandrasekhar S. and Miller J. 1974 Mon. Not. Roy. astr. Soc. 167, 63.

Kozlowski M., Jaroszynski M. and Abramowicz M.A. 1978 Astron. Astrophys. 63, 209.

Lifshitz E.M. and Pitaevskii L.P. 1981 Physical Kinetics (Pergamon Press, Oxford).

Muchotrzeb B. and Paczyński B. 1982 Acta Astron. 31, 1.

Seguin F.H. 1975 Ap. J. 197, 745.

Shapiro S.L. and Tenkolsky S.A. 1983 Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects (Wiley Interscience, New York).

Synge J.L. and Schild A 1959 Tensor Calculus (Toronto).

Table I

The change in sense of the inward and outward directions for dynamical effects of rotation in the Kerr geometry.

	F	R_+	R_-
$r > r^+_{ph}$	positive	positive	positive
$r = r^+_{ph}$	positive	zero *	zero
$r_\infty < r < r^+_{ph}$	positive	positive	negative
$r = r_\infty$	zero	positive	negative
$r^-_{ph} < r < r_\infty$	negative	positive	negative
$r = r^-_{ph}$	negative	zero	zero +
$r < r^-_{ph}$	negative	negative	negative

* $\tilde{R}_+$ has its minimum here

\+ $\tilde{R}_-$ has its maximum here

FIGURE CAPTIONS

Fig.1 The physical definition of the outward direction.

Fig.2 The reversal in sense of the outward and inward directions at the circular photon orbit $r = 3\tilde{M}$. Circles r = const. are shown by solid lines and photon paths by broken lines. The outward direction is defined in exactly the same was as in Fig.1.

Fig.3 Keplerian orbits around a Schwarzschild black hole. The gravitational force is shown by the broken line. The centrifugal force, for various angular momenta is shown by the solid lines. Angular momenta corresponding to the circular photon orbit and marginally stable orbit are labelled by (PH) and (MS), respectively.

Fig.4 The marginally stable orbit locates athe minimal value of the curvature radius. (Imagine the surface $\tilde{\mathcal{R}}(r)$ as surface of a rigid bowl.)

Fig.5 (Taken from Kozlowski, Jaroszynski and Abramowicz, 1978): Shear magnitude for ℓ = const. accretion flows in various Kerr space-times vs. the "proper" radius $\hat{r} = \int g_{rr}^{1/2}\, dr$. The inner edges of the corresponding accretion disks are marked by small circles. Note that shear there and at the horizon is non-zero.

Fig.6 Microscopic mass flow through a surface S'.

Fig.7 The location of the two photon orbits and the circle having infinite geodesic curvature radius in various Kerr geometries.

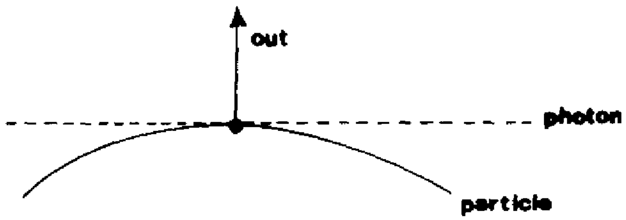

Fig.1

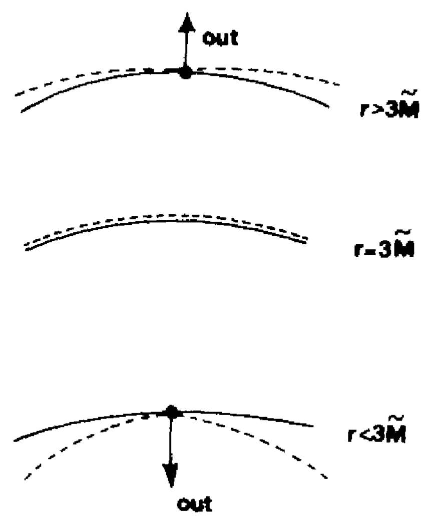

Fig.2

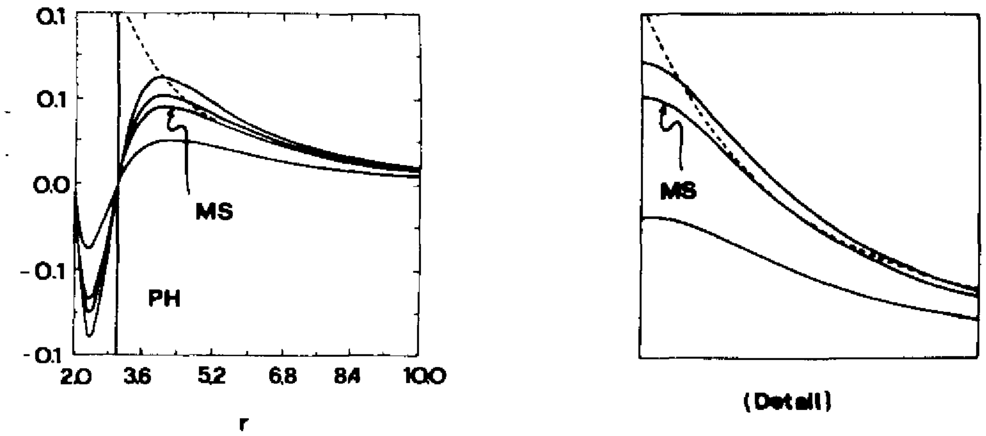

Fig.3

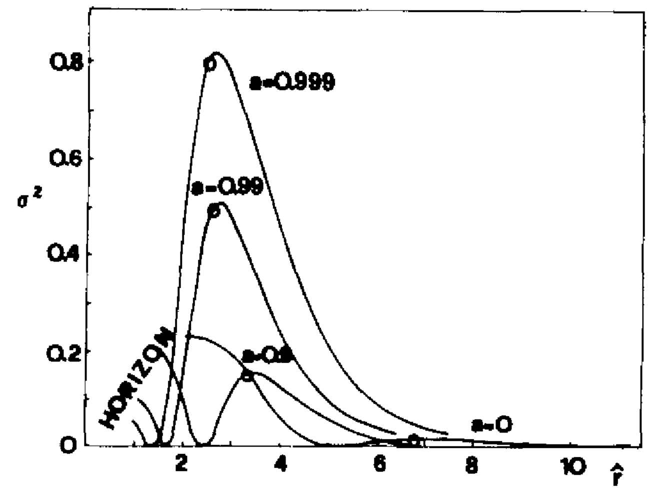

Fig.5

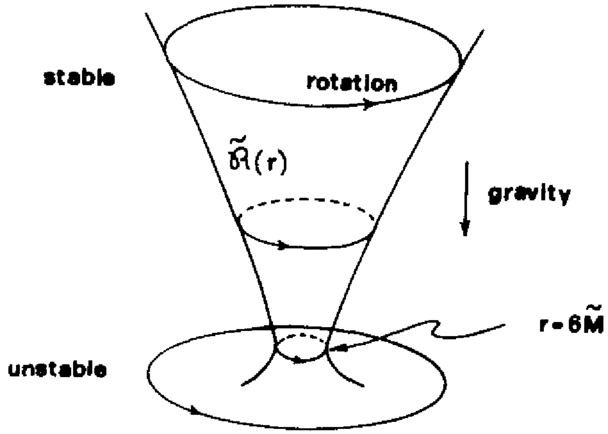

Fig.4

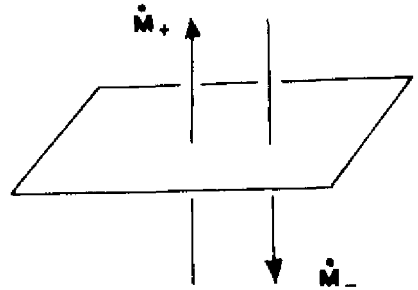

Fig.6

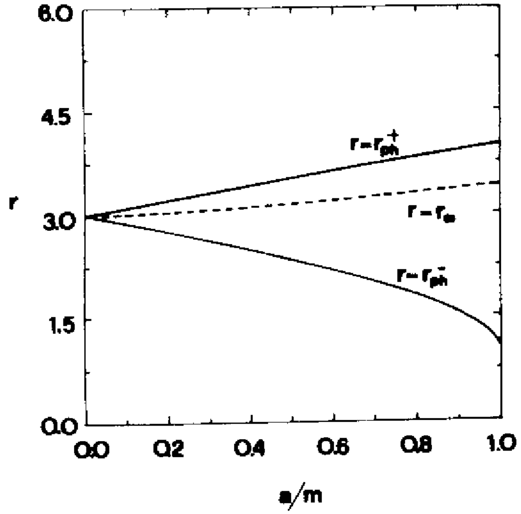

Fig.7

Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica