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THE PHYSICS OF HIGH CURRENT BEAMS

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ABSTRACT

An outline is presented of paraxial charged particle optics in the presence of self-fields arising from the space-charge and current carried by the beam. Solutions of the envelope equations for beams with finite emittance are considered for a number of specific situations, with the approximation that the density profile of the beam is uniform with a sharp edge, so that the focusing remains linear. More realistic beams are then considered, and the problems of matching, emittance growth and stability are discussed.

An attempt is made to emphasize physical principles and physical ideas rather than to present the detailed mathematical techniques required for specific problems. The approach is a tutorial one, and several 'exercises' are included in the text. Most of the material is treated in more depth in the author's forthcoming book, (Ref. 6).

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TABLE OF CONTENTS

1 INTRODUCTION

In 1982 a course of lectures on High Current Beam Dynamics was presented at INS by I Hofmann, (Ref. 1). The material in the present report covers some of the same ground, but the approach is descriptive rather than mathematical, with emphasis on basic physical ideas. The aim is to present a wider perspective on some of the beam properties, rather than to provide material for making specific calculations.

First, it is important to ask what is meant by the term 'high current'. This might seem very different to a conventional accelerator engineer, a klystron designer, and a physicist investigating pulsed light ion beams for inertial confinement fusion, or 'intense relativistic electron beams' (IREB) which require neutralization before they can propagate. A low current beam could easily be defined as one in which the interaction between particles, or the particles and their environment, can be neglected. Just when this interaction becomes important, of course, depends on the particular application. Nevertheless, within the framework of beam physics familiar to accelerator builders there are some convenient criteria that can be established. As the beam current at a given energy, and within a given configuration, is increased, several effects can occur. We first consider the case when there are no background charges to cause neutralization or scattering. First, the collective self-field effects associated with space-charge and beam current can upset the focusing. A familiar example is the Q-shift that provides a limit to the current that can be injected into a synchrotron. A second effect is the onset of collective instabilities associated with the finite plasma frequency of the beam, considered as a medium. These can be of two types, intrinsic instabilities arising from the phase-space distribution of the particles in thebeam, and instabilities caused by coupling with the environment. In nearly all particle accelerator applications it is these latter that are important. (The former are, however, considered by Hofmann in his lectures, (Ref. 1) in the context of the very demanding requirements for heavy ion fusion). Effects of the type considered so far can be studied formally by use of the Vlasov equation, though in many cases this approach, which can obscure the essential physics, is not required.

A further type of interaction, which involves collisions between individual particles, is outside the scope of Liouville's theorem and the Vlasov approach. Perhaps surprisingly, it is not normally important in high current beams, but is of most practical interest in the 'Touschek effect' in, electron storage rings and the 'Boersch effect' in electron microscopy and electron beam lithography; it will not be considered further here.

In this report the term 'high-current' refers to a beam in which the forces arising from the self-fields are not negligible compared with the external focusing forces. Most of the discussion is within the constraints imposed by the linear paraxial approximation. As discussed later, this implies that, for non-relativistic particles, the difference of potential between the edge of the beam and its axis, multiplied by the charge q, must be much less than the kinetic energy of the beam particles, $q\phi \ll \frac{1}{2}m_0\beta^2c^2$. If this is not true the behaviour is non-linear and if $q\phi \gtrsim \frac{1}{2}m_0\beta^2c^2$ a 'virtual cathode' is formed and the beam does not propagate. (For relativistic particles, owing to the self magnetic field, the criterion is less severe).

When $q\phi$ is of the same order as $\frac{1}{2}m_0\beta^2c^2$ these constraints can be made less severe by reducing the potential ϕ by introducing a neutralizing charge of opposite sign in the beam. Such beams, discussed for example in the book by Miller (Ref. 2), might be called 'ultra-high current beams'. They will not be discussed here.

2 BEAMS WITH UNIFORM TRANSVERSE CHARGE DENSITY

To retain linearity of transverse focusing forces in the presence of self fields, it is necessary to use beams that have a transverse distribution of density that is uniform, and a cross-section that is circular or elliptical. As a simple example we consider a beam of circular cross-section in a uniform focusing channel, matched so that the beam radius is independent of z, the distance along the channel. In practice, a quadrupole channel in which the phase change of the betatron oscillation per period is small can be well approximated by a uniform channel. Formally this can be expressed as the smooth approximation, in which the two channels focus with the same betatron wavelength.

There are two possible distributions that satisfy the condition of uniform transverse density. The first is that of Kapchinskij and Vladimirskij (Ref. 3), known as the 'K-V distribution'. For an elliptical beam, with semi axes a and b and emittances ε_x and ε_y, the particles at a given value of z are uniformly distributed on the 4-dimensional hyperellipsoidal shell

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{x'^2a^2}{\varepsilon_x^2} + \frac{y'^2b^2}{\varepsilon_y^2} = 1 \qquad (1)$$

where the prime denotes d/dz. (Sometimes it is convenient to use time as independent variable, in which case $\dot{x} = \beta c x'$). This distribution is not realistic of course, but it provides a convenient 'zero order' approximation. For a circular beam a = b and eqn. (1) becomes

$$\frac{x^2 + y^2}{a^2} + \frac{a^2}{\varepsilon^2}\left(x'^2 + y'^2\right) = 1 \qquad (2)$$

This distribution has the following properties. First, the projections of the trajectories on the xz and yz planes are sinusoidal, with wavelength /2π

$$\lambda = \frac{a^2}{\varepsilon} \qquad (3)$$

In this case λ is the familiar lattice function β.

Projections of the trajectories on the xx' and xy planes are shown in Fig. 1a. In the xy plane they are ellipses such that the sum of the squares of the semi-axes are constant. This implies that for every particle the sum of kinetic and potential energy is the same. Also, at any point xy all particles have the same speed, and the angular distribution of velocity is constant, as shown in Fig. 1b.

Defining the temperature roughly as

$$kT_\perp = \tfrac{1}{2}\gamma m_0 \beta_\perp^2 c^2 \qquad (4)$$

it is easy to show that $\beta_\perp^2 c^2$ decreases from a maximum on the axis to zero at the edges in a manner proportional to $a^2 - r^2$, as illustrated in Fig. 1c.

No proof has been given of the initial statement that the density distribution in the xy plane is uniform. This is perhaps most directly shown by geometrical integration; see for example Lapostolle, Ref. 4. It is interesting to see the result as an extension of the theorem of Archimedes, which shows geometrically that the area $4\pi r^2$ of the surface a sphere is equal to that of the curved surface a cylinder of the same radius, and height equal to the diameter of the sphere. Projecting this surface radially on the cylinder axis gives a distribution uniform from −r to +r, and zero outside. In the K-V distribution the corresponding projection is from 4 to 2 dimensions, rather than from 3 to 1.

The second distribution that has a uniform density distribution is one in which all the particles make helical orbits about the axis, with angular velocity proportional to radius. This is represented by a twisted two dimensional surface in xx'yy' space. The flow is laminar; this means that within a volume element all particles have the same velocity. Viewed as a plasma, the temperature kT is zero. The emittance, however, which is defined in terms of the projection of the trajectories on the xx' or yy' planes is not zero, since across the beam there is a range of x' for each value of x.

Exercise 1: find the emittance of a beam of radius a and energy γ for which the angular momentum of the particles at the beam edge is p_Θ.

3 THE PERVEANCE CONCEPT

For both the K-V and helical distributions it is straightforward to find the self-fields. The outward repulsive electric field may be found in terms of the charge density nq and radial position,

$$E_r = \frac{\pi r^2 nq}{2\pi r\varepsilon_0} = \frac{rnq}{2\varepsilon_0} \tag{5}$$

If βc is the velocity of beam, then there will be magnetic field arising from the current

$$B_\theta = \frac{\mu_0 \pi r^2 nqc}{2\pi r} = \frac{rnq\beta}{2\varepsilon_0 c} \tag{6}$$

If the beam radius is a, and the number of particles per unit length is N then the force on a particle at radius r in the beam is:

$$\begin{aligned} F_r &= qE_r - q\beta c B_r \\ &= \frac{Nq^2 r}{4\pi\varepsilon_0 a^2}(1-\beta^2) \end{aligned} \tag{7}$$

Setting this equal to $\gamma m_0 r$, and writing $r = \beta^2 c^2 r$ yields the contribution to r, the derivative of the radial gradient of the particle orbit:

$$r'' = \frac{2Nr_c}{\beta^2\gamma}(1-\beta^2)\frac{r}{a^2} = \frac{2Nr_c}{\beta^2\gamma^3}\frac{r}{a^2} \tag{8}$$

where r_c is the classical particle radius $r_c = q^2/4\pi\varepsilon_0 m_0 c^2$.

It is convenient to define the dimensionless perveance

$$K = \frac{2Nr_c}{\beta^2\gamma^3} \tag{9}$$

This is related to the more familiar non-relativistic perveance $k = I/V^{3/2}$ when $\gamma \sim 1$, $K \approx 15{,}000\ k$. For electrons a perveance of 1 'microperv', $k = 10^{-6}$, corresponding for example to 10 kV and 1 amp, is equivalent to $K = 0.015$.

An example of the physical meaning of a beam with $K = 1$ is illustrated in Fig. 2. A non-relativistic electron beam is extracted from a planar cathode by means of a transparent grid. It is confined by an 'infinite' magnetic field to a close fitting conducting tube of radius a. Consider now the path of integration 1 2 3 4 shown, where the radius $1 \rightarrow 2 \ll 2 \rightarrow 3$. The grid and tube are at the same potential, so that $\int_3^4 E d\ell = -\int_4^1 E d\ell$. For small K, the potential depression $q \int_4^3 E_r\, dr = V_1 - V_4 = \frac{1}{2} m_0 c^2(\beta_1{}^2 - \beta_4{}^2)/q$. Inserting the value of E_r from eqn. (5), and setting $N = \pi a^2 n$ yields

$$\frac{Nq^2}{4\pi\varepsilon_0} = \frac{1}{2} m_0 (\beta_1{}^2 - \beta_4{}^2)\, c^2 \tag{10}$$

whence

$$K = \frac{2Nr_c}{\beta_1{}^2} = 1 - \frac{\beta_4{}^2}{\beta_1{}^2} \tag{11}$$

This formula is only true when $\beta_4 \ll \beta_1$, since as β_4 decreases the charge density increases, making E_r larger. Nevertheless, as β_4 approaches zero, and a virtual cathode appears on the axis, K is of order unity. It is also interesting to note that as K approaches unity, the stored electrostatic energy in the beam becomes approximately equal to the kinetic energy of the electrons, (Ref. 5). Note also that $K = 1$ can be written $a\omega_p = 2\beta c$, (again in the non-relativistic limit), where ω_p is the plasma frequency, defined as $(nq^2/m\varepsilon_0)^{\frac{1}{2}}$. The distance $2\pi\beta c/\omega_p$ travelled by the beam in a time corresponding to one cycle of plasma oscillations is then πa.

If the beam is partly neutralized by charges of the opposite sign, where f is the neutralization fraction, that the electrostatic force is reduced by the factor 1-f; the definition of the perveance can then be extended

$$K = \frac{2Nr_c}{\beta^2\gamma}\left(1-\beta^2-f\right) \qquad (12)$$

If $f = 1-\beta^2 = 1/\gamma^2$ the self-electric and magnetic fields cancel; then $K = 0$, and an initially parallel beam remains so. (Scattering, of course, is neglected in this simple model). For an electron beam the electrons move parallel, and the neutralizing ions oscillate transversely in the potential well of the electrons. It is interesting to make a Lorentz transformation into a frame moving with the velocity of the electron beam. In this frame the electron density is decreased by the Lorentz factor γ, while the ion density is increased by the same amount, this increases f by a factor of γ^2 to become unity. This means that the electrons and ions have the same density; there is no electric field, so that electrons remain stationary, whereas the ions experience an inward force arising from the self-magnetic field associated with the ion current. This is illustrated in Fig. 3. (This is of course a somewhat artificial calculation, in any practical beam f will not be uniform across the beam cross-section).

If $f > 1/\beta^2$, the perveance becomes negative; in particular, when the beam is completely neutralized, $K = -2Nr_c/\beta^2\gamma$, and the beam pinches. This is further discussed in the next section. Finally we note that the perveance K can be written in several alternative forms. For $f = 0$, in terms of the Alfven current

$$I_A = \left(\frac{4\pi\varepsilon_0 m_0 c^3}{q}\right)\beta\gamma \qquad (13)$$

K may be expressed as

$$K = \frac{2Nr_c}{\beta^2\gamma^3} = \frac{2I}{\beta\gamma^2\, I_A} = \frac{\omega_p{}^2 a^2}{2\beta^2\gamma^2 c^2} \qquad (14)$$

where ω_p is the plasma frequency of the beam measured in the moving frame,

$$\omega_p{}^2 = ne^2/\varepsilon_0\gamma m_0 .$$

4 THE PARAXIAL ENVELOPE EQUATION WITH SELF-FIELDS

At the edge of the beam $r = a$, so that eqns. (8) and (9) may be combined to give

$$a'' = K/a \qquad (15)$$

This is one of the terms, in different notation, of the so-called K-V envelope equations, first presented by Kapchinskij and Vladimirskij in 1959, (Ref. 3). These may be derived in several ways, (see for example, Ref. 6) and the results are quoted below for an elliptical beam with semi-axes a and b

$$a'' + \kappa_x(z)a - \frac{\varepsilon_x^{\;2}}{a^3} - \frac{2K}{a+b} = 0$$
$$b'' + \kappa_y(z)a - \frac{\varepsilon_y^{\;2}}{a^3} - \frac{2K}{a+b} = 0 \qquad (16)$$

The discussion which follows is intended to illustrate the physical meaning of the various terms, examined from different points of view. For this it is sufficient to put a = b and $\varepsilon_x = \varepsilon_y$ to give

$$\underset{1}{a''} + \underset{2}{\kappa(z)a} - \underset{3}{\frac{\varepsilon^2}{a^3}} - \underset{4}{\frac{K}{a}} = 0 \qquad (17)$$

The terms have been numbered for further reference. Several interesting special cases can be studied by taking various combinations of terms.

Case 1, terms 1 and 2

This represents the motion of a single particle in a focusing channel. For the special case of uniform focusing, where $\kappa(z)$ is independent of z, the motion is harmonic. If λ_0 is the wavelength of the particle oscillation, $\kappa = 1/\bar{\lambda}_0^{\,2}$.

Case 2, terms 2 and 3

This represents a matched beam in a uniform channel (a' = 0). Writing κ in terms of $\bar{\lambda}_0$ yields

$$\varepsilon = a/\bar{\lambda}_0^{\,2} \qquad (18)$$

as in eqn (3).

Case 3, terms 2 and 4

This represents a uniform beam with laminar flow. The condition is that

$$K = (a/\bar{\lambda}_0)^2$$

Clearly this must be less than unity, as noted before.

Case 3, terms 3 and 4

Since these terms have the same sign there can only be a balance when K is negative. The condition for this is

$$\varepsilon^2 = - Ka^2 \tag{19}$$

Exercise 2

For a completely neutralized non-relativistic beam, ($f=1$, $\gamma-1<<1$), with the definition of temperature as in eqn. (4) show that

$$I^2 \approx 2NkT \tag{20}$$

This is the Bennett pinch relation, true also in plasma columns when $\beta_\perp{}^2 >> \beta_\parallel{}^2$. It is usually proved by setting the magnetic pressure of the self-field, $B^2/2\mu_0$, equal to the gas pressure, nkT. It can be written in the form

$$\beta_\perp{}^2 \;/\; \beta_\parallel{}^2 = Nr_c \tag{21}$$

$Nr_c << 1$ neutral beam, $Nr_c >> 1$ plasma

There is not a smooth transition as Nr_c passes through unity; for Nr_c between about 1 and 1000 there are two superimposed streams, a plasma plus a 'runaway' beam. Of course such systems are complicated, and such topics as collisions, radiation, and the electric field required to maintain the stream in a steady state (if there is one) must be taken into account.

It is interesting to note that both 'optical' and 'hydrodynamic' descriptions of beams are possible. The former which is used here, is appropriate to accelerator beams. Nevertheless, in deriving the paraxial envelope equation, an alternative approach may be made. The forces on a small volume element of the beam can be considered. This is acted upon by the focusing force and self-forces in the same way as a single charge. For a non-laminar beam, however, there will be an additional force arising from the pressure gradient in the beam. This pressure can be defined formally in terms of the pressure tensor, even though there are no collisions. It is shown in Ref. 7 that this gives rise to the emittance term in eqns. (16) and (17).

Case 4, terms 2, 3 and 4

This represents a matched beam with emittance and space-charge. We now introduce α, the 'acceptance' of a channel of radius a. This is the emittance of a low intensity beam (with $K \approx 0$) that produces a beam of radius a. Setting $a'' = 0$ as before for a matched beam, the equations relating a, λ, K, ε (with finite K) and α (with $K = 0$) are

$$\frac{a}{\lambda_0^2} - \frac{\varepsilon^2}{a^3} - \frac{K}{a} = 0 \qquad (22a)$$

$$\frac{a}{\lambda_0^2} - \frac{\alpha^2}{a^3} = 0 \qquad (22b)$$

from which it follows that

$$\frac{a^4}{\varepsilon^2 \lambda_0^2} = 1 + \frac{Ka^2}{\varepsilon^2} = \frac{\alpha^2}{\varepsilon^2} \qquad (23)$$

We can now define λ, (without subscript) as the oscillation wavelength in the presence of self-fields, and write

$$\frac{\lambda_0}{\lambda} = \frac{\varepsilon}{\alpha} = \frac{\sigma}{\sigma_0} = \left(1 + \frac{Ka^2}{\varepsilon^2}\right)^{-\frac{1}{2}} \qquad (24)$$

where σ and σ_0 are the betatron phase changes over some arbitrary length. This is illustrated in Fig. 4. Although this theory has been derived for a uniform focusing channel, it is a good approximation for a periodic channel, where σ_0, which can be taken as the phase-shift per period, is small. From eqn. (24) it is evident that Ka^2/ε^2 is an important parameter. For $K = 0$, $\sigma/\sigma_0 = 1$. For $\varepsilon = 0$, $\sigma/\sigma_0 = 0$, and the beam particles move parallel to the axis. When $K = \varepsilon^2/a^2$, $\sigma/\sigma_0 = 1/\sqrt{2}$; this may be conveniently be taken as the divide between 'emittance dominated' and 'self-field dominated' beams. It may be noted that in paraxial scaling when a is changed, ε^2/K must be changed in the same ratio. In such scaling, phase-shifts are preserved, but the angle that trajectories make with the axis can be varied.

Exericse 3.

In systems that are not paraxial, (for example when aberrations, end effects etc are taken into account) the angles that trajectories make with the axis must remain constant if true scaling is to be achieved. This may be important in some experiments. What are the scaling laws in this case?

Exercise 4

Using eqn. (4) again, and defining the Debye length λ_D as

$$\lambda_D^2 = \frac{kT}{\varepsilon_0 n q^2}$$

show that the Debye length equals the beam radius when $Ka^2/\varepsilon^2 \approx 1$. What does this imply when a beam is regarded as a plasma? (Not a simple question, since the K-V distribution is rather singular. See discussion by Hofmann in Ref. 8).

Exercise 5

Equation (24) can be written in many ways, by considering the 2nd and 4th terms show that it is equivalent to

$$\begin{aligned} I &= \left(\frac{2\pi\varepsilon_0 m_0 c^3}{q}\right) \beta^3\gamma^3 \frac{\alpha^2}{\varepsilon^2} \left(1 - \frac{\varepsilon^2}{\alpha^2}\right) \\ &= \tfrac{1}{2}\, I_A\ \beta^2\gamma^2 \frac{\alpha^2}{a^2} \left(1 - \frac{\varepsilon^2}{\alpha^2}\right) \end{aligned} \tag{25}$$

where I_A is the Alfven current, (eqn. 13).

This formula was derived in Ref. 9 by Reiser. He also analysed the much more complicated problem of propagation in a periodic channel, where, for large values of σ_0, the approximation used here is not valid.

Cases 5 and 6, Terms 1 and 3, and 1 and 4

Equation (17) is also useful for discussing beams in free space. Terms 1 and 3 describe a hyperbolic emittance waist, and terms 1 and 4 describe the classical 'space-charge spreading curve', or alternatively a space-charge waist. The two equations are

$$a^3\, a'' = \varepsilon^2 \tag{26a}$$

$$a\, a'' = K \tag{26b}$$

The solutions are well known; the first is a hyperbola, with asymptotic angle $\theta_e = \varepsilon/a_0$, where a_0 is the radius of the waist; the second cannot be expressed in analytical form, but near the waist can be fitted to a hyperbola with asymptotic angle $\theta_s = K^{\frac{1}{2}}$. **Exercise 6** : prove this. Note that this is independent of the value of a_0, the radius of the waist. Beyond the waist the curve spreads more rapidly than a hyperbola. These curves are shown in Fig. 5.

Using the emittance and self-field angle θ_e and θ_s eqn. (24) can be written in yet another way,

$$1 + \frac{Ka^2}{\varepsilon^2} = 1 + \frac{\theta_s^2}{\theta_e^2} \tag{27}$$

Exercise 7

Show that when self-fields and emittance are both present, the asymptotic angle is $(\theta_e^2 + \theta_s^2)^{\frac{1}{2}}$.

Exercise 8.

Find the envelope equation for a beam with $\varepsilon = 0$ and negative K.

Exercise 9

In an emittance waist all trajectories cross the axis, in a space-charge waist none does. Sketch trajectories in a) real space, b) phase space near a waist when $Ka^2/\varepsilon^2 = 1$.

In the discussion so far a channel with uniform focusing has been assumed. For a straight channel a force towards the axis which is proportional to r can only be provided by an E_r or B_θ field. Neither of these can be produced, however, without charges or currents within the space for the beam. A fair approximation can nevertheless be provided by a quadrupole array (electrostatic or magnetic) if σ_0 is small. If the axis is curved, then a focusing force proportional to the distance from the axis can be provided by a betatron field when $n = (1-n)$, or $n = \frac{1}{2}$.

A further type of focusing, used for example in microwave tubes, employs a B_z field which can either be uniform or periodic. In such a field the force is perpendicular to the transverse velocity, and for a completely uniform field there is no unique physically defined axis. If, however, the beam is observed not in the Laboratory frame but a frame rotating with the Larmor frequency

$$\Omega_L = \frac{1}{2}\,\omega_c = -\,\frac{qB_z}{\gamma m_0} \tag{28}$$

then the particle motion is the same as it would be in a radial electric field about the axis of rotation.

$$E_r = \tfrac{1}{2} r \Omega_L B_z$$

This 'Larmor transformation' is used in electron optics to de-couple the radial and angular motion, (see Ref. 6 Appendix 1).

A very simple example is provided by considering a single electron accelerated from a cathode outside the magnetic field, as sketched in Fig. 6. In the lab frame it moves in a helix through the axis; in the Larmor frame it moves in a plane, with wavelength $\lambda = \beta c/\Omega_L$.

Exercise 10

Prove that the canonical angular momentum in the laboratory frame is equal to the mechanical angular momentum in the Larmor frame.

An interesting example of flow in a solenoidal field is 'Brillouin flow', (Ref. 10). This is particularly simple for a non-relativistic beam, but becomes complicated when relativistic effects appear. In the lab frame a uniform density beam with radius independent of z requires a balance between centrifugal, Lorentz, and space-charge forces. In the usual notation

$$m_0 r \dot{\theta}^2 = - \frac{nq^2 r}{2\varepsilon_0} - qB_z r\dot{\theta} \tag{30}$$

Introducing the plasma frequency ω_p, this may be written

$$\dot{\theta}^2 + \tfrac{1}{2}\,\omega_p^{\,2} - 2\dot{\theta}\Omega_L = 0 \tag{31}$$

Choosing Ω_L and ω_p such that $\dot{\theta} = \Omega_L$ yields

$$\Omega_L^{\,2} = \tfrac{1}{2}\,\omega_p^{\,2} \tag{32}$$

In the Larmor frame the magnetic field is transformed to a radial electric field which is just balanced by the space-charge force. The trajectories are straight lines parallel to the axis.

It is consistent to measure emittance in the Larmor frame. For Brillouin flow $\varepsilon = 0$.

Exercise 11

Find the xx' plane emittance of a beam of radius a described by eqn. (31) such that $\dot{\theta} \neq \Omega_L$.

It is interesting to note that the xx' plane emittance is not zero, but the flow is laminar, so that the beam temperature is zero. This is an important example that shows that, despite rough calculations made earlier (e.g., exercise 2), emittance and temperature are conceptually different.

5 SOME CONSEQUENCES OF NON-UNIFORM TRANSVERSE CHARGE DENSITY

The discussion so far has been restricted to idealized models for the beam. Although these are unrealistic in a practical sense, they do illustrate the meaning of the main physical parameters. A clear understanding of these models forms the basis for a discussion of realistic beams. In practice neither beams nor emittance diagrams are uniform with a sharp circular or elliptical boundary. More parameters are required to describe them. Further, when self-fields are large, nonlinearities are necessarily introduced in the transverse motion.

Several generalizations are necessary. First, a new definition of emittance is required. A convenient one is the r.m.s. emittance, multiplied by a factor four. We denote this here by $\bar{\varepsilon} = 4 \quad \varepsilon$, where angle brackets refer to r.m.s. values.

$$\bar{\varepsilon} = 4 \left(x^2 \quad x'^2 \quad - \quad xx' \; ^2 \right)^{\frac{1}{2}} \tag{33}$$

The factor 4 makes $\bar{\varepsilon}$ equal to the emittance ε defined earlier for a uniformly filled ellipse. With this definition it was shown by Sacherer, (Ref. 11) that the K-V equations (16) are still true, provided that a and b are replaced by 2 a and 2 b . In the absence of self-fields, the system is still linear and $\bar{\varepsilon}$ is invariant. In the presence of self-fields, however, ε is no longer invariant. Before discussing the implications of this fact, we consider the concept of matching.

6 THE CONCEPT OF MATCHING, AND SOME EXAMPLES OF MATCHED BEAMS

Before discussing the behaviour of realistic high current beams in focusing channels it is necessary to examine in more detail the concept of matching, and introduce some definitions. The idea of a uniform beam matched to a uniform channel was discussed in section 4. Provided that the beam has a K-V distribution described by eqn. 2 and the paraxial envelope eqn. (17) is satisfied when a" = 0, the beam propagates without change of radius, and may be termed matched. If the beam is introduced into a channel for which x" $\neq 0$, the radius oscillates periodically, as described by the envelope equation. For a periodic rather than a uniform channel, the matched solution is that with the same periodicity as the focusing channel. A non-matched beam will show an additional periodicity superposed on that of the channel.

For a beam with a K-V distribution the concept of matching is simple. By choosing the correct initial conditions it is always possible to find a matched solution, uniform or periodic. For non K-V distributions in the presence of self-fields, however, the situation is more complicated. We consider first a uniform focusing channel which provides linear focusing for a simple particle; this corresponds to a parabolic potential well. Although an arbitrarily large number of matched solutions can be found, in general a beam with a given transverse distribution function in xx' space cannot be matched to a uniform channel. An example is provided by a beam with Gaussian profile and Gaussian transverse velocity distribution. If a beam of this form is introduced into a uniform channel, the transverse distribution will change as it moves along the channel. There may be an oscillatory component, and (as we see later) the beam profile will become rather more uniform. The oscillatory behaviour is minimized if the beam is 'r.m.s. matched'. This means that the beam has the same value of r.m.s. emittance as a K-V beam that would be matched.

It was noted above that a beam with Gaussian distribution in x and x' cannot be matched to a uniform focusing system when self-fields are important. By assuming a Gaussian distribution in x', however, corresponding to a finite transverse temperature, it is possible to find a distribution in x that will be matched. This is a classic 'atmosphere' problem, of a Maxwellian gas in a parabolic potential well, with the additional feature that the gas is charged and therefore modifies the potential. Poisson's equation $\nabla^2\phi + nq = 0$ needs to be solved simultaneously with the Boltzmann relation $n = n_0 e^{-e\phi/kT}$. This leads to an integro-differential equation that can only be solved numerically. The form of the solution is shown in Fig. 7; the shape changes from rectangular to Gaussian as the transition is made from space-charge to emittance dominated beams. The rectangular distribution is just a K-V distribution with zero emittance, and the Gaussian solution is equivalent to the standard atmosphere problem for a neutral gas held in a parabolic potential well.

Another matched distribution that is often used as a model for charged beams is the 'waterbag', corresponding to a uniform density of points within a sharp boundary in xx'yy' space. This has been studied by Lapostolle (Refs 12,6) who shows that the radial profile can be expressed in terms of Bessel functions; the shape varies from rectangular for a space-charge dominated beam to parabolic for an emittance dominated beam. For an emittance dominated beam the bounding surface of the 'bag' in xx'yy' space is a hyper-ellipsoid.

As already stated, an arbitrarily large number of matched distributions can be found in a uniform focusing channel. For a periodic channel, on the other hand none is known except the K-V distribution. Whether strictly periodic non-linear distributions exist is still a matter of some controversy.

7 VARIATION OF BEAM PROFILE AND R.M.S. EMITTANCE ALONG A FOCUSING CHANNEL

In the last few years there has been considerable interest in the propagation of space-charge dominated beams along periodic focusing channels. This arose in connection with proposals for inertial confinement fusion driven by heavy ions. For efficient transport, where losses must be kept to a very low value, it is important to know first, how the emittance and beam profile change along the channel and second, whether the beams are stable. Although these two questions are inter-related, we consider only the first in this section. Three approaches have been made to this problem, computational, theoretical and experimental. Although much has been learned, the situation is still not completely clear. Furthermore, it is not always possible to distinguish sharply between an instability that saturates very quickly because of non-linearity, and a change in the distribution function arising because it does not initially represent an equilibrium.

Computational studies of space-charge dominated beams have shown that non-uniform initial distributions, such as a Gaussian, tend very rapidly to become more uniform as they travel along the focusing channel. (This is illustrated in Fig. 8, taken from Ref. 13). This is not surprising, if one regards such a beam as a plasma. A uniform focusing channel can be replaced in principle by a rigid cylinder of uniform charge density, that is transparent to the beam. Both give rise to a parabolic potential well about the axis. In equilibrium the beam plus background charge form a nearly uniform neutral plasma, which has a boundary layer equal to λ_D where the density falls almost to zero, the Debye shielding length. For a strongly space-charge dominated beam this is much less than the beam radius, a feature illustrated in Fig. 7.

Although it is not possible to calculate how the beam evolves, the emittance growth can be correlated with the way that the profile changes along the channel; since the beam may be expected to approximate to a flat distribution, a limit to the emittance growth can be found. This may be done by introducing the 'non-linear field energy', a term introduced by Struckmeier et al in 1984, (Ref. 13), and examined in more detail in a paper by Wangler et al, (Ref. 14), which includes references to earlier work and ideas. We consider different distributions with the same r.m.s. emittance and current, and consider their transverse energy. This consists of three parts, kinetic, potential, and (for a non-relativistic beam), electrostatic. For such beams the electrostatic energy, (cut off at some convenient radius greater than the beam radius) is a function of the beam shape. It may be found by simple volume integration of E_r^2, and is found to be a minimum for a uniform K-V beam, and to increase as the beam becomes non-uniform. It is much larger for a Gaussian than for a parabolic profile. The difference between chis energy and that for a K-V distribution, known as the non-linear field-energy, denoted by U.

It can be shown that the change ΔU in U and the change in ε as the beam moves along the channel can be related

$$\frac{\varepsilon}{\varepsilon_i} = \left[1 - \frac{\Delta U}{2w_0}\left(\frac{\sigma_0^2}{\sigma_i^2} - 1\right)\right]^{\frac{1}{2}} \qquad (34)$$

where σ_i is the initial value of σ, the betatron phase change per period of the K-V distribution with the same value of ε, and w_0 is a normalizing energy, equal to the electrostatic energy of the equivalent K-V beam between the axis and the beam edge. There is an additional assumption in deriving this formula that the r.m.s. beam radius remains constant; this is found from computations to be a good approximation when σ/σ_0 is small.

This formula, together with the observation that distributions tend to become more uniform, shows how the emittance of an initially mismatched beam evolves. Computations show that, for $\sigma_0 < 60°$, the value of ε grows rapidly to about the value given by eqn. (34), and then oscillates. This is illustrated in Fig. 9, taken from Ref. 13. Although these computations have been done for a periodic channel, this behaviour is to be expected for a uniform channel also. Interesting insights into the mechanism by which this initial growth occurs, and the distance that it takes to develop, are given in a paper by Anderson, Ref. 15.

In this section the physical ideas underlying the emittance growth in space-charge dominated beams of non-uniform density profiles has been outlined. Initial conditions correspond to those of a matched K-V beam with the same value of ε. Further details may be found in Refs. 13 and 14, and the many references therein.

8 THE STABILITY OF SPACE CHARGE DOMINATED BEAMS

So far the problem of beam stability has not been discussed. Before doing this we note that most instabilities of importance in accelerator design arise from interaction between the beam and the walls, and would not exist if the beam were in a smooth perfectly conducting tube. One exception to this is the negative mass instability, which can be thought of as arising from the fact that the beam is a plasma, and since the mass is negative, so is the square of the plasma frequency.

A second class of instability depends only on the velocity distribution in the beam. A well-known longitudinal instability of this type is the two-stream instability; a velocity distribution in a particular direction that does not increase monotonically from a maximum to zero is often (but not always) unstable. Examination of the K-V distribution shows that it is of this type. At a given point the transverse component of velocity $v_\perp$ of all particles has the same amplitude, but is uniformly distributed in angle, (Fig. 1b). If the velosity component in one direction is considered, then the distribution has the form $(1-v^2/v^2{}_{max})^{-\frac{1}{2}}$, which has two well defined maxima. Instability might therefore be expected, and indeed it is found to occur if $\sigma/\sigma_0 > 0.4$, corresponding to $Ka^2/\varepsilon^2 \approx 5$. This physical explanation was supplied by Hofmann; a more formal discussion may be found in Ref. 8, and by Lapostolle in Ref. 4.

In practice, this instability would saturate very rapidly, and merely modify the transverse profile and velocity distribution function slightly. From the previous section, since a K-V distribution has the lowest value of U, the non-linear field energy, this change would absorb energy and decrease the emittance. Computations in Ref. 14 show that this is indeed what happens. This might at first sight seem paradoxical, but Liouville's theorem is still obeyed. The r.m.s. emittance is merely an average, and does not contain information about the local phase space density.

Exercise 12

Find a simple non-linear channel, into which is injected a low current beam simple xx' distribution, such that ε decreases monotonically along the channel. (See Ref. 6, Appendix 5).

The discussion so far refers to a channel which is uniform, or periodic with small phase shift per section. When the phase shift becomes, greater than $\pi/3$, parametric resonances driven by the structure become important. These are discussed in the following section.

9 INSTABILITIES AND RESONANCES IN A PERIODIC CHANNEL

When the periodicity of the focusing channel is such that σ_0 exceeds about $\pi/3$, parametric coupling between the structure and natural modes of oscillation of the beam become possible. These were identified for a K-V beam in a study by Hofmann et al., and found in numerical simulations by Haber (Ref. 15). Typically the phase-space diagrams develop 'arms' typical of a sub-harmonic resonance, and then settle down to a beam with increased emittance that continues gradually to increase. The physical mechanism in this case arises from phase-space filamentation.

Very many instabilities are predicted analytically, using a linearized Vlasov analysis described in Ref. 15, but only a few have been identified computationally. As a general rule, instabilities become serious as σ_0 approaches $\pi/2$, and it is now accepted that for beams with small σ/σ_0, σ_0 should be kept below $\pi/3$ if efficient transport through long channels is to be achieved. This type of instability can occur with non K-V beams, but as expected, computations show it to be less serious. An example of this from Ref. 13 is illustrated in Fig. 10.

Attempts have been made to identify specific resonances experimentally, using beams of caesium ions and electrons, (Refs 16 and 17). These beams have been produced from uniform cathodes, and so initially they are uniform in real space but Gaussian in velocity space; this configuration cannot strictly be matched into a linear focusing channel. Such experiments are not easy, and no specific resonances have been conclusively identified. Nevertheless, propagation deteriorates markedly as σ_0 approaches $\pi/2$, as may be seen from Fig. 11, taken from Ref. 16.

Concerning the topics of this section and the previous one, several featurs have been clarified in recent years, but much still remains to be learned, particularly with regard to the behaviour of practical systems where very efficient transmission is needed.

No mention has been made in this report of beams with a distribution of velocities in the z-direction, nor of bunched beams. More understanding, using some of the concepts discussed here, has been obtained concerning the emittance growth in bunched beams in linear accelerators, but again many details are still not clear. Further progress in this rather difficult field is to be expected.

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FIGURE CAPTIONS

Fig. 1 a) Projections on a transverse (xy) plane of the trajectories of the beam particles for a uniform matched K-V beam in a uniform focusing channel. Projections on a plane through the beam axis are sinusoidal.

b) Distribution of velocities at a point in the beam. All particles have the same value of $v_\perp$, but the distribution is uniformly distributed in angle.

c) Distribution of $v_\perp$ across the beam radius.

Fig. 2 Diagram to illustrate the potential depression that develops on the axis of an electron beam. For a beam launched into a close fitting tube through a grid at the same potential, $\int_1^4 E d\ell = - \int_3^4 E d\ell$.

Fig. 3 Diagram to illustrate a partly neutralized zero perveance electron beam, which transforms to an ion beam with negative perveance in the Lorentz frame in which the electrons are at rest.

Fig. 4 Diagram to illustrate the meaning of a σ (without space-charge) and σ_0 (with space-charge). In a uniform channel the distance defining σ_0 is arbitrary, in a periodic channel it is taken as the length of one period.

Fig. 5 Emittance and space-charge waists.

Fig. 6 Electron trajectories in laboratory and Larmor frames of an electron entering a uniform B_z field from a cathode outside the field.

Fig. 7 Variation of the transverse density of a beam with transverse Maxwellian velocity distribution in a uniform focusing channel. The shape changes from uniform to Gaussian as the temperature is increased. (More details in Ref. 6, Section 4.6.2).

Fig. 8 Results of particle-in-cell computations after one period of focusing channel, illustrating how space-charge dominated beams tend to become more uniform. Initial distributions are K-V, waterbag, parabolic, conical and Gaussian. $\sigma_0 = 60$ and $\sigma = 15°$. From Ref. 13.

Fig. 9 Emittance growth of r.m.s. matched beams with several initial distributions for $\sigma_0 = 60°$, $\sigma = 25°$. From Ref. 13.

Fig. 10 As Fig. 9, but with $\sigma_0 = 90°$, $\sigma = 41\delta$. The much greater emittance growth can be seen.

Fig. 11 This figure, together with the caption (slightly modified) is taken from Ref. 16. Plotted are calculated σ values for stable and apparently stable beams for various σ_0. Filled-in symbols represent beams with the same current and emittance at the beginning and end of the lattice. Hollow symbols mark σ values derived from beams reproducing ε and current over at least the last 10 periods for $\sigma_0 = 100°$. Circles mark σ values derived using full beam distribution emittance. Triangles mark calculations using central 95% current of the phase space distribution. The shaded region marks the calculated instability of the envelope equations. Curve A marks the region of equivalent σ attainable at injection with our limited source emittance.

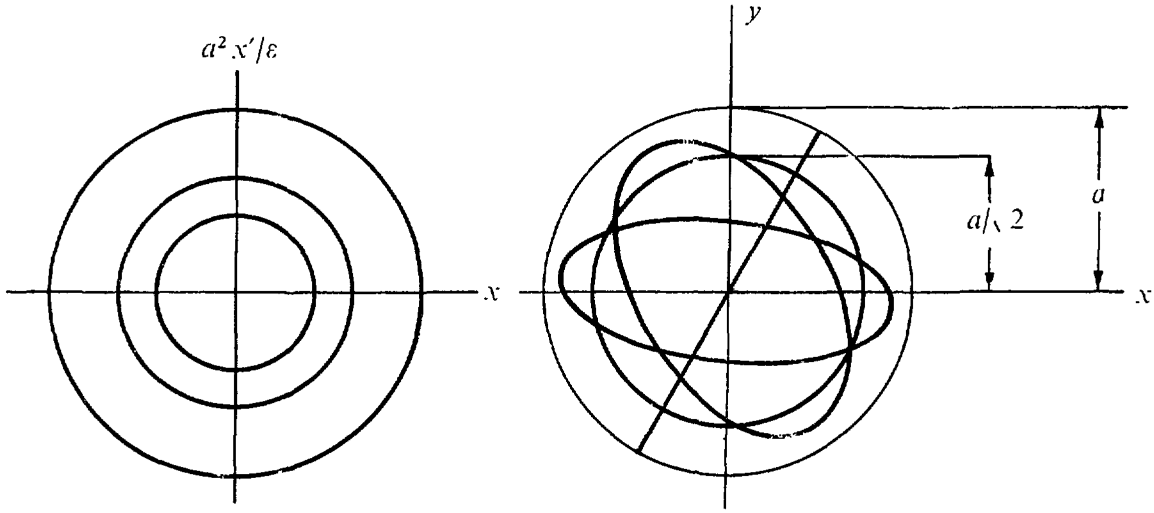


Fig. 1a

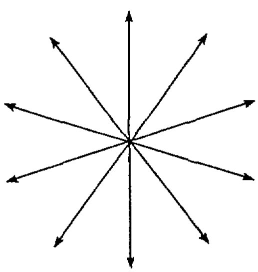


Fig. 1b

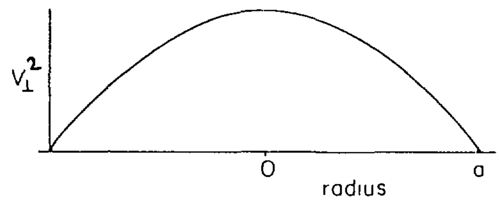


Fig. 1c

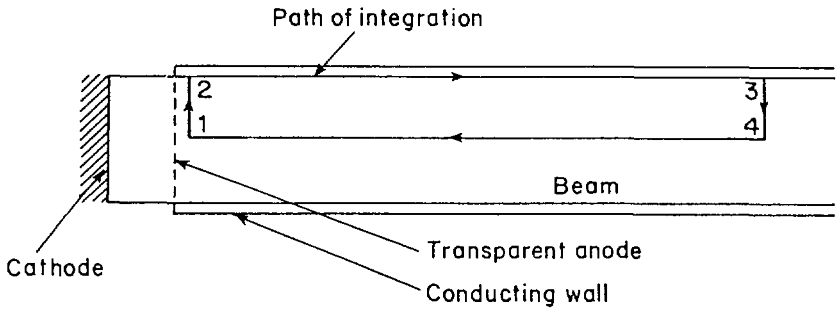

Path of integration
2
3
1
4
Beam
Cathode
Transparent anode
Conducting wall

Fig. 2

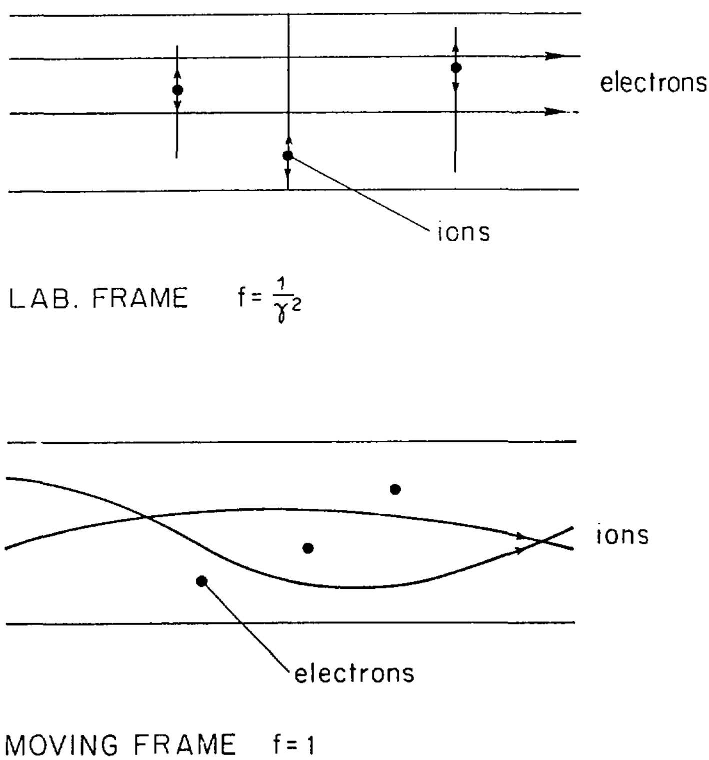

electrons
ions
LAB. FRAME $f = \frac{1}{\gamma^2}$
ions
electrons
MOVING FRAME f = 1

Fig. 3

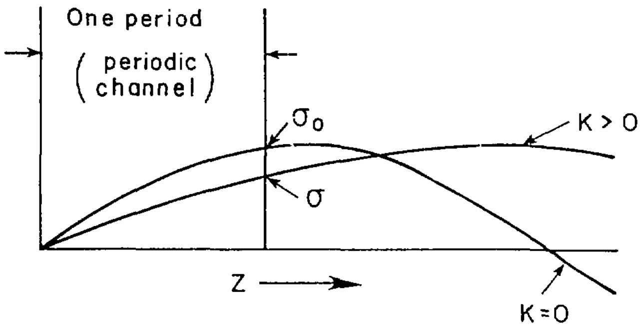

One period
periodic
channel
σ_0
K > 0
σ
Z
K = 0

Fig. 4

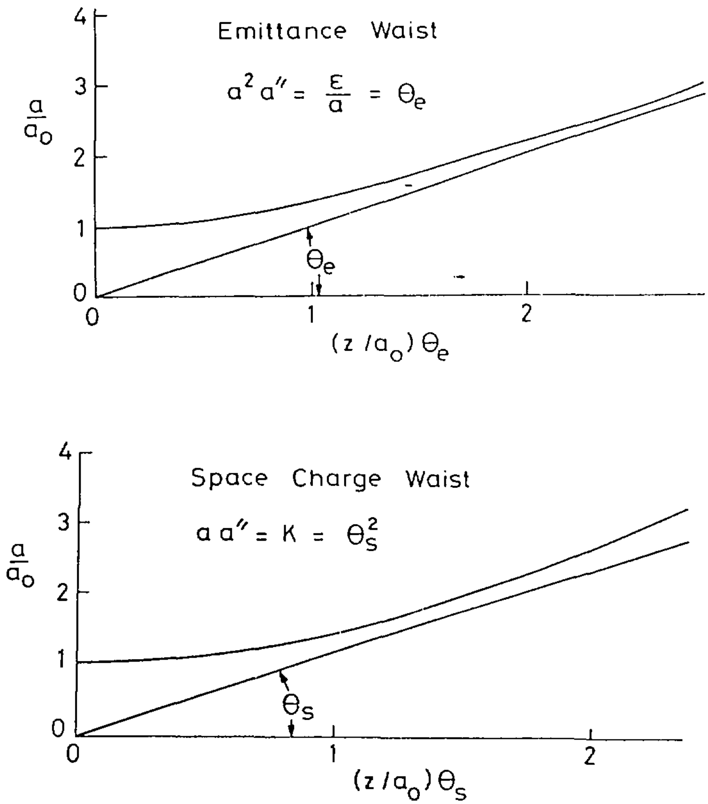

Emittance Waist
$a^2 a'' = \frac{\varepsilon}{a} = \Theta_e$
$\frac{a}{a_o}$
Θ_e
$(z/a_o)\,\Theta_e$
Space Charge Waist
$a\,a'' = K = \Theta_s^2$
$\frac{a}{a_o}$
Θ_s
$(z/a_o)\,\Theta_s$

Fig. 5

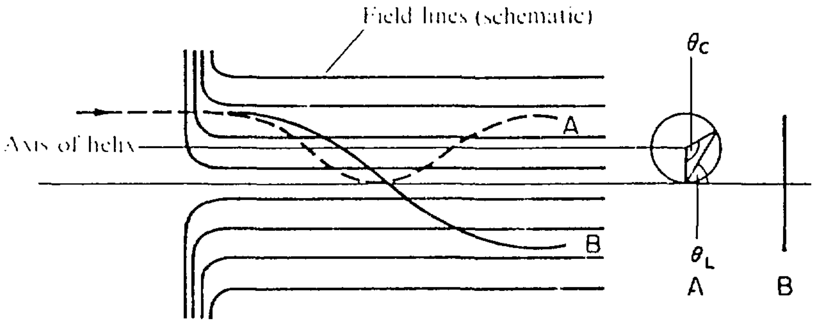

Field lines (schematic)
Axis of helix
A
B
θ_C
θ_L
A
B

Fig. 6

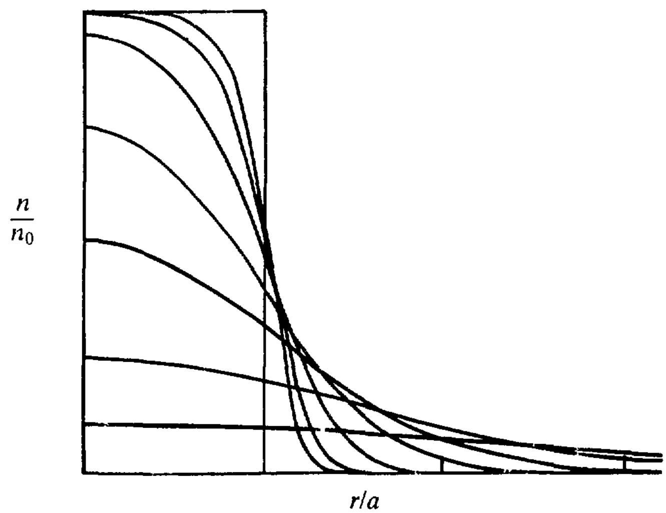

$\frac{n}{n_0}$
r/a

Fig. 7

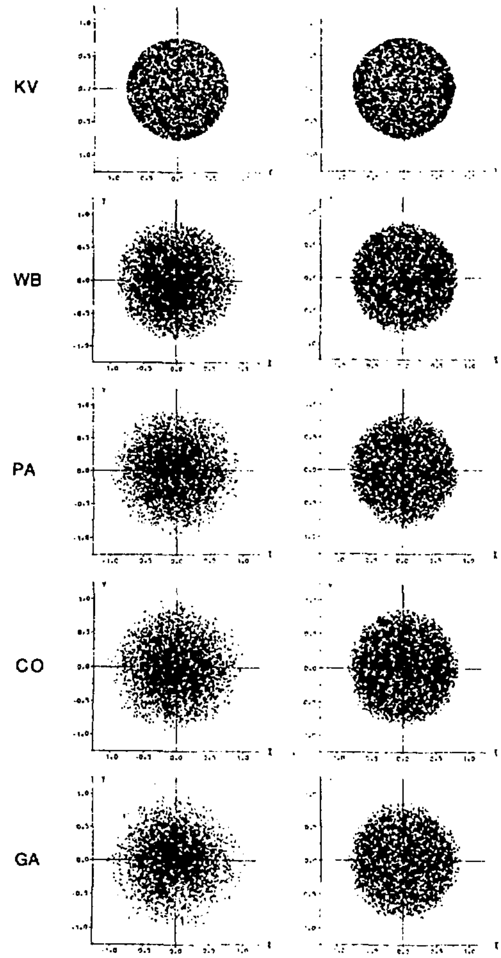

KV
WB
PA
CO
GA

Fig. 8

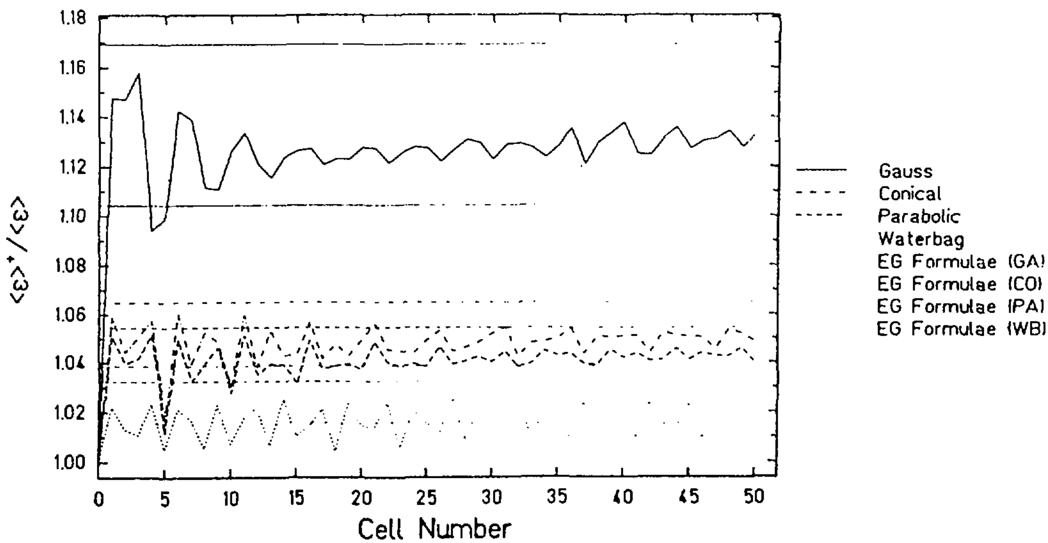
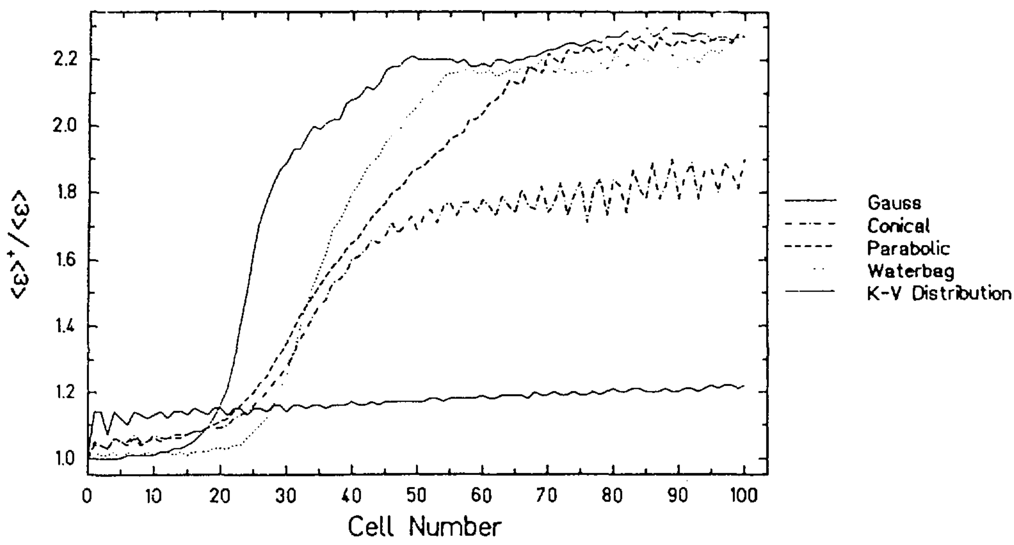
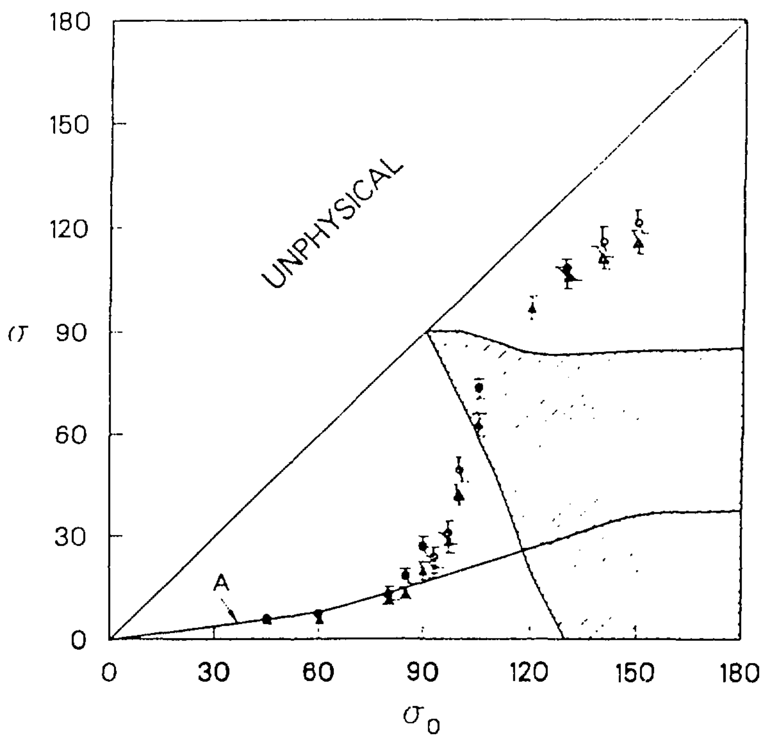

1.18
1.16
1.14
1.12
1.10
1.08
1.06
1.04
1.02
1.00
$\langle 3 \rangle_* / \langle 3 \rangle$
0
5
10
15
20
25
30
35
40
45
50
Cell Number
Gauss
Conical
Parabolic
Waterbag
EG Formulae (GA)
EG Formulae (CO)
EG Formulae (PA)
EG Formulae (WB)

Fig. 9



2.2
2.0
1.8
1.6
1.4
1.2
1.0
0
10
20
30
40
50
60
70
80
90
100
Cell Number
Gauss
Conical
Parabolic
Waterbag
K-V Distribution

Fig. 10



180
150
120
90
60
30
0
σ
UNPHYSICAL
A
0
30
60
90
120
150
180
σ0

Fig. 11