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Open quantum systems and the two-level atom interacting with a single mode of the electromagnetic field

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Abstract: On the basis of Lindblad theory of open quantum systems we obtain new optical equations for the system of two-level atom interacting with a single mode of the electromagnetic field. The conventional Bloch equations in a generalized form with field phases are obtained in the hypothesis that all the terms are slowly varying in the rotating frame. An analytical expression for the input-output characteristics of an optical bistable Farby-Perot resonator is obtained which includes a big atomic detuning as in the McCall-Gibbs expression of the dispersive optical bistability and an absorbtion as in the Agrawal - Carmichael - Hermann expression of the absorbtive-dispersive optical bistability.

1. Introduction

In a previous paper [1] the damped harmonic oscillator, in the Schrodinger, Heisenberg and Weyl-Wigner-Moyal representations, has been analysed on the basis of Lindblad theory of open quantum systems [2]. It was shown that various master equations for the damped quantum oscillator used in the literature for the description of the damped collective modes in deep inelastic collisions are particular cases of the Lindblad equation.

In this paper we use the same method for the quantum system of a two-level atom interacting with a single mode of the electromagnetic field. Such a system has been the object of detailed theoretical investigations in the last twenty years and it is not exaggerated to say that it lies at the heart of modern quantum optics [3-4]. Usually this system is described by the Maxwell equations coupled with the optical Bloch equations [5],

$$\frac{du}{dt} + \gamma_{\perp} u + \gamma_{\perp}\delta v = 0 \tag{1.1a}$$

$$\frac{dv}{dt} + \gamma_{\perp} v - \gamma_{\perp}\delta u - \chi w = 0 \tag{1.1b}$$

$$\frac{dw}{dt} + \gamma_{\parallel}(w-w^{0}) + \chi v = 0 \tag{1.1c}$$

where u, v, w depend on density matrix elements

$$u = \rho_{21}e^{i\omega t} + \rho_{12}e^{-i\omega t} \tag{1.2a}$$

$$v = i(\rho_{21}e^{i\omega t} - \rho_{12}e^{-i\omega t}) \tag{1.2b}$$

$$w = \rho_{22} - \rho_{11}, \tag{1.2c}$$

χ is the Rabi frequency, $\gamma_{\perp}$ and $\gamma_{\parallel}$ the decay rates for the off-diagonal and diagonal density matrix elements and $\delta = (\omega_o - \omega)/\gamma_\perp$ is the atomic detuning. These equations has been obtained from the original Bloch equations derived by Feynman and Vernon [6], by considering two relaxation processes: a) a dephasing of the electric dipole moment described by the phenomenological parameter $\gamma_\perp$ and, b) a decay of the population, described by the phenomenological parameter $\gamma_\parallel$.

In the present paper, due to the fact that the two-level atom is described by the Pauli spin matrix σ_z and the electromagnetic field by an harmonic oscillator, we open the system with all three Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (1.3)$$

and the coordinate q and momentum p operators.

In this way, by applying the Lindblad method, we obtain a master equation depending on the phenomenological coefficients which describe the opening of the system. We show that these coefficients satisfy fundamental constraints.

From this master equation we derive equations for the expectation values $<\sigma_x>$, $<\sigma_y>$, $<\sigma_z>$ and the corresponding equations in the rotating frame where the electric field is a slowly varying function. The new variables u, v, w (the Bloch vector components) can be obtained from σ_x, σ_y and σ_z by the unitary transformation

$$u = Tr(\rho R\sigma_x R) , \quad v = Tr(\rho R\sigma_y R) , \quad w = Tr(\rho R\sigma_z R) \qquad (1.4)$$

where

$$R = \begin{pmatrix} 0 & e^{\frac{i\omega t}{2}} \\ e^{-\frac{i\omega t}{2}} & \end{pmatrix} \qquad (1.5)$$

Comparing the new equations with the conventional optical Bloch equations we obtain additional terms describing the following three effects: (a) a resonance frequency shift of the atom $\omega \to \omega_0 - e$; (b) an additional coupling of the polarization with the population valid also at equilibrium; (c) a zero point polarization. The terms corresponding to the zero point polarization and to the coupling of the polarization with the population due to the opening of the system are rapidly varying in the rotating frame. Neglecting these terms we obtain the optical Bloch equations in a form where the electric field phases are given explicitly. Using these equations we study the optical bistability of a Fabry-Perot resonator filled with a two-level atomic system. For the steady state we obtain a system of equations with an approximate analitical solution, which includes the approximate expressions previously obtained by Agrawal, Carmichael and Hermann [7,24] and McCall and Gibbs [8].

2. The quantum master equation

We consider the electric field $\vec{E}$ described by an harmonic oscillator with the coordinate q and the momentum p:

$$\vec{E} = \frac{\vec{e}}{\sqrt{2\varepsilon_0 V}}(\omega q + p), \tag{2.1}$$

where ω is the frequency of the electromagnetic field, V the quantization volume and ε_0 the electric permitivity. The operators q and p obey the comutation relation $[q,p] = i\hbar$. If $|1\rangle$ and $|2\rangle$ are the fundamental and respectively the excited states of a two-level atom we can define the three hermitian operators

$$\sigma_x = |2\rangle\langle 1| + |1\rangle\langle 2| \tag{2.2a}$$

$$\sigma_y = \frac{|2\rangle\langle 1| - |1\rangle\langle 2|}{i} \tag{2.2b}$$

$$\sigma_z = |1\rangle\langle 1| - |2\rangle\langle 2| \tag{2.2c}$$

which can be represented by the matrices (1.3) and satisfy the commutation relations

$$[\sigma_x,\sigma_y] = 2i\sigma_z,\ [\sigma_z,\sigma_x] = 2i\sigma_y,\ [\sigma_y,\sigma_z] = 2i\sigma_x \tag{2.3}$$

Considering the electric dipole interaction, the hamiltonian of the quantum system takes the form

$$H = H_o - \bar{\mu}\sigma_x(p+\omega q) \tag{2.4a}$$

$$\bar{\mu} = \frac{\vec{\mu}\vec{e}}{\sqrt{2\varepsilon_o V}} \tag{2.4b}$$

where $\bar{\mu}$ is the electric dipole momentum and

$$H_o = \frac{1}{2}(-\hbar\omega_o\sigma_z + p^2 + \omega^2 q^2) \tag{2.5}$$

In agreement with Lindblad's theorem, the density matrix of the quantum system satisfies the following equation of motion

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \frac{1}{2\hbar}\sum_n([V_n\rho,V_n^+] + [V_n,\rho V_n^+]), \tag{2.6}$$

where the operators V_n are of the form

$$V_n = a_n p + b_n q + A_n\sigma_y + B_n\sigma_x + C_n\sigma_z \quad n = 1,\ 2,\ 3,\ 4,\ 5 \tag{2.7}$$

We should like to mention that the coefficients a_n and b_n, which open the electromagnetic field, are related to the diffusion and the friction coefficients which appear in the description of the damped harmonic oscillator [1] and, as we shall show in the following, the A_n and B_n coefficients which open the atom and the interaction are related to the decay and dephasing rates of the atom and the coefficient C_n which open only the interaction are related to the new effects mention at the end of the first section.

Neglecting the spontaneous interaction of an atom with the considered mode of the electromagnetic field, from (2.6) we obtain the following quantum master equation:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_o,\rho] + \frac{i\bar{\mu}}{\hbar}[\sigma_x(p+\omega q),\rho] + \frac{i\lambda}{2\hbar}([p,\rho q+q\rho]-[q,\rho p+p\rho]) +$$
$$+\frac{i\Lambda_{xy}}{2}([\sigma_y,\rho\sigma_x+\sigma_x\rho] - [\sigma_x,\rho\sigma_y+\sigma_y\rho]) +$$
$$+\frac{i\Lambda_{yz}}{2}([\sigma_z,\rho\sigma_y+\sigma_y\rho] - [\sigma_y,\rho\sigma_z+\sigma_z\rho]) +$$
$$+\frac{i\Lambda_{zx}}{2}([\sigma_x,\rho\sigma_z+\sigma_z\rho] - [\sigma_z,\rho\sigma_x+\sigma_x\rho]) -$$
$$-\frac{D_{qq}}{\hbar^2}[p,[p,\rho]] - \frac{D_{pp}}{\hbar^2}[q,[q,\rho]] + \frac{D_{qp}}{\hbar^2}([q,[p,\rho]] + [p,[q,\rho]]) -$$
$$-\Gamma_{zx}[\sigma_y,[\sigma_y,\rho]] - \Gamma_{yz}[\sigma_x,[\sigma_x,\rho]] - \Gamma_{xy}[\sigma_z,[\sigma_z,\rho]] +$$
$$+D_{xy}([\sigma_x,[\sigma_y,\rho]]+[\sigma_y,[\sigma_x,\rho]]) + D_{yz}([\sigma_y,[\sigma_z,\rho]]+[\sigma_z,[\sigma_y,\rho]]) +$$
$$+D_{zx}([\sigma_z,[\sigma_x,\rho]] + [\sigma_x,[\sigma_z,\rho]]) \qquad (2.8)$$

where the phenomenological quantities have the expressions

$$D_{qq} = \frac{\hbar}{2}\sum_n a_n a_n^*, \quad D_{pp} = \frac{\hbar}{2}\sum_n b_n b_n^*,$$

$$2D_{pq} = -\frac{\hbar}{2}\sum_n (a_n b_n^* + a_n^* b_n), \quad \lambda = \frac{1}{2i}\sum_n (a_n b_n^* - a_n^* b_n),$$

$$\Gamma_{zx} = \frac{1}{2\hbar}\sum_n A_n A_n^*, \quad \Gamma_{yz} = \frac{1}{2\hbar}\sum_n B_n B_n^*, \quad \Gamma_{xy} = \frac{1}{2\hbar}\sum_n C_n C_n^* \qquad (2.9)$$

$$2D_{xy} = -\frac{1}{2\hbar}\sum_n (A_n B_n^* + A_n^* B_n), \quad \Lambda_{xy} = \frac{1}{2i\hbar}\sum_n (A_n B_n^* - A_n^* B_n),$$

$$2D_{zx} = -\frac{1}{2\hbar}\sum_n (B_n C_n^* + B_n^* C_n), \quad \Lambda_{zx} = \frac{1}{2i\hbar}\sum_n (B_n C_n^* - B_n^* C_n),$$

$$2D_{yz} = -\frac{1}{2\hbar}\sum_n (C_n A_n^* + C_n^* A_n), \quad \Lambda_{yz} = \frac{1}{2i\hbar}\sum_n (C_n A_n^* - C_n^* A_n)$$

An important consequence of the expressions (2.9) are the following fundamental constrains:

$$D_{qq} \geq 0 \quad , \quad D_{pp} \geq 0 \tag{2.10a}$$

$$D_{pp}D_{qq} - D_{pq}^2 \geq \frac{\hbar^2\lambda^2}{4} \tag{2.10b}$$

$$\Gamma_{zx} \geq 0, \ \Gamma_{yz} \geq 0, \ \Gamma_{xy} \geq 0 \tag{2.11a}$$

$$\Gamma_{zx}\Gamma_{yz} - D_{xy}^2 \geq \frac{1}{4}\Lambda_{xy}^2 \tag{2.11b}$$

$$\Gamma_{yz}\Gamma_{xy} - D_{zx}^2 \geq \frac{1}{4}\Lambda_{zx}^2 \tag{2.11c}$$

$$\Gamma_{xy}\Gamma_{zx} - D_{yz}^2 \geq \frac{1}{4}\Lambda_{yz}^2 \tag{2.11d}$$

Indeed, the inequalities (2.10a) and (2.11a) follow directly from the definitions (2.9) and the inequalities (2.10b) and (2.11b-d) from the Schwartz inequality:

$$(\mathrm{Re}\sum_n a_n^* b_n)^2 + (\mathrm{Im}\sum_n a_n^* b_n)^2 \leq \sum_n |a_n|^2 \sum_{n'} |b_{n'}|^2$$

The expressions of the expectation values for the field and atomic operators are the following:

$$\langle q(t)\rangle = \mathrm{Tr}(\rho(t)q) \tag{2.12a}$$

$$\langle p(t)\rangle = \mathrm{Tr}(\rho(t)p) \tag{2.12b}$$

$$\Delta_{qq}(t) = \mathrm{Tr}(\rho(t)q^2) - \langle q(t)\rangle^2 \tag{2.12c}$$

$$\Delta_{pp}(t) = \mathrm{Tr}(\rho(t)p^2) - \langle p(t)\rangle^2 \tag{2.12d}$$

$$\Delta_{pq}(t) = \mathrm{Tr}(\rho(t)\frac{pq+pq}{2}) - \langle p(t)\rangle\langle q(t)\rangle \tag{2.12e}$$

$$<\sigma_x(t)> = Tr\big(\rho(t)\sigma_x\big) \qquad (2.13a)$$

$$<\sigma_y(t)> = Tr\big(\rho(t)\sigma_y\big) \qquad (2.13b)$$

$$<\sigma_z(t)> = Tr\big(\rho(t)\sigma_z\big) \qquad (2.13c)$$

$$\Delta_{xx}(t) = Tr\big(\rho(t)\sigma_x^2\big) - <\sigma_x(t)>^2 \qquad (2.13d)$$

$$\Delta_{yy}(t) = Tr\big(\rho(t)\sigma_y^2\big) - <\sigma_y(t)>^2 \qquad (2.13e)$$

$$\Delta_{zz}(t) = Tr\big(\rho(t)\sigma_z^2\big) - <\sigma_z(t)>^2 \qquad (2.13f)$$

$$\Delta_{xy}(t) = Tr\Big(\rho(t)\frac{\sigma_x\sigma_y+\sigma_y\sigma_x}{2}\Big) - <\sigma_x(t)><\sigma_y(t)> \qquad (2.13g)$$

$$\Delta_{yz}(t) = Tr\Big(\rho(t)\frac{\sigma_y\sigma_z+\sigma_z\sigma_y}{2}\Big) - <\sigma_y(t)><\sigma_z(t)> \qquad (2.13h)$$

$$\Delta_{zx}(t) = Tr\Big(\rho(t)\frac{\sigma_z\sigma_x+\sigma_x\sigma_z}{2}\Big) - <\sigma_z(t)><\sigma_x(t)> \qquad (2.13i)$$

For the positive mapping $V_n \to Tr(\rho V_n)$, results the inequality:

$$Tr\big(\rho V_n^+ V_n\big) \geq Tr\big(\rho V_n^+\big)\,Tr\big(\rho V_n\big) \qquad (2.14)$$

From (2.9), (2.13) and (2.14) one obtains the following inequalities:

$$D_{qq}\Delta_{pp}(t) + D_{pp}\Delta_{qq}(t) - 2D_{qp}\Delta_{qp}(t) \geq \frac{\lambda\hbar^2}{2} \qquad (2.15)$$

$$\Gamma_{zx}\Delta_{yy}(t) + \Gamma_{yz}\Delta_{xx}(t) - 2D_{xy}\Delta_{xy}(t) \geq \Lambda_{xy}<\sigma_z(t)> \qquad (2.16a)$$

$$\Gamma_{xy}\Delta_{zz}(t) + \Gamma_{zx}\Delta_{yy}(t) - 2D_{yz}\Delta_{yz}(t) \geq \Lambda_{yz}<\sigma_x(t)> \qquad (2.16b)$$

$$\Gamma_{yz}\Delta_{xx}(t) + \Gamma_{xy}\Delta_{zz}(t) - 2D_{zx}\Delta_{zx}(t) \geq \Lambda_{zx}<\sigma_y(t)> \qquad (2.16c)$$

From (2.10) and (2.15) we can see that the four field phenomenological coefficients λ, D_{qq}, D_{pp}, D_{qp}, of the master equation (2.b) satisfy four inequalities. At the same time, the nine atomic phenomenological coefficients Λ, Γ, D satisfy the nine inequalities (2.11) and (2.16).

3. Expectation value equations

From the master equation (2.8), the following field equations are obtained

$$\frac{d\langle p\rangle}{dt} = -\omega^2\langle q\rangle + \omega\bar{\mu}\langle\sigma_x\rangle \tag{3.1a}$$

$$\frac{d\langle q\rangle}{dt} = -\lambda\langle q\rangle + \langle p\rangle - \bar{\mu}\langle\sigma_x\rangle \tag{3.1b}$$

From the above equations we can see that the field expectation values depend on the friction coefficient λ and do not depend on diffusion terms. In order to have a physical interpretation for λ we derive the equation satisfied by the electric field $\langle E\rangle$ expection value which must be in agreement with Maxwell equations. From (3.1) one obtains

$$\frac{d^2}{dt^2}(\langle p\rangle + \langle q\rangle) + \lambda\Big[\frac{d}{dt}(\langle p\rangle+\omega\langle q\rangle) + \omega(\langle p\rangle-\omega\langle q\rangle)\Big] +$$

$$+ \omega^2(\langle p\rangle+\omega\langle q\rangle) = 2\omega^2\mu\langle\sigma_x\rangle \tag{3.2}$$

Taking into account the expression of the vector potential

$$\vec{A} = \frac{\vec{e}}{\sqrt{2\varepsilon_0 V}}\left(q - \frac{p}{\omega}\right) \tag{3.3}$$

and the eq. (3.2) the following system of equations is obtained:

$$\frac{d^2\langle E\rangle}{dt^2} + \lambda\left(\frac{d\langle E\rangle}{dt} - \omega^2\langle A\rangle\right) + \omega^2\langle E\rangle = \frac{\mu}{\varepsilon}\,\frac{\omega^2\langle\sigma_x\rangle}{V} \tag{3.4a}$$

$$\langle E\rangle = -\frac{d\langle A\rangle}{dt} \tag{3.4b}$$

Considering a near resonance case and neglecting the decay of the system ($\langle A\rangle \sim e^{-i\omega t}$), the expressions (3.4) lead to the equation

$$\frac{d^2\langle E\rangle}{dt^2} + 2\lambda\,\frac{d\langle E\rangle}{dt} + \omega^2\langle E\rangle = \frac{\mu}{\varepsilon}\,\frac{\omega^2\langle\sigma_x\rangle}{V} \tag{3.5}$$

This equation can be compared with the classical field equation obtained from Maxwell eqs:

$$\frac{d^2E}{dt^2} + \alpha' c\frac{dE}{dt} + \omega^2 E = -\frac{\mu}{\epsilon}\frac{\partial^2 S}{dt^2} \qquad (3.6)$$

where α' is the absorbiton coefficient of the electromagnetic waves and S the macroscopic polarization. From (3.5) and (3.6) we obtain a simple physical interpretation of the quantum friction coefficient.

$$\lambda = c\,\frac{\alpha'}{2} \qquad (3.7a)$$

The equations (3.5) and (3.6) can be identified when

$$S = \frac{\langle\sigma_x\rangle}{V} \quad , \quad \langle\sigma_x\rangle \sim \sin(\omega t) \qquad (3.7b)$$

Really, in the classical equation (3.6) the polarization has been considered as a quantity oscillating with the field frequency ω.

The electromagnetic field dispersion is given by the expression:

$$\Delta_{EE} = \langle E^2\rangle - \langle E\rangle^2 = \frac{1}{2\epsilon V}\left(\Delta_{pp} + \omega^2\Delta_{qq} + 2\omega\Delta_{qp}\right) \qquad (3.8)$$

From the master equation (2.8) we obtain the equations:

$$\frac{d\Delta_{qq}}{dt} = -2\lambda\Delta_{qq}(t) + 2\Delta_{qp}(t) + 2D_{qq} \qquad (3.9a)$$

$$\frac{d\Delta_{pp}}{dt} = -2\lambda\Delta_{pp}(t) - 2\omega^2\Delta_{qp}(t) + 2D_{pp} \qquad (3.9b)$$

$$\frac{d\Delta_{qp}}{dt} = -\omega^2\Delta_{qq}(t) + \Delta_{pp}(t) - 2\lambda\Delta_{qp}(t) + 2D_{pq} \qquad (3.9c)$$

This system of equations can be solved [1]. With the notations

$$X(t) = \begin{pmatrix} \omega\Delta_{qq}(t) \\ \frac{1}{\omega}\,\Delta_{pp}(t) \\ \Delta_{qp}(t) \end{pmatrix} \tag{3.10}$$

the solution of (3.9) takes the form:

$$X(t) = (Te^{Kt}T)\,X(0) + T(e^{Kt}-1)K^{-1}TD \tag{3.11}$$

where

$$T = \frac{1}{2i\omega}\begin{pmatrix} i\omega & -i\omega & 2\omega \\ -i\omega & i\omega & 2\omega \\ -\omega & -\omega & 0 \end{pmatrix} \tag{3.12}$$

$$K = \begin{pmatrix} -2(\lambda-i\omega) & 0 & 0 \\ 0 & -2(\lambda+i\omega) & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \tag{3.13}$$

From (3.11) we obtain the equilibrium value of the electromagnetic field dispersion

$$X(\infty) = -(TK^{-1}T)\,D = -R^{-1}D \tag{3.14}$$

where

$$R = \begin{pmatrix} -2\lambda & 0 & 2\omega \\ 0 & -2\lambda & -2\omega \\ -\omega & \omega & -2\lambda \end{pmatrix} \tag{3.15a}$$

$$D = \begin{pmatrix} 2\omega D_{qq} \\ \frac{2}{\omega}\,D_{pp} \\ 2D_{pq} \end{pmatrix} \tag{3.15b}$$

$$R^{-1} = \frac{-1}{4\lambda(\lambda^2+\omega^2)}\begin{pmatrix} -2\lambda^2+\omega^2 & \omega^2 & 2\omega\lambda \\ \omega^2 & 2\lambda^2+\omega^2 & -2\omega\lambda \\ -\omega\lambda & \omega\lambda & 2\lambda^2 \end{pmatrix} \tag{3.15c}$$

From (3.14) and (3.15) we can give a physical interpretation to the quantum diffusion coefficients:

$$D_{qq} = \lambda\Delta_{qq}(\infty) - \Delta_{qp}(\infty) \tag{3.16a}$$

$$D_{pp} = \lambda\Delta_{pp}(\infty) + \omega^2\Delta_{qp}(\infty) \tag{3.16b}$$

$$D_{pq} = \frac{1}{2}(\omega^2\Delta_{qq}(\infty) - \Delta_{pp}(\infty) + 2\lambda\Delta_{qp}(\infty)) \tag{3.16c}$$

From the expressions (3.16) and (2.15), we obtain the following inequality for the field variances:

$$\Delta_{qq}(\infty)\Delta_{pp}(\infty) - \Delta_{qp}^2(\infty) \geq \frac{\hbar^2}{4} \tag{3.17}$$

From the master equation (2.8) and the expressions (2.13a-c) the following equations for the atomic operator expectation values are obtained:

$$\frac{d\langle\sigma_x(t)\rangle}{dt} = -4(\Gamma_{zx}+\Gamma_{xy})\langle\sigma_x(t)\rangle + (\omega_o-4D_{xy})\langle\sigma_y(t)\rangle - 4D_{zx}\langle\sigma_z(t)\rangle + 4\Lambda_{yz} \tag{3.18a}$$

$$\frac{d\langle\sigma_y(t)\rangle}{dt} = -(\omega_o-4D_{xy})\langle\sigma_x(t)\rangle - 4(\Gamma_{yz}+\Gamma_{xy})\langle\sigma_y(t)\rangle - 4D_{yz}\langle\sigma_z(t)\rangle + 4\Lambda_{zx} + \frac{2\bar{\mu}}{\hbar}\,\mathrm{Tr}\left[(\omega q+p)\,\sigma_z\rho\right] \tag{3.18b}$$

$$\frac{d\langle\sigma_z(t)\rangle}{dt} = -4D_{zx}\langle\sigma_x(t)\rangle - 4D_{yz}\langle\sigma_y(t)\rangle - 4(\Gamma_{zx}+\Gamma_{yz})(\langle\sigma_z(t)\rangle-\sigma_z^0) - \frac{2\bar{\mu}}{\hbar}\,\mathrm{Tr}\left[(\omega q+p)\,\sigma_y\rho\right] \tag{3.18c}$$

where

$$\sigma_z^0 = \frac{\Lambda_{xy}}{\Gamma_{zx}+\Gamma_{yz}} \tag{3.19}$$

In the following we define the quantity

$$\bar{\chi} = \frac{2\bar{\mu}}{\hbar}\,\mathrm{Tr}\left[(\omega q+p)\,\rho\right] \tag{3.20}$$

and the electric field amplitude $\mathcal{E}$

$$\langle E\rangle = \frac{1}{2}(E e^{-i\omega t} + E^{*} e^{i\omega t}) \qquad (3.21)$$

Taking into account (2.1) and (2.4b) $\bar{\chi}$ takes the form

$$\bar{\chi} = \chi e^{-i\omega t} + \chi^{*} e^{i\omega t} \qquad (3.22a)$$

$$\chi = \frac{\mu E}{\hbar} \qquad (3.22b)$$

where $|\chi|$ is the Rabi frequency. Considering "the mean field approximation" the equations (3.18) become:

$$\frac{d\langle\sigma_x(t)\rangle}{dt} = -4(\Gamma_{zx}+\Gamma_{xy})\langle\sigma_x(t)\rangle + (\omega_o-4D_{xy})\langle\sigma_y(t)\rangle-4D_{zx}\langle\sigma_z(t)\rangle+4\Lambda_{yz} \qquad (3.23a)$$

$$\frac{d\langle\sigma_y(t)\rangle}{dt} = -(\omega_o-4D_{xy})\langle\sigma_x(t)\rangle-4(\Gamma_{yz}+\Gamma_{xy})\langle\sigma_y(t)\rangle-4D_{yz}\langle\sigma_z(t)\rangle+$$

$$+ 4\Lambda_{zx} + \bar{\chi}\langle\sigma_z(t)\rangle \qquad (3.23b)$$

$$\frac{d\langle\sigma_z(t)\rangle}{dt} = -4D_{zx}\langle\sigma_x(t)\rangle-4D_{yz}\langle\sigma_y(t)\rangle-4(\Gamma_{zx}+\Gamma_{yz})(\langle\sigma_z(t)\rangle-\sigma_z^0) -$$

$$- \bar{\chi}\langle\sigma_y(t)\rangle \qquad (3.23c)$$

The equations (3.18) and (3.23) are nonlinear because, in agreement with (3.22), the quantity $\bar{\chi}$ is proportional with the electric field amplitude E which, as a solution of the Maxwell equations, contains the polarization $\langle\sigma_x\rangle$. The equations (3.18) and (3.23) give also the variances Δ_{xx}, Δ_{yy}, Δ_{zz}, Δ_{xy}, Δ_{yz}, Δ_{zx}, because $\langle\sigma_x^2\rangle = \langle\sigma_y^2\rangle = \langle\sigma_z^2\rangle = 1$ and $\langle\sigma_x\sigma_y+\sigma_y\sigma_x\rangle = \langle\sigma_y\sigma_z+\sigma_z\sigma_y\rangle = \langle\sigma_z\sigma_x+\sigma_x\sigma_z\rangle = 0$. As a result, we obtain expressions of the form: $\frac{d}{dt}\Delta_{xx} = -2\langle\sigma_x\rangle\frac{d\langle\sigma_x\rangle}{dt}$, etc.

4. The Maxwell-Bloch equations

Taking into account (1.3) we find the following expressions of the expectation values of the atomic operators:

$$\langle\sigma_x\rangle = \rho_{12} + \rho_{21} \qquad (4.1a)$$

$$\langle\sigma_y\rangle = i(\rho_{12}-\rho_{21}) \qquad (4.1b)$$

$$\langle\sigma_z\rangle = \rho_{11} - \rho_{22} \qquad (4.1c)$$

From (4.1) and (1.2) we find the expression of the Bloch vector as a function of $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, $\langle\sigma_z\rangle$

$$u = \langle\sigma_x\rangle\cos\omega t - \langle\sigma_y\rangle\sin\omega t \qquad (4.2a)$$

$$v = -\langle\sigma_x\rangle\sin\omega t - \langle\sigma_y\rangle\cos\omega t \qquad (4.2b)$$

$$w = -\langle\sigma_z\rangle \qquad (4.2c)$$

The inverse transformation of (4.2) is

$$\langle\sigma_x\rangle = u\cos\omega t - v\sin\omega t \qquad (4.3a)$$

$$\langle\sigma_y\rangle = -u\sin\omega t - v\cos\omega t \qquad (4.3b)$$

$$\langle\sigma_z\rangle = -w \qquad (4.3c)$$

From (3.25), (4.2) and (4.3) we obtain:

$$\frac{du}{dt} + (\omega_o'-\omega)v + (\gamma_\perp'\cos^2\omega t+\gamma_\perp''\sin^2\omega t)u +$$

$$+ \frac{1}{2}(\gamma_\perp''-\gamma_\perp')\sin(2\omega t)v + (\gamma_2\sin\omega t - \gamma_1\cos\omega t)w$$

$$- \frac{1}{2i}(\chi-\chi^*+\chi^* e^{2i\omega t}-\chi e^{-2i\omega t})w + \lambda_2\sin\omega t - \lambda_1\cos\omega t = 0 \qquad (4.4a)$$

$$\frac{dv}{dt} - (\omega_o'-\omega)u + (\gamma_\perp'\sin^2\omega t+\gamma_\perp''\cos^2\omega t)v +$$

$$+ \frac{1}{2}(\gamma_\perp''-\gamma_\perp')\sin(2\omega t)u + (\gamma_2\cos\omega t + \gamma_1\sin\omega t)w -$$

$$- \frac{1}{2}(\chi+\chi^*+\chi^* e^{2i\omega t}+\chi e^{-2i\omega t})w + \lambda_2\cos\omega t + \lambda_1\sin\omega t = 0 \qquad (4.4b)$$

$$\frac{dw}{dt}+\gamma_{\parallel}(w-w^{0})+(-\gamma_{1}\cos\omega t+\gamma_{2}\sin\omega t)u+(\gamma_{1}\sin\omega t+\gamma_{2}\cos\omega t)v\ +$$

$$+\ \frac{1}{2i}(\chi-\chi^{*}+\chi^{*}e^{2i\omega t}-\chi e^{-2i\omega t})u\ +\ \frac{1}{2}(\chi+\chi^{*}+\chi^{*}e^{2i\omega t}+\chi e^{-2i\omega t})v\ =\ 0$$

(4.4c)

where

$$\gamma'_{\perp} = 4(\Gamma_{zx}+\Gamma_{xy}) \quad , \quad \gamma''_{\perp} = 4(\Gamma_{yz}+\Gamma_{xy}) \tag{4.5a}$$

$$\gamma_{\parallel} = 4(\Gamma_{zx}+\Gamma_{yz}) \tag{4.5b}$$

$$\omega' = \omega_{0} - 4D_{xy} = \omega_{0} - \delta \tag{4.5c}$$

$$\gamma_{1} = 4D_{zx} \quad , \quad \gamma_{2} = 4D_{yz} \tag{4.5d}$$

$$\lambda_{1} = 4A_{yz} \quad , \quad \lambda_{2} = 4A_{zx} \tag{4.5e}$$

At the same time, the expressions (4.3) can be written in the form:

$$\langle\sigma_{x}(t)\rangle = \frac{1}{2}(S e^{-i\omega t} + S^{*}e^{i\omega t}) \tag{4.6a}$$

$$\langle\sigma_{y}(t)\rangle = \frac{1}{2i}(S e^{-i\omega t} - S^{*}e^{i\omega t}) \tag{4.6b}$$

$$\langle\sigma_{z}(t)\rangle = -w \tag{4.6c}$$

where:

$$S = u - iv \tag{4.7}$$

In the following we shall show that if we assume that u, v and w and implicitly S are slowly varying in time amplitudes we can obtain the Maxwell-Bloch equations. Indeed taking the mean values in time of the equations (4.4) we obtain the eqs.:

$$\frac{du}{dt} + \gamma_{\perp}u + \gamma_{\perp}\delta v - \frac{1}{2i}(\chi-\chi^{*})w = 0 \tag{4.8a}$$

$$\frac{dv}{dt} + \gamma_{\perp}v - \gamma_{\perp}\delta u - \frac{1}{2}(\chi+\chi^{*})w = 0 \tag{4.8b}$$

$$\frac{dw}{dt} + \gamma_{\parallel}(w-w^{0}) - \frac{i}{2}\left[\chi(u+iv) - \chi^{*}(u-iv)\right] = 0 \tag{4.8c}$$

where

$$\gamma_\perp = \frac{1}{2}(\gamma'_\perp+\gamma''_\perp) = \frac{1}{2}(\Gamma_{zx} + \Gamma_{yz} + \Gamma_{xy}) \qquad (4.8d)$$

$$\delta = (\omega_0'-\omega)/\gamma_\perp \qquad (4.8e)$$

which are nothing else than the optical Bloch equations.

We should like to mention that the expressions (4.8d) and (4.8b) leads to the well known inequalities

$$2\gamma_\perp \geq \gamma_\parallel . \qquad (4.9)$$

If in (3.25) we choose the phase of the field amplitude equal to zero, χ becomes real and the equations (4.8) take the more familiar form [2]

$$\frac{du}{dt} + \gamma_\perp u + \gamma_\perp \delta v = 0 \qquad (4.10a)$$

$$\frac{dv}{dt} + \gamma_\perp v - \gamma_\perp \delta u - \chi w = 0 \qquad (4.10b)$$

$$\frac{dw}{dt} + \gamma_\parallel (w-w^0) + \chi v = 0. \qquad (4.10c)$$

Taking into account the expressions (4.7) and (4.3c) the equations (4.8) lead to the equations:

$$\frac{dS}{dt} + \gamma_\perp(1+i\delta)S = i\chi\langle\sigma_z\rangle \qquad (4.11a)$$

$$\frac{d\langle\sigma_z\rangle}{dt} + \gamma_\parallel(\langle\sigma_z\rangle-\sigma_z^0) = \frac{1}{2}(S\chi^*-S^*\chi), \qquad (4.11b)$$

which for a system of atoms take the form:

$$\frac{dS}{dt} + \gamma_\perp(1+i\delta)S = i\,\frac{\mu}{\hbar}\,EN \qquad (4.12a)$$

$$\frac{dN}{dt} + \gamma_\parallel(N-N^0) = i\,\frac{\mu}{\hbar}\,\frac{1}{2}(SE^*-S^*E) \qquad (4.12b)$$

where $N = n\langle\sigma_z\rangle$, $N^0 = n\sigma_z^0$, n being the atom density and S the macroscopic polarization.

Denoting $\delta = \delta_A/\gamma_\perp$, $i\chi = 2\bar{g}\langle A\rangle$, $\sigma_z^0 = -\frac{N}{2}$, $S = \langle R^-\rangle$,

$S^+ = \langle R^+\rangle$, $\sigma_z = R_3$, we can write the equations (4.12a-b) in the form

$$\langle \dot R^- \rangle = 2\bar g \langle A\rangle\langle R_3\rangle - (\gamma_\perp + i\delta_N)\langle R^-\rangle \qquad (4.12a')$$

$$\langle \dot R_3\rangle = -\bar g(\langle A\rangle\langle R^+\rangle + \langle A^+\rangle\langle R^-\rangle) - \gamma_\parallel(\langle R_3\rangle + \frac{N}{2}) \qquad (4.12b')$$

previously derived by Bonifacio and Lugiato [9].

The problem of the interaction of the laser radiation with a system of atoms can be solved coupling the equations (4.12) with the Maxwell equations which lead to the classical field equations

$$\frac{\partial^2 \vec E}{dt^2} + a' \frac{c}{\eta}\frac{\partial \vec E}{\partial t} + \frac{c^2}{\eta^2}\nabla\times(\nabla\times\vec E) = -\frac{\mu}{\epsilon_0\eta^2}\frac{\partial^2\vec P}{dt^2} \qquad (4.13)$$

where $\vec E$ is the classical electric field, a' the absorbtion coefficient of some perturbative transitions, η the refractive index for these perturbative transitions and $\vec P$ the macroscopic polarization.

In the case of the plan waves

$$E = \frac{1}{2}(Ee^{-i\omega t} + E^* e^{i\omega t}) \qquad (4.14a)$$

$$E = E_+ e^{ikz} + E_- e^{-ikz} \qquad (4.14b)$$

(4.13) lead to the field equations:

$$\frac{\partial E_-}{dt} - \frac{c}{\eta}\left(\frac{\partial E_-}{\partial z} - \frac{a'}{2} E_-\right) = igS_{-1} \qquad (4.15a)$$

$$\frac{\partial E_+}{dt} + \frac{c}{\eta}\left(\frac{\partial E_+}{\partial z} + \frac{a'}{2} E_+\right) = igS_{+1} \qquad (4.15b)$$

where S_+ and S_- are the first order coefficients of the polarization S in its Fourier expansion with the coordinate z, and the coupling coefficient g has the expression:

$$g = \frac{\omega\mu}{2c} \tag{4.16}$$

5. Optical Bistability

The optical bistability predicted by Szöke et al. [10] and experimentally discovered by Gibbs et al. [11], in present reached a huge practical importance due to its applicability in optical computing [12-13]. As a result, many works has been dedicated to obtain expressions of the input-output characteristics [14-15]. McCall gave a first analysis of this phenomenon using Maxwell-Bloch equations [16] and Bonifacio and Lugiato obtained a simple analytical solution [17-18]. Bonifacio and Lugiato also showed that a simple analytical solution for an optical bistable Fabry-Perot resonator, taking into account all spatial effects is not possible and proposed the model of the ring cavity [19]. However, P. Meystre analysed the applicability of the mean-field approximation for a Fabry-Perot cavity, introduced by Bonifacio and Lugiato, and proposed the description of the propagation effects with a system of two coupled nonlinear differential equations [20]. Agrawal, Carmichael and Hermann obtained improved equations for the description of the absorbtive-dispersive optical bistability and solved them analytically [21-24], [7].

In this section, using the optical Bloch equations in the form (4.8) we obtain a set of coupled nonlinear equations which lead to an analytical solution, taking into account a big atomic detuning as in the expression (5.17) in [6] derived for the dispersive optical bistability and the influence of the absorbtion as in the expressions (35) in [24] or (74) in [7].

We consider a Fabry-Perot resonator with the length l and a system of atoms inside with the density n (Fig. 1). E_I is the electric field amplitude of the laser beam incident on the mirror M_2, E_- and E_+ the slowly varying amplitudes in the resonator and $E_T = \sqrt{T}\, E_-$ the amplitude of the transmitted field. In this case we consider the amplitude E of the electric field in the cavity of the form:

$$E = E_- e^{-ikz} - E_+ e^{ikz} \tag{5.1}$$

To obtain the bistability characteristic we consider a field amplitude equation obtained from (4.13):

$$\frac{\partial E}{\partial t} + \frac{1}{2}\,\frac{c}{\eta}\left[\frac{1}{ik}\,\frac{\partial^2 E}{\partial z^2} + (\alpha' - ik)\,E\right] = igS\,, \tag{5.2}$$

In the steady state, from (4.12) we obtain

$$N = \frac{N^0}{1 + |\epsilon|^2} \tag{5.3a}$$

$$S = (1+\delta^2)^{-\frac{1}{2}} N^0 (\gamma_{\parallel}/\gamma_{\perp})^{\frac{1}{2}} (i+\delta) \frac{\epsilon}{1+|\epsilon|^2} \tag{5.3b}$$

where

$$\epsilon = \frac{E}{E_s}\ , \quad E_s = \frac{\hbar}{\mu}\sqrt{\gamma_{\perp}\gamma_{\parallel}(1+\delta^2)} \tag{5.4}$$

From (5.3) and (5.2) in the steady state we obtain a nonlinear field equation

$$\frac{d^2\epsilon}{d\zeta^2} + \left[\varkappa^2 + \varkappa\left(\overline{a_0}\,\frac{i+\delta}{1+|\epsilon|^2} + i\overline{a'}\right)\right]\epsilon = 0 \tag{5.5}$$

where $\kappa = k\ell$, $\bar{\alpha}' = \alpha'\ell$, $\zeta = z/\ell$ and

$$\bar{\alpha}_o = \frac{\eta N^o \omega \mu^2 \ell}{c\hbar\epsilon(1+\delta^2)\gamma_\perp} = \alpha_o \ell \tag{5.6}$$

In agreement with (5.1), we write the boundary conditions for a Fabry-Perot cavity in the form:

$$\epsilon_+(0) = \sqrt{R}\,\epsilon_-(0) \tag{5.7a}$$

$$\epsilon_-(1)e^{-i\phi} = \sqrt{R}\,\epsilon_+(1)e^{i\phi} + Ty \tag{5.7b}$$

where $\epsilon_+ = E_+/E_s$, $\epsilon_- = E_-/E_s$, $\epsilon_I = E_I/E_s$, $y = \epsilon_I/\sqrt{T}$ and ϕ is the cavity detuning and we have chosen the phase for $\epsilon_-(0)$ real. To solve the equation (5.5) with the boundary conditions (5.7) we consider the Fourier expansion of the nonlinear term in (5.5):

$$\frac{1}{1+|\epsilon|^2} = \frac{1}{1+|\epsilon_-|^2+|\epsilon_+|^2-2|\epsilon_-||\epsilon_+|\cos 2\kappa\zeta} =$$

$$= a_o + a_1 e^{i2\kappa\zeta} + a_1 e^{-i2\kappa\zeta} + \ldots \tag{5.8}$$

From (5.5) and (5.8) we obtain the McCall equations [25] in the form:

$$\frac{d\epsilon_-}{d\zeta} - \frac{\alpha_o(1-i\delta)}{2}(a_o\epsilon_- - a_1\epsilon_+) = 0 \tag{5.9a}$$

$$\frac{d\epsilon_+}{d\zeta} + \frac{\alpha_o(1-i\delta)}{2}(a_o\epsilon_+ - a_1\epsilon_-) = 0 \tag{5.9b}$$

From (5.8) we calculate the McCall coefficients as Fourier terms

$$a_o = \frac{1}{(1+f^2+b^2-2fb)^{1/2}(1+f^2+b^2+2fb)^{1/2}} \tag{5.10a}$$

$$a_1 = \frac{1}{2fb}\left(\frac{1 + f^2 + b^2}{(1 + f^2 + b^2 - 2fb)^{1/2}(1 + f^2 + b^2 + 2fb)^{1/2}} - 1\right) \quad (5.10b)$$

where $f = |\epsilon_-|$, $b = |\epsilon_+|$. Considering $\epsilon_- = fe^{i\theta_-}$, $\epsilon_+ = be^{i\theta_+}$, $\theta = \theta_+ - \theta_-$ the equations (5.9) become:

$$\frac{1}{f}\frac{df}{d\zeta} - \frac{\alpha_o}{2}(a_o - a_1 \frac{b}{f} \cos\theta - \delta a_1 \frac{b}{f} \sin\theta) = 0 \qquad (5.11a)$$

$$\frac{1}{b}\frac{db}{d\zeta} + \frac{\alpha_o}{2}(a_o - a_1 \frac{f}{b} \cos\theta - \delta a_1 \frac{f}{b} \sin\theta) = 0 \qquad (5.11b)$$

$$\frac{d\theta_-}{d\zeta} - \frac{\alpha_o}{2}(\delta(a_o - a_1 \frac{b}{f} \cos\theta) + a_1 \frac{b}{f} \sin\theta) = 0 \qquad (5.11c)$$

$$\frac{d\theta_+}{d\zeta} - \frac{\alpha_o}{2}\left[\delta(a_o - a_1 \frac{f}{b} \cos\theta) + a_1 \frac{f}{b} \sin\theta\right] = 0 \qquad (5.11d)$$

Using the expressions (5.3), (5.4) and (5.10) one can show that the equations (5.11) are equivalent with the equations (2.7) in [7] when the nonlinear detuning $\theta = 0$.

At the same time, the boundary conditions lead to the expressions

$$|y|^2 = \frac{f_1^2 + Rb_1^2 - 2\sqrt{R}\, f_1 b_1 \cos(\theta_1 + 2\nu)}{T^2} \qquad (5.12a)$$

$$b(0) = \sqrt{R}\, f(0) = \sqrt{R}\, x \qquad (5.12b)$$

$$\theta(0) = 0 \qquad (5.12c)$$

where $f_1 = f(1)$, $b_1 = b(1)$.

In principle from eqs. (5.11) and (5.12) the bistability characteristics $|y(x)|$ can be obtained. In the following we obtain an approximate solution. When f, $b << 1$, $a_o = 1$, $a_1 = 0$ the terms in θ can be neglected. When $f = b$, the stetic medium is almost saturated and we can approximate $\cos\theta = 1$, $\sin\theta = 0$. As a result eqs. 5.11 become

$$\frac{1}{f}\frac{df}{d\zeta} - \frac{a_o}{2}(a_o - a_1) = 0 \qquad (5.13a)$$

$$\frac{1}{b}\frac{db}{d\zeta} + \frac{a_o}{2}(a_o - a_1) = 0 \qquad (5.13b)$$

$$\frac{d\theta}{d\zeta} - \frac{2\delta}{f}\frac{df}{d\zeta} = 0 \qquad (5.13c)$$

and considering the switching when the field in the cavity is about the saturation field $f \approx b \sim 1$, from (5.10) we approximate:

$$a_o - a_1 = \frac{1}{2fb}\left(1 - \left(\frac{1 + f^2 + b^2 - 2fb}{1 + f^2 + b^2 + 2fb}\right)^{\frac{1}{2}}\right) =$$

$$= \frac{1}{2fb}\left(1 - \frac{1 - fb/(1 + f^2 + b^2)}{1 + fb/(1 + f^2 + b^2)}\right) = \frac{1}{1 + f^2 + b^2 + fb} \qquad (5.14)$$

From (5.13) and (5.12b) we obtain the simple expression $fb = \sqrt{R}\, x^2$ and from (5.12-14) we obtain:

$$|y|^2 = \frac{1}{T^2}\left(f_1^2 + R^2\frac{x^4}{f_1^2} - 2Rx^2\cos(\theta_1 + 2\phi)\right) \qquad (5.15a)$$

$$\theta_1 = 2\delta\, \ln\left(\frac{f_1}{x}\right) \qquad (5.12b)$$

$$(1 + \sqrt{R}\, x^2)\ln\left(\frac{f_1}{x}\right) + \frac{1}{2}(f_1^2 - x^2) + \frac{Rx^4}{2}\left(\frac{1}{x^2} - \frac{1}{f_1^2}\right) = \frac{\bar{a}_o}{2} \qquad (5.15c)$$

In the mean field approximation when $f_1 = x$, these expressions yield:

$$|y|^2 = x^2\left\{\frac{1}{T^2}\left[1 + R^2 - 2R\cos\left(\frac{4C\delta}{1+3x^2} - 2\phi\right)\right] + \right.$$

$$\left. + \frac{2C}{1+3x^2} + \left(\frac{4C}{1+3x^2}\right)^2\right\} \qquad (5.16)$$

where the parameter C corresponds to Bonifacio cooperativity C_B, $C = C_B/(1+\delta^2) = \bar{\alpha}_0/4T$. In order to explicitate the dependence on the atomic detuning we renormalise the saturation field $E_s = E_s'\sqrt{1+\delta^2}$, $x = f(0) = \mathcal{E}_-(0) = E_-(0)/E_s = x/\sqrt{1+\delta^2}$, $x' = E_-(0)/E_s'$, $|y| = y'/\sqrt{1+\delta^2}$, $y' = |E_I/E_s'\sqrt{T}|$.

In this case, the expression (5.16) becomes

$$y'^2 = x'^2\left\{\frac{1}{T^2}\left[1 + R^2 - 2R\cos\left(\frac{4C_B T\delta}{1+\delta^2+3x'^2} - 2\phi\right)\right] + \frac{8C_B}{1+\delta^2+3x'^2} + \left(\frac{4C_B}{1+\delta^2+3x'^2}\right)^2\right\} \qquad (5.17)$$

This expression can be compared with the analitical expression previously obtained by Agraval, Carmichael and Hermann [7,24]

$$y'^2 = x'^2\left[\left(1 + \frac{4C_B}{1+\delta^2+3x'^2}\right)^2 + \left(\frac{4C_B\delta}{1+\delta^2+3x'^2} - \frac{2\phi}{T}\right)^2\right] \qquad (5.18)$$

and with the expression for a pure-dispersive bistability previously obtained by McCall-Gibbs [8]

$$y'^2 = x'^2\,\frac{1 + R^2 - 2R\cos(2\beta-2\phi)}{T^2} \qquad (5.19)$$

where $\beta \sim \delta$. In comparison with (5.18) and (5.19) the expression (5.17) essentially takes into account the influence of a big atomic detuning as in (5.19) or the influence of the absorbtion as in (5.18).

6. The atom-field equations and the fundamental constraints

With the notations (4.5) the atom-field equations

(3.23) can be written in the form:

$$\frac{d\langle\sigma_x\rangle}{dt} + \gamma'_\perp\langle\sigma_x\rangle - (\omega_o - s)\langle\sigma_y\rangle + \gamma_1\langle\sigma_z\rangle - \lambda_1 = 0 \qquad (6.1a)$$

$$\frac{d\langle\sigma_y\rangle}{dt} + \gamma''_\perp\langle\sigma_y\rangle + (\omega_o - s)\langle\sigma_x\rangle + (\gamma_2 + \overline{\chi})\langle\sigma_z\rangle - \lambda_2 = 0 \qquad (6.1b)$$

$$\frac{d\langle\sigma_z\rangle}{dt} + \gamma_\parallel(\langle\sigma_z\rangle - \sigma_z^o) + \gamma_1\langle\sigma_x\rangle + (\gamma_2 - \overline{\chi})\langle\sigma_y\rangle = 0 \qquad (6.1c)$$

In order to compare the above system of equations with the optical Bloch equations (4.10) we shall rewrite these equations in the $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, $\langle\sigma_z\rangle$ amplitudes by using the transformation (4.2).

$$\frac{d\langle\sigma_x\rangle}{dt} = -\gamma_\perp\langle\sigma_x\rangle + (\omega_o - s)\langle\sigma_y\rangle \qquad (6.2a)$$

$$\frac{d\langle\sigma_y\rangle}{dt} = -(\omega_o - s)\langle\sigma_x\rangle - \gamma_\perp\langle\sigma_y\rangle \qquad (6.2b)$$

$$\frac{d\langle\sigma_z\rangle}{dt} = -\gamma_\parallel(\langle\sigma_z\rangle - \sigma_z^o) \qquad (6.2c)$$

We can see that the difference between the two systems of equations consists mainly in the terms related with the opening of the interaction term in the Hamiltonian (2.4a). Indeed choosing $C_n = 0$ and $A_n = -B_n$ the two systems are identical and the only difference consists in the fact that the fundamental constraint $2\gamma_\perp \geq \gamma_\parallel$ becomes $2\gamma_\perp = \gamma_\parallel$. We conclude that the simple generalization of the optical Bloch equations from the constraint $2\gamma_\perp = \gamma_\parallel$ to the constraint $2\gamma_\perp > \gamma_\parallel$ must be accompanied with the introduction of the new terms which appear in the equations (6.1). Evidently these terms indicate an additional coupling of the polarization with the population. The equations for the polarization $\langle\sigma_x\rangle$

and $\langle\sigma_y\rangle$ have a dependence on the population $\langle\sigma_z\rangle$ and at equilibrium a zero point polarization due to the terms λ_1 and λ_2.

We should like to mention again that all these coefficients must satisfy the fundamental constraints (2.10) and (2.11). Unfortunately, up to now only the optical Bloch equations have been used for the description of the present data. Consequently, in the following we shall discuss possible new constraints on measurable parameters $\gamma_\perp$ and $\gamma_\parallel$.

Using the expressions (3.18) and (4.5 a-c), the inequality (2.11b) becomes

$$\sigma_z^o \leq \left[1 - \frac{(\gamma'_\perp - \gamma''_\perp)^2 + 4s^2}{\gamma_\parallel^2}\right]^{\frac{1}{2}} \tag{6.3}$$

In the case of a weak interaction when the atom is in the ground state $\sigma_z^o = 1$, from the above relation we obtain $\gamma'_\perp = \gamma''_\perp$ and $s = 0$. A frequency shift $s < \frac{\gamma_\parallel}{2} < \gamma_\perp$ appear only when $\sigma_z^o < 1$. At the same time, the inequalities (2.11 c-d) become

$$\gamma_1\left(\gamma_\perp - \frac{\gamma_\parallel}{2}\right) - \gamma_2^2 \geq \frac{\lambda_1^2}{4} \tag{6.4.a}$$

$$\gamma_2\left(\gamma_\perp - \frac{\gamma_\parallel}{2}\right) - \gamma_1^2 \geq \frac{\lambda_2^2}{4} \tag{6.4b}$$

which lead to the inequalities

$$\gamma_\perp - \frac{\gamma_\parallel}{2} > \gamma_1 \tag{6.5a}$$

$$\gamma_\perp - \frac{\gamma_\parallel}{2} > \gamma_2 \tag{6.5b}$$

Considering a coherent electromagnetic field defined to minimize the Heisenberg uncertainty relation

$$\Delta_{qq}\Delta_{pp} \approx \hbar^2/4, \tag{6.6}$$

from (3.17) we obtain $\Delta_{qp} = 0$. Consequently the relations (3.16 a-b) yield

$$D_{qq} = \lambda\Delta_{qq} \tag{6.7a}$$

$$D_{pp} = \lambda D_{pp}. \tag{6.7b}$$

From (6.7) and (2.10b) we obtain $D_{qp} = 0$ and as a result, from (3.16c), we obtain that for the coherent light in the steady-state

$$\Delta_{pp} = \omega^2\Delta_{qq} \tag{6.8}$$

From (6.8), (2.15), (3.8) and (2.9) we obtain

$$\Delta_{EE}V \geq \left(\sum_n a_n a_n^*\right)^{-1} \frac{\hbar\lambda}{2\epsilon} \tag{6.9}$$

On the other hand (6.4), (6.5) and (2.9) yield

$$\sum_n b_n b_n^* = \omega^2 \sum_n a_n a_n^* \tag{6.10}$$

A possible solution, with complex parameters a_n and b_n, is

$$b_n = -i\omega a_n. \tag{6.11}$$

Consequently, from the definition of λ, (2.5) we obtain

$$\sum_n a_n a_n^* = \frac{\lambda}{\omega} \tag{6.12}$$

which allows to write the relation (6.9) in the form

$$\epsilon\Delta_{EE}V \geq \frac{\hbar\omega}{2} \tag{6.13}$$

or

$$\Delta_{NN} \geq \frac{1}{2} \tag{6.14}$$

where by $\Delta_{NN} = \frac{\varepsilon\Delta_{EE}}{\hbar\omega}$ we define the variance of the number of photons for the coherent light.

Conclusions

We have obtained a master equation for the quantum system of an atom interacting with a single mode of the electromagnetic field. From this equation we have obtained a set of equations for the expectation values of the field and the atomic variables. From the field equations we found that the absorbtion of the electromagnetic waves is a quantum friction while the dispersion of the electromagnetic field is a quantum diffusion. From the equations of the atom we found that the atomic decay and the dipole dephasing rates are due to the quantum diffusion while the equilibrium population is a result of the quantum friction and the quantum diffusion.

In comparison with the conventional optical Bloch equation, the new equations obtained indicate three additional effects: (a) a small frequency shift $s < \gamma_{\parallel}$; (b) an additional coupling of the polarization with the population valid also at equilibrium; (c) a zero point polarization. We should like to mention that the last two effects are purely quantum effects. They appear only due to the opening of the interaction term in the Hamiltonian (2.4a) with the terms $C_n\sigma_z$ in the operators V_n. Even in the absence of the field, due to the zero point vibrations of the harmonic oscillator we have always a non zero interaction term which leads to the coupling of the polarization with the population and to a zero point polarization. In the rotating frame the new terms become rapidly varying. The optical Bloch equations are obtained in a generalized form where the field phases appear explicitly by neglecting these terms. For an optical bistabile Fabry-Perot resonator, these equations lead to a system of two coupled nonlinear differential equations. If the nonlinear detuning θ is neglected these equations are identical with the equations derived by Agrawal and Carmichael. An analytical input-output characteristics of an optical bistabile Fabry-Perot resonator is obtained which for a small nonlinear detuning $\theta \ll 1$, leads to the expression of Agrawal, Carmichael and Hermann and which in the absence of the absorption leads to the expression of McCall and Gibbs. At the same time, we believe that the neglected, rapidly varying in the rotating frame, terms of the polarization and of the population, could significantly alter the bistability characteristics obtained from the Maxwell-Bloch equations.

Calculating the variances of the electric field we have found that the corresponding equations are not coupled with equations for the atomic variables. However, the opening of the system with the atomic operators increases the number of terms in the quantum diffusion coefficients $D_{qq} = \frac{\hbar}{2} \sum_{n=1}^{N} a_n a_n^*$, $D_{pp} = \frac{\hbar}{2} \sum_{n=1}^{N} b_n b_n^*$ and consequently increases the broadening of the spectral lines, broadening which is independent of the atomic variables or variances.

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