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COMPUTATION OF CONDITIONAL WIENER INTEGRALS BY THE COMPOSITE APPROXIMATION FORMULAE WITH WEIGHT

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1. INTRODUCTION

Functional integration method (e.g. [1]) is popular in various branches of contemporary science, particularly in quantum mechanics, field theory, probability theory, statistical physics [2]. It is widely employed in solving integral and differential equations, equations with functional derivatives, in studying asymptotics of the eigenvalues of differential operators, etc. Numerical functional integration is one of the promising means of calculations in the modern quantum physics [3]. Due to employment of this method in combination with the powerful computing techniques it became possible to solve many problems which for a long time were inaccessible for numerical study, e. g. problems concerned the topological structure of vacuum in quantum gauge theories. These problems require the computations outside of perturbation theory. One of the ways of numerical evaluation of functional integrals is Monte Carlo method of lattice computations. Introducing a space-time lattice one can replace the evaluation of functional integrals by the computation of ordinary ones of a high dimension ($\geqslant 10^5$). This approach causes however some serious problems both theoretical and technical, such as the problem of continuum limit, the dependence of result on the lattice spacing and on the means of discretization, the low speed of convergence of approximations. Besides that, this method guarantees the convergence and gives the error estimate only in probabilistic sence and needs too much computer time and memory. Thus the development of effective numerical methods to be different from lattice Monte Carlo simulations is of great importance. Some significant results in this area have been obtained recently [4].

One of the most perspective approaches is the creation of approximation formulae which are exact on a given class of functionals. In the framework of this approach we have derived [5] for the functional integrals with respect to Gaussian measures in separable Frechét spaces some new approximation formulae exact on a class of polynomial functionals of a given degree. One of the useful features of these formulae is the absence of discretization of paths. Besides that, due to high speed of convergence the employment of the derived formulae replaces the evaluation of functional integral by the evaluation of an ordinary one of a low dimension, thus allowing the use of the deterministic methods (Gaussian quadrature, Tchebyshev, etc.) and leading to a significant economy of computer time and memory.

We used the derived approximation formulae in particular case of conditional Wiener measure in our computations of Feynman path integrals in Euclidean quantum mechanics [6]. We computed the energies of low-lying levels of certain quantum mechanical systems, the wave function, the propagator. We have shown that our formulae present the way of the effective numerical investigation of topological effects concerned the instanton solutions. The computation of topological charge, topological succeptibility, θ-vacua energy [7] allowed us to search the validity bounds of the dilute instanton gas approximation.

In the present paper we derive and study some new approximation formulae for conditional Wiener integrals with the weight. One of the advantages of these formulas over the approximation formulae that we have derived formerly consists in the fact that the new formulae being more precise can be used at the same time for the large class of functionals that is of high importance for the practical applications. The employment of the formulae is illustrated with the numerical examples. The comparison of our numerical results with the values obtained by the other authors that used the Monte Carlo simulation indicates the higher efficiency of our method.

2. CONSTRUCTION OF THE APPROXIMATION FORMULAS

We shall study the conditional Wiener integral

$$\int_C P[x]\, F[x]\, d_w x$$

considering it as a Lebesgue integral with respect to a conditional Wiener measure $d_w x$. The measure $d_w x$ is a Gaussian measure defined on a space $C = \{x(t) \in C[0,1]\ ;\ x(0) = x(1) = 0\}$ by the correlation function $B(t,s) = \min\{t,s\} - ts$ and by the mean value $m(t) = 0$. $F[x]$ is a real functional defined on C ; $P[x]$ is a weight functional

$$P[x] = \exp\left\{ \int_0^1 [\, p(t)\, x^2(t) + q(t)\, x(t)]\, dt \right\} \quad ;$$

$t, s \in [0,1]$; $p(t)$, $q(t) \in C[0,1]$. For this kind of integrals we have derived the family of approximation formulae of the arbitrary degree of accuracy dependent on a natural parameter m.

<u>Theorem 1</u> [8].
Let $K(s)$ be the solution of the differential equation

$$(1-s)\, K'(s) - (1-s)^2 K^2(s) - 3K(s) - 2p(s) = 0\ , \quad s \in [0,1] \qquad (1)$$

with initial condition

$$K(1) = -\frac{2}{3}\, p(1) \;;$$

$$W(t) = \exp\left\{ \int_0^t (1-s)\, K(s)\, ds \right\};$$

$$a(t) = \int_0^t L(s)\, ds - \frac{1-t}{W(t)} \int_0^t K(s)\, W(s) \left[\int_0^s L(u)\, du \right] ds$$

$$L(t) = \int_0^t \left[K(s) W(s) H(s) - q(s) \right] ds + c\,; \quad H(t) = \int_t^1 q(s) \frac{1-s}{W(s)}\, ds$$

and the constant c is determined by the condition

$$\int_0^1 L(s)\, ds = 0 .$$

Then the approximation formula

$$\int_C P[x]\, F[x]\, d_w x = [W(1)]^{-\frac{1}{2}} \exp\left\{ \frac{1}{2} \int_0^1 L^2(t)\, dt \right\} \times \tag{2}$$
$$\times\, 2^{-m} \int_{-1}^{1} \dots \int_{-1}^{1} F[\Psi_m(v,\cdot) + a(\cdot)]\, dv_1 \dots dv_m \; + \; \mathcal{R}_m(F),$$

where

$\Psi_m(v,\cdot) = \sum_{k=1}^{m} c_k^{(m)}\, \Psi(v_k,\cdot)\;;$ $[c_k^{(m)}]^2,\ k = 1,2,\dots,m$ are the roots of

$$Q_m(z) = \sum_{k=0}^{m} (-1)^k z^{m-k}/k! \;\;;$$

$$\Psi(z,\cdot) = f(z,\cdot) - \sigma(z,\cdot)\,, \qquad \sigma(z,t) = \begin{cases} \operatorname{sign}(z), & t \le |z| \\ 0, & t > |z| \end{cases}$$

$$f(z,t) = \operatorname{sign}(z)\, \frac{1-t}{W(t)} \left[1 + \int_0^{\min\{|z|,t\}} K(s)\, W(s)\, ds \right]$$

is exact for every polynomial functional of degree $\le 2m+1$.

Remark. The estimate of the remainder $\mathcal{R}_m(F)$ is given in [5]. Formula (2) gives the good approximation of the exact value when $F[x]$ is closed to the polynomial functional of degree $\le 2m+1$. More precise approximations can be achieved for the large class of functionals if one uses the "composite approximation formulae". We have obtained these formulae [9] for a functional integral with respect to Gaussian measure without weight. In the case of conditional Wiener measure the derived formulae are written as follows:

Theorem 2 [9].

Suppose $\theta_m(v,t) = \sum_{K=1}^{m} c_K^{(m)} \rho(v_K,t)$; $\rho(\tau,t)=\begin{cases} -t\,\mathrm{sign}(\tau), & t\le|\tau| \\ (1-t)\,\mathrm{sign}(\tau), & t>|\tau| \end{cases}$

$$\theta_{m,n}(v,t) = \sum_{K=1}^{n} \frac{2}{K\pi} \sin(K\pi t) \sum_{j=1}^{m} c_j^{(m)} \mathrm{sign}(v_j) \cos(K\pi v_j) .$$

Then the following approximation formula

$$\int_C F[x]\, d_w x \approx (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\{-\tfrac{1}{2}(u,u)\} \times$$
$$\times 2^{-m} \underbrace{\int_{-1}^{1} \dots \int_{-1}^{1}}_{m} F[\theta_m(v,\cdot) - \theta_{m,n}(v,\cdot) + U_n(u,\cdot)]\, dv\, du , \tag{3}$$

where

$$U_n(u,t) = \sum_{K=1}^{n} \frac{\sqrt{2}}{K\pi} u_K \sin(K\pi t)$$

is exact for every polynomial functional of degree $\le 2m+1$.

Remark. The sufficient conditions of the convergence of approximations (3) to the exact value and the estimate of the remainder are given in [9]. Combining the methods of computation of functional integrals elaborated in [8] and [9] we obtain the new formulae with weight. These formulae acquire the advantages of the composite approximation formulae.

Theorem 3.

Under the conditions of theorems 1 and 2, the approximation formula

$$\int_C P[x]\, F[x]\, d_w x = (2\pi)^{-\frac{n}{2}} [W(1)]^{-\frac{1}{2}} \exp\left\{ \int_0^1 L^2(t)\, dt \right\} \times \tag{4}$$
$$\times \int_{\mathbb{R}^n} \exp\{-\tfrac{1}{2}(u,u)\}\, 2^{-m} \underbrace{\int_{-1}^{1} \dots \int_{-1}^{1}}_{m} \Phi[\theta_m(v,\cdot) - \theta_{m,n}(v,\cdot) + U_n(u,\cdot)]\, dv\, du + $$
$$+ R_{m,n}(F),$$

where

$$\Phi[x] = F[\varphi(x) + a(\cdot)], \tag{5}$$
$$\varphi(x(t)) = x(t) - \frac{1-t}{W(t)} \int_0^t K(s) W(s) x(s)\, ds ,$$

is exact for every polynomial functional of degree $\le 2m+1$.

Proof.

We employ the linear transformation $x(t) \longmapsto y(t)$ given by the relation $y = x + \hat{A} x$, where

$$\hat{A}\,x(t) = (1-t)\int_0^t K(s)x(s)\,ds\,, \qquad K(s) \in C[0,1]\ .$$

This transformation maps the space C onto itself in one - to - one correspondence [10]. Applying this transformation to the integral with the weight $P[x]$ we obtain

$$\int_C P[y]\,F[y]\,d_w y = D\int_C P[x+\hat{A}x]\,F[x+\hat{A}x]\ \times$$
$$\times\ \exp\left\{\frac{1}{2}\int_0^1 [(1-t)^2K^2(t) + 3K(t) - (1-t)K'(t)]\,x^2(t)\right\} d_w x\,.$$

where D is the Fredholm determinant $D = \exp\left\{\frac{1}{2}\int_0^1 (1-s)\,K(s)\,ds\right\}$. After some more transformations, assuming the condition of theorem 1 holds, we obtain in the right-hand side of the last equality the conditional Wiener integral without weight. We apply for this integral our composite approximation formula. The assertion of theorem 3 follows now from theorem 2 due to the continuity of $\varphi(x)$ and $a(t)$ and to the linearity of $\varphi(x)$.

3. CONVERGENCE OF APPROXIMATIONS

Now we shall study the convergence of approximations obtained using the formula (4) to the exact value of integral as $n \to \infty$,

Theorem 4.

Let $F[x]$ be a continuous functional on C satisfying the condition

$$|F[x]| \leq G(A(x,x)) \quad \text{for almost all } x \in C.$$

Here $G(\tau)$ is a nondecreasing positive function, $A(x,x)$ is a non-negative quadratic functional satisfying

$$A(x,x) = \sum_{k=1}^{\infty} 2\gamma_k k^2 \pi^2 \left[\int_0^1 x(t)\sin(k\pi t)\,dt\right]^2 ;$$

$$\sum_{k=1}^{\infty} \gamma_k < \infty\ ; \quad \gamma_k \geq 0, \quad k = 1,2,\dots$$

$$A(\varphi(x),\varphi(x)) \leq \beta A(x,x) \quad \text{for almost all } x \in C\ ;$$

$$\underbrace{\int_{-1}^{1}\dots\int_{-1}^{1}}_{m}\int_C G(2\beta A(\theta_m,\theta_m) + 2\beta A(x,x) + 2A(a,a))\,d_w x\,dv < \infty. \quad (6)$$

Then the remainder of the formula (4) $R_{m,n}(F)$ approaches zero, when $n \to \infty$.

Proof.

Since
$$|\Phi[x]| \le G\left(A(\varphi(x)+a,\ \varphi(x)+a)\right)$$
and
$$A(x+y,\ x+y) \le 2A(x,x)+2A(y,y)$$
there holds
$$|\Phi[x]| \le \widetilde{G}(A(x,x)) \quad \text{, where} \quad \widetilde{G}(z) = G(2\beta z + 2A(a,a)), \tag{7}$$
$\widetilde{G}(z)$ is a nondecreasing positive function. It means that the sufficient condition of $\mathcal{R}_{m,n}(F) \to 0$ is
$$\underbrace{\int_{-1}^{1}\cdots\int_{-1}^{1}}_{m}\int_{C} \widetilde{G}(A(\theta_m,\theta_m)+A(x,x))\, d_w x\, dv < \infty$$
according to theorem 4 of paper [9]. But this condition is satisfied due to (6) and (7). Thus the proof of the theorem is complete.

We will find now the estimate of the remainder of the composite approximation formula with the weight.

Theorem 5.

Suppose the quadratic functional $A(x,x)$ satisfies the conditions of theorem 4 and the functional $F[x]$ can be expressed in the form
$$F[x+x_o] = P_{2m+1}[x] + z_{2m+1}[x,x_o],$$
where $P_{2m+1}[x]$ is a polynomial functional of degree $\le 2m+1$, $x_o \in C$ and
$$|z_{2m+1}[x,x_o]| \le [A(x,x)]^{m+1}\{c_1 \exp[c_2 A(x+x_o,\ x+x_o)] +$$
$$+ c_3 \exp[c_2 A(x_o,x_o)]\},$$
$$c_1, c_2, c_3 > 0, \quad 1-4\beta c_2 \gamma_k \ge \delta > 0, \quad k=1,2,\dots$$
Then for the remainder of the composite formula (4) there holds the estimate
$$|\mathcal{R}_{m,n}(F)| \le \widetilde{D}\,\beta^{m+1}\left(\xi_m + (2m)^{m+1}\eta_m\right)\left(\sum_{k=n+1}^{\infty}\gamma_k\right)^{m+1},$$
where
$$\widetilde{D} = [W(1)]^{-\frac{1}{2}} \exp\left\{\frac{1}{2}\int_0^1 L^2(t)\,dt\right\},$$
ξ_m, η_m are positive constants dependent on m.

Proof.

It follows from the condition of the theorem that
$$\Phi[x+x_o] = P_{2m+1}[\varphi(x)] + z_{2m+1}[\varphi(x),\ \varphi(x_o)+a],$$

where $P_{2m+1}[\varphi(x)]$ is a polynomial functional of degree $\leq 2m+1$ due to the linearity of $\varphi(x)$.

$$|\tau_{2m+1}[\varphi(x), \varphi(x_0)+a]| \leq \beta^{m+1}[A(x,x)]^{m+1} \times$$
$$\times\{d_1 \exp[d_2 A(x+x_0, x+x_0)] + d_3 \exp[d_2 A(x_0, x_0)]\},$$

$$d_1 = c_1 \exp\{2c_2 A(a,a)\}; \quad d_2 = 2c_2\beta; \quad d_3 = c_3 \exp\{2c_2 A(a,a)\}.$$

Therefore, the functional $\Phi[x]$ satisfies the conditions of theorem 5 of paper [9] on the estimate of the remainder of the composite formula without weight. The employment of this theorem completes the proof.

Remark. The constants ξ_m and η_m are determined by the coefficients d_1, d_2, d_3 in accordance with the theorem 5 of [9].

Let us consider now the particular case of formulas (4) when the coefficients of the weight functional $P[x]$ are constant: $p(t) \equiv p$; $q(t) \equiv q$. One can often find this case in the practical applications. The composite formula with weight acquires now the form

$$\int_C P[x]\,F[x]\,d_w x = \sqrt{\frac{\sqrt{2p}}{\sin\sqrt{2p}}}\,\exp\left\{\frac{q^2}{2p\sqrt{2p}}\left[\tan\sqrt{\frac{p}{2}} - \sqrt{\frac{p}{2}}\right]\right\} \times$$
$$\times(2\pi)^{-\frac{n}{2}}\int_{R^n}\exp\{-\tfrac{1}{2}(u,u)\}\,2^{-m}\underbrace{\int_{-1}^{1}\cdots\int_{-1}^{1}}_{m}\Phi[\theta_m(v,\cdot)-\theta_{m,n}(v,\cdot)+U_n(u,\cdot)]\,dv\,du + {}$$
$$+ R_{m,n}(F), \qquad (8)$$

$-\infty < p < \frac{\pi^2}{2}$, $\Phi[x]$ is defined by (5). The function $a(t)$ in this case is expressed explicitly

$$a(t) = q\cdot\left(p\cos\sqrt{\tfrac{p}{2}}\right)^{-1}\cdot \sin\sqrt{\tfrac{p}{2}}\,t\cdot\sin\sqrt{\tfrac{p}{2}}(1-t)$$

that is significant for the practical applications. In order to study the properties of formula (8) we shall prove that the transformation $\varphi(x)$ is bounded on C in L_2 - norm.

Theorem 6.

For the transformation $\varphi(x)$ given by (5) and (1) with the constant coefficient $p(t) \equiv p > 0$ there holds the estimate

$$\|\varphi(x)\|^2_{L_2} \leq \beta\,\|x\|^2_{L_2}, \quad x \in C, \qquad (9)$$

where $\beta = 1 + w + 2\sqrt{w} \equiv \beta_1$; $\qquad w = \frac{(2p)^2}{27} \cdot \frac{(1+\sqrt{\frac{p}{2}})^2}{(1-\frac{p}{3})^4}$.

Proof.

It follows from (1) with $p(t) \equiv p$ that

$$\varphi(x(t)) = x(t) - \Psi(x(t))$$

where

$$\Psi(x(t)) = \sin\sqrt{2p}(1-t)\int_0^t z(s)x(s)ds \; ; \qquad z(s) = \frac{\lambda\cos\lambda - \sin\lambda}{(1-s)\sin^2\lambda} ,$$

$$\lambda = \lambda(s) = \sqrt{2p}(1-s).$$

We will show that $\int_0^1 [\Psi(x(t))]^2 dt \le w \int_0^1 x^2(t)dt$.

From inequality

$$\left[\int_0^t z(s)x(s)ds\right]^2 \le t\int_0^t z^2(s)x^2(s)ds \le \int_0^t z^2(s)x^2(s)ds , \qquad 0 \le t \le 1$$

after the change of the order of integration with respect to t and S we get

$$\|\Psi(x)\|^2_{L_2} \le \int_0^1 z^2(s)x^2(s)\int_S^1 \sin^2\sqrt{2p}(1-t)\,dt\,ds = \int_0^1 x^2(s)\omega(s)\,ds ,$$

where

$$\omega(S) = \sqrt{\frac{p}{2}}\left(\frac{\lambda\cos\lambda - \sin\lambda}{\lambda\sin^2\lambda}\right)^2\left(\lambda - \frac{\sin 2\lambda}{2}\right) .$$

Using the properties of trigonometric functions by some transformations we find that

$$0 \le \omega(s) \le \frac{\sqrt{2p}}{27}\lambda^3\frac{(1+\frac{\lambda}{2})^2}{(1-\frac{\lambda^2}{6})^4} \le w = \frac{(2p)^2}{27}\frac{(1+\sqrt{\frac{p}{2}})^2}{(1-\frac{p}{3})^4} \quad \text{for } 0 \le s \le 1 .$$

Using the Hölder inequality we obtain now

$$\int_0^1 [x(t) - \Psi(x(t))]^2 dt \le (1 + 2\sqrt{w} + w)\int_0^1 x^2(t)dt$$

that completes the proof of the theorem.

Remark. In [5] we have got the estimate (9) with

$\beta = 2 + \frac{2}{3}\gamma^2 p^2(\frac{\gamma}{3}+1)^2 \equiv \beta_2$, $\gamma = \frac{\sqrt{2p}}{\sin\sqrt{2p}}$. The new estimate with $\beta = \beta_1$ is more ingenious. It works better when p is small.

Now we can formulate the following corollary of theorem 5.

Corollary.

Suppose the functional $F[x]$ can be expressed in the form

$$F[x+x_0] = P_{2m+1}[x] + r_{2m+1}[x, x_0], \qquad x_0 \in C,$$

$P_{2m+1}[x]$ is a polynomial functional of degree $\leqslant 2m+1$, and

$$|r_{2m+1}[x,x_0]| \leqslant \Big(\int_0^1 x^2(t)\,dt\Big)^{m+1} \cdot \Big\{c_1 \exp\Big[c_2 \int_0^1 (x+x_0)^2 dt\Big] + c_3 \exp\Big[c_2 \int_0^1 x_0^2 dt\Big]\Big\},$$

$$c_1, c_2, c_3 > 0; \qquad 1 - \frac{4c_2\beta}{\pi^2} \geqslant \delta > 0.$$

Then there holds the following estimate of the remainder of the composite formula (8)

$$|R_{m,n}(F)| \leqslant \tilde{D}\,\beta^{m+1}\big(\xi_m + (2m)^{m+1}\eta_m\big)\cdot \frac{1}{n^{m+1}}, \qquad \beta = \min\{\beta_1, \beta_2\}.$$

Particularly, it follows from this estimate that the convergence of approximations obtained using the composite formulas with the weight for $n \to \infty$ has the order $O(n^{-m-1})$.

4. NUMERICAL EXAMPLES

We shall consider first the pedagogical example of computation of the conditional Wiener integral $I = \int_C P[x]\,F[x]\,d_W x$ with the weight

$$P[x] = \exp\Big\{\int_0^1 [p x^2(t) + q\,x(t)]\,dt\Big\}$$

and $F[x] = (P[x])^{-1}$. The exact value of the integral does not depend on p and q in this case and is equal to 1. The results of our computation of this integral by the "elemantary" formula with the weight (2) for $m=1$ and $m=2$ as well as by the "composite" formula (8) with $n=m=1$ are shown in Figs. 1 and 2.

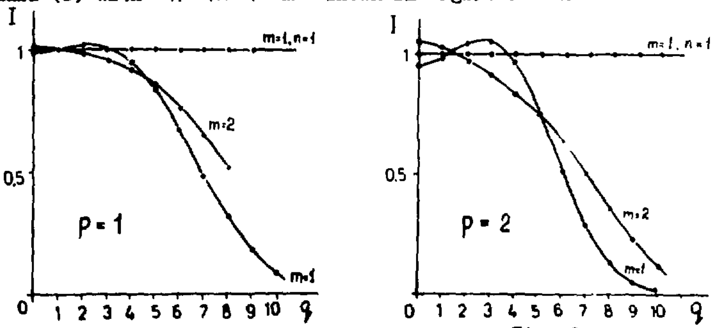

Fig. 1. Fig. 2.

The good accuracy of approximations obtained using the "elementary" formula is achieved at the relatively small values of p and q. This fact illustrates the statement that the formula (2) can be successfully employed only in the case when $F[x]$ is closed to the polynomial functional of degree $\leqslant 2m+1$. Using the "elementary" formula with weight for $m=1$ one has to evaluate the 3-dimensional integral, while for $m=2$ the dimension of integral equals 4. In the case of the "composite" formula, $n=m=1$, it is necessary to evaluate the 4-dimensional integral. We compute these integrals using the Gaussian quadratures. The results presented in Figs. 1 and 2 show that the "composite" formula with weight gives the more accurate values than the "elementary" formula of the same dimension. The results computed using (2) and (8) for $p=1$ and large q are represented in Fig. 3 in a large scale. The figures show that the composite formulae can give the good approximations even if $F[x]$ is not closed to the polynomial functional of degree $\leqslant 2m+1$.

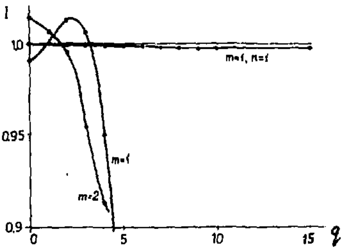

Fig. 3.

We shall illustrate now the use of our formulae with examples of simple quantum - mechanical models characterized by the Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dX^2} + V(X), \qquad X \in (-\infty, \infty).$$

We shall study the energy E_0 of the ground state and the propagator $G(\tau) = \langle 0 | X(0) X(\tau) | 0 \rangle$. The expressions for the principal quantities of Euclidean quantum mechanics in the form of functional integrals with respect to the conditional Wiener measure are given in [6]. The basis for the computation of these quantities is the Green function $\langle X | e^{-HT} | X \rangle = Z(X,T)$:

$$Z(X,T) = \frac{1}{\sqrt{2\pi T}} \int_C \exp\left\{-T\int_0^1 V(\sqrt{T}\,x(t)+X)\,dt\right\} d_W x. \qquad (10)$$

Consider the inharmonic oscillator with the potential $V(X)=\frac{1}{2}X^2+gX^4$. The numerical values computed using the composite approximation formulas with various n and m are listed in Tables 1 and 2. The overall CPU time of computation of E_0 and $G(0)$ per point g has been ca. 30 s for $n=m=1$, ca. 3.5 min. for $n=1$, $m=2$ and

Table 1.

g	E_0 [5]		E_0 (this work)				E_0^* [12]
	T	$n=m=1$	T	$n=1, m=2$	T	$n=2, m=1$	
0.1	4.5	0.570	2.5	0.550	7.5	0.557	0.559146
0.2	4.0	0.616	2.1	0.601	3.5	0.602	0.602405
0.5	3.0	0.707	1.6	0.692	4.0	0.693	0.696176
1.0	2.5	0.832	1.1	0.816	3.6	0.796	0.803771

Table 2.

g	$G(0)$ [5]		$G(0)$ (this work)				$G^{(N)}(0)$ [11]		$G^*(0)$ [11]
	T	$n=m=1$	T	$n=1, m=2$	T	$n=2, m=1$	$N=4$	$N=20$	
0.1	4.5	.419	2.5	.411	6.5	.412	.433±.16	.409±.06	.4125
0.2	4.0	.377	2.5	.359	5.0	.367	-	-	-
0.5	3.0	.313	2.0	.301	4.0	.305	.296±.07	.293±.04	.3058
1.0	2.5	.263	2.0	.257	3.3	.256	.269±.08	.267±.08	.2571

Table 3.

μ	E_0 [16]		E_0 (this work)				E_0^* [13]
	T	$n=m=1$	T	$n=1, m=2$	T	$n=2, m=1$	
- 1	2.0	1.371	3.1	1.337	3.5	1.338	1.3389
0	3.0	0.556	2.0	0.528	3.0	0.530	0.5302
1	4.0	0.632	3.2	0.577	4.5	0.657	0.5689
2	4.5	1.144	1.1	1.151	1.6	1.144	1.1448

ca. 40 s. for $n=2$, $m=1$. Exact values [11, 12] are denoted by E_0^*, $G^*(0)$. For comparison we cite the results of the paper [11], obtained via 10 simulations of 3000 paths each by the evaluation of N-fold integral using Monte Carlo method. These results are denoted $G^{(N)}(0)$. Reported CPU time is 10×25 s per point g for $N=4$ and 10×17 min per point g for $N=20$. The system with the double-well potential $V(X) = \frac{1}{2}(X^2-\mu)^2$ is of interest because it provides the convenient framework for studying the tunneling effects concerned the instanton solutions. Our results are listed in Table 3. CPU time is the same as for the inharmonic oscillator.

E_0^* denotes the exact value [13]. Since the results of the other authors are reported in a diagram form we compare them with our data in Fig. 4 (our results are given by the dots). The circles show the results [14] of lattice Monte Carlo computations. The squares represent the results of [15] obtained evaluating the N-fold integral via averaging over 10 Monte Carlo iterations on lattice with $N=303$ points and spacing $\varepsilon = 0.25$. The solid line corresponds to E_0^*, the dashed line denotes the dilute instanton gas approximation.

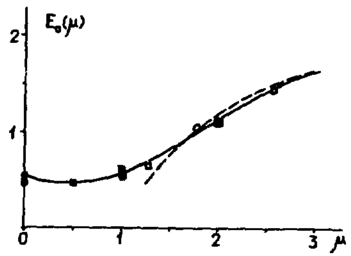

Fig. 4.

Our programs are written in standard Fortran and implemented on the CDC - 6500 computer. In [11] the computations were performed on the Vax 780 computer. In [14] and [15] this information as well as on CPU, is not given.

5. CONCLUSIONS

Due to limitation of space we were not able to discuss the results and the wide spectrum of application in detail. One can find some of the applications in [5 - 7]. The method of computation of functional integrals based on the derived approximation formulae has some important advantages over the Monte Carlo simulation method. The use of these formulae yields the more precise results while requiring the evaluation of the ordinary integrals with essentially lower dimensions. Our computations show in general a CPU time shorter by an order for the same accuracy of results. This method provides the approximations with the deterministic accuracy control. Due to the absence of lattice discretization the problems concerned the finiteness of the lattice spacing [14] do not appear. We have shown also that the use of the "composite" formulas is more advisable than the use of the "elementary" ones. Our work in the approximation theory of functional integration is stimulated by the recent advances of measure theory in quantum field theory. We are working now in the study of the functional integrals with respect to a Gaussian measure in a two - dimensional quantum field theory with polynomial interaction of boson fields. These results will be published elsewhere.

REFERENCES

1. Gelfand I.M., Yaglom A.M. J. Math. Phys., 1960, 1, p.48.
2. Kazmanishvili A.S. Functional Integration as a Method of Solving the Physical Problems. Naukova Dumka, Kiev, 1987 (in Russian).
3. Glimm J., Jaffe A. Quantum Physics. A Functional Integral Point of View. Springer, New York, 1981.
4. Egorov A.D., Sobolevskii P.I., Yanovich L.A. Approximate Methods of Computation of Functional Integrals. Nauka i Technika, Minsk, 1985, (in Russian).
5. Greguš M., Lobanov Yu.Yu., Sidorova O.V., Zhidkov E.P. J. Comp. and Appl. Math., 1987, 20, p.247.
6. Lobanov Yu.Yu., Zhidkov E.P. JINR, E2 - 87 - 5o7, Dubna, 1987.
7. Lobanov Yu.Yu., Zhidkov E.P. JINR, P2 - 86 - 433, Dubna, 1986 (in Russian).
8. Lobanov Yu.Yu., Sidorova O.V., Zhidkov E.P. JINR, P11 - 84 - 775, Dubna, 1984 (in Russian).
9. Lobanov Yu.Yu., Sidorova O.V., Zhidkov E.P. JINR, P11 - 83 - 867, Dubna, 1983 (in Russian).
10. Lobanov Yu.Yu., Sidorova O.V., Zhidkov E.P. In: JINR Rapid Communications, No.4 - 84, Dubna, 1984, p.28 (in Russian).
11. Cahill K., Reeder R. Phys. Lett., 1984, 136B, p.77.
12. Biswas S.N. et al. J. Math. Phys., 1973, 14, p.1190.
13. Blankenbecler R. et al. Phys. Rev., 1980, D21, p.1055.
14. Shuryak E.V., Zhirov O.V. Nucl. Phys., 1984, B242, p.393.
15. Creutz M., Freedman B. Ann. of Phys., 1981, 132, p.427.
16. Lobanov Yu.Yu., Sidorova O.V., Zhidkov E.P. JINR, P11 - 85 - 765, Dubna, 1985 (in Russian).

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Лобанов Ю.Ю. и др. E11-88-233
Вычисление континуальных интегралов по условной мере Винера с помощью составных приближенных формул с весом

Получены новые приближенные формулы для континуальных интегралов с весом по условной мере Винера. Формулы точны на классе функциональных многочленов произвольной заданной степени. Доказана сходимость аппроксимаций, получаемых по этим формулам, к точному значению интеграла, найдена оценка остаточного члена формул. Результаты иллюстрируются численными примерами. Преимущества этих формул над методом Монте-Карло расчетов на решетке демонстрируются на примере вычисления некоторых величин в евклидовой квантовой механике.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1988

Lobanov Yu.Yu. et al. E11-88-233
Computation of Conditional Wiener Integrals by the Composite Approximation Formulae with Weight

New approximation formulae with weight for the functional integrals with conditional Wiener measure are derived. The formulae are exact on a class of polynomial functionals of a given degree. The convergence of approximations to the exact value of integral is proved, the estimate of the remainder is obtained. The results are illustrated with numerical examples. The advantages of the formulae over lattice Monte Carlo method are demonstrated in computation of some quantities in Euclidean quantum mechanics.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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