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# COHERENT BEAM-BEAM EFFECT

by

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Abstract

The stability of the coherent beam-beam effect between rigid bunches is studied analytically and numerically for a linear force by evaluating eigenvalues. For a realistic force, the stability if investigated by following the bunches for many revolutions.



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## 1. Introduction

The coherent beam-beam interaction between the rigid bunches of two counter-rotating beams in an $e^+e^-$ storage ring has been studied by Rees and Ritson[1] and by Piwinski[2]. Rees and Ritson set up a system of coupled linear equations for the centre-of-mass motion. They use it to find the coherent tune shift for one particular mode of the coherent oscillations, and conclude that the tune shift is inversely proportional to the tune split between the two beams. Piwinski treats the stability of k identical bunches in each beam, colliding in 2k identical crossing points around the machine. The case of equal tunes gives a closed expression for the threshold of the instability $\Delta Q$, as a function of the tune Q. It behaves like a sawtooth function, with zeros at all tunes which are multiples of ½.

We have carried out a study under the same conditions as Piwinski[2], using a linear approximation to the beam-beam force. Our conclusions concerning the threshold of the instability are somewhat different: it is still a sawtooth function, but its zeros occur only at integral tunes. We have also investigated how this behaviour is changed by splitting the tunes of the $e^+$ and $e^-$ beams and by different phase advances between the crossing points.

The linear approximation applies only to oscillation amplitudes which are small compared to the beam sizes. Therefore the behaviour at large amplitudes was studied by following bunches around the machine for many revolutions, using realistic non-linear forces.

## 2. Mathematical model - linear forces

### 2.1 Kicks and transformations

There are k equidistant bunches in each of two counter-rotating beams in a storage ring. They collide in 2k crossing points; between the crossing points there are 2k machine sectors. The positions $x_{mn}$ and the slopes $x'_{mn}$ refer to the $m^{th}$ bunch in the $n^{th}$ beam, where $m = 1 \ldots k$ and $n = 1, 2$.

A revolution in thr storage ring is described by an alternating sequence of 2k kicks which describe the interaction between the counter-rotating beams, and 2k transformations through the sectors of the machine. This sequence is schematically indicated in Fig. 1 for the case of three bunches in each beam. This is the smallest number of bunches with all the general features. For complete generality, we allow for different bunch populations, different phase advances in the sectors, and different crossing points. In the $s^{th}$ crossing point, the interaction occurs between the $m^{th}$ bunch of beam 1 and the $n^{th}$ bunch of beam 2.

In linear approximation the change in the slope is proportional to the difference in the position of the two bunches. This law can be expressed in the form of a 4 × 4 matrix:

$$\begin{pmatrix} x_{m1} \\ x'_{m1} \\ x_{n1} \\ x'_{n2} \end{pmatrix}_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta_{m1s} & 1 & -\delta_{m1s} & 0 \\ 0 & 0 & 1 & 0 \\ -\delta_{n2s} & 0 & \delta_{n2s} & 1 \end{pmatrix} \begin{pmatrix} x_{m1} \\ x'_{m1} \\ x_{n2} \\ x'_{n2} \end{pmatrix}_i = \begin{pmatrix} A_{m1s} & B_{m1s} \\ B_{n2s} & A_{n2s} \end{pmatrix} \begin{pmatrix} x_{m1} \\ x'_{m1} \\ x_{n2} \\ x'_{n2} \end{pmatrix}_i \qquad (1)$$

In (1), A and B are 2 × 2 matrices. In normalized phase space in which free betatron oscillations are described by circles and which we shall use, the parameter $\delta_{mns}$ is simply related to the linear beam-beam tune shift:

$$\delta_{mns} = 4\pi \, \Delta Q_{mns} \qquad (2)$$

The $m^{th}$ bunch in the first beam collides with the $n^{th}$ bunch in the second beam in two diametrically opposite crossing points, as can be seen from Fig. 1. Hence, there are only $2k^2$ coefficients $\delta_{mns}$.

The transformation through the $s^{th}$ sector is a simple rotation in phase space by an angle $\mu_{sn}$:

$$\begin{pmatrix} x_{mn} \\ x'_{mn} \end{pmatrix}_f = \begin{pmatrix} \cos\mu_{sn} & -\sin\mu_{sn} \\ \sin\mu_{sn} & \cos\mu_{sn} \end{pmatrix} \begin{pmatrix} x_{mn} \\ x'_{mn} \end{pmatrix}_i = R_{sn} \begin{pmatrix} x_{mn} \\ x'_{mn} \end{pmatrix}_i \quad . \qquad (3)$$

Again, the pairing between bunches and sectors follows from Fig. 1.

The indices m, n, s are cyclic indices, m and n are in the range $1 \leq m \leq k$, $1 \leq n \leq 2$, and s is in the range $1 \leq s \leq 2k$.

## 2.2 Matrices

All the positions $x_{mn}$ and slopes $x'_{mn}$ can be grouped into a column vector V of 4k elements. Its transpose $\tilde{V}$ is given by

$$\tilde{V} = (x_{11}, x'_{11}, x_{21}, x'_{21}, \ldots, x_{k1}, x'_{k1}, x_{12}, x'_{12}, \ldots x_{k2}, x'_{k2}) \qquad (4)$$

The k simultaneous kicks can all be described by matrices of the form (1). All these matrices can be arranged in a single 4k × 4k matrix operating on the vector V. Similarly, all the 2k simultaneous rotations of the form (3) can be arranged in a single 4k × 4k matrix operating on the vector V.

The kick and rotation matrices for half a revolution are shown in Table I for the case of three bunches in each beam. Inspection of Table I shows how the matrices for the second half of a revolution and/or for more than three bunches can be obtained. The rules are:

i) Successive kick matrices are obtained by increasing the third index of the A and B matrices by one in the upper half of the matrix, and by reducing it in the lower half. In addition, the columns of the B matrices are cyclically shifted, to the right in the upper half matrix, and to the left in the lower half matrix.

ii) Successive rotation matrices are obtained by increasing the first index of the R matrices by one in the upper half, and reducing it by one in the lower half of the rotation matrix.

It can be demonstrated directly that all the matrices in Table I are symplectic.

## 3. Results of the linear theory

### 3.1 Eigenvalues for a perfect machine

The stability behaviour of the bunch motion is determined by the eigenvalues of the matrix for a full revolution. Instability occurs if at least one eigenvalue $\lambda$ is larger than one in absolute value. Since all the matrices involved in the revolution matrix are symplectic, the same holds for the revolution matrix. The eigenvalues of a symplectic matrix occur in reciprocal pairs. As is the case for linear coupling resonances[3], there are three possibilities:

i) Two eigenvalues form a complex conjugate pair and are on the unit circle;

ii) Four eigenvalues, so that two eigenvalues are the complex conjugate of the other two, and also that two eigenvalues are the reciprocal of the other two; none of them is on the unit circle;

iii) Two real eigenvalues form a reciprocal pair.

Only possibilities ii) and iii) give rise to an instability.

For the special case that all $\delta_{mns}$ are the same, and that all $\mu_{sn}$ are the same, the revolution matrix is the product of two identical half-revolution matrices. Hence, it is sufficient to investigate the eigenvalue behaviour of a

TABLE I. Explicit form of kick and rotation matrices for three bunches

| kick matrix | rotation matrix |
|---|---|
| $\begin{vmatrix} A_{111} & & & B_{111} & & \\ & A_{213} & & & B_{213} & \\ & & A_{315} & & & B_{315} \\ B_{121} & & & A_{121} & & \\ & B_{223} & & & A_{223} & \\ & & B_{325} & & & A_{325} \end{vmatrix}$ | $\begin{vmatrix} R_{11} & & & & & \\ & R_{31} & & & & \\ & & R_{51} & & & \\ & & & R_{62} & & \\ & & & & R_{22} & \\ & & & & & R_{42} \end{vmatrix}$ |
| $\begin{vmatrix} A_{112} & & & & B_{112} & \\ & A_{214} & & & & B_{214} \\ & & A_{316} & B_{316} & & \\ & & B_{126} & A_{126} & & \\ B_{222} & & & & A_{222} & \\ & B_{324} & & & & A_{324} \end{vmatrix}$ | $\begin{vmatrix} R_{21} & & & & & \\ & R_{41} & & & & \\ & & R_{61} & & & \\ & & & R_{52} & & \\ & & & & R_{12} & \\ & & & & & R_{32} \end{vmatrix}$ |
| $\begin{vmatrix} A_{113} & & & & & B_{113} \\ & A_{215} & & B_{215} & & \\ & & A_{311} & & B_{311} & \\ & B_{125} & & A_{125} & & \\ & & B_{221} & & A_{221} & \\ B_{323} & & & & & A_{323} \end{vmatrix}$ | $\begin{vmatrix} R_{31} & & & & & \\ & R_{51} & & & & \\ & & R_{11} & & & \\ & & & R_{42} & & \\ & & & & R_{62} & \\ & & & & & R_{22} \end{vmatrix}$ |

half-revolution matrix. Piwinski[2] has derived a closed expression for the threshold of the instability, in terms of the linear beam-beam tune shift $\Delta Q$. His results, shown in Fig. 2 for k = 1 and Fig. 3 for k = 3, behave like a sawtooth function; the branches have $\Delta Q = 0$ at all tunes Q which are a multiple of k. The sawtooth pattern is periodic in the tune Q with period k.

We have studied the eigenvalue behaviour numerically and found that the threshold of the instability is given by the branches drawn as full lines in Figs. 2 and 3. The additional branches found by Piwinski and drawn as dashed lines correspond to eigenvalues $\lambda = \pm i$ for half a revolution matrix. Even for

$\Delta Q$ slightly above these branches all eigenvalues are on the unit circle, and no instability occurs.

### 3.2 Eigenvalues for split tunes

To study the effect of split tunes, we have evaluated the eigenvalues under the condition that all bunches have equal populations and that all crossing points are identical, but the tunes of the two beams, $Q \pm \delta$, are different. The stability threshold found for the case $k = 1$ and $\delta = 0.1$ is shown in Fig. 4. Another example with $k = 3$ and $\delta = 0.2$ is shown in Fig. 5. Comparing Figs. 4 and 5 to Figs. 2 and 3 indicates that splitting the tunes of the two beams introduces a more complicated fine structure into the instability threshold, but does not necessarily improve it.

### 3.3 Eigenvalues for unequal half revolutions

So far, the discussion has been limited to the case where the transformation for a full revolution is the product of two identical half-revolution transformations. If this restriction is removed, the stability is determined by the eigenvalues of the full revolution matrix.

The simplest case is that of a machine with two equal bunches, one in each beam, which meet in two identical crossing points, and with phase advances $2\pi(\frac{1}{2}Q \pm \delta Q)$ for the two arcs joining the crossing points. The stability in this case was investigated numerically and analytically.

The result of the calculation is shown in Figs. 6 and 7 for two values of $\xi_\perp$. There is a difference in the behaviour of the branch starting at $Q = 1$, $\Delta Q = 0$ which is a threshold for instability, and the branch starting at $Q = 0.5$, $\Delta Q = 0$, which is a stopband of finite width. For values of $\Delta Q$ above the threshold the motion is always unstable, while for values of $\Delta Q$ above the stopband the motion is again stable up to the integral threshold. The width of the stopband increases with increasing $\Delta Q$ and $\delta Q$.

The stopbands arise when the four eigenvalues of the revolution matrix are all complex but not on the unit circle. The edges of the stopband are given by

$$\Delta Q = \frac{\mathrm{ctn}\,\pi\,\zeta}{\pi(1 + \cos 4\pi\delta Q)}\left(1 \pm \sqrt{\frac{1 - \cos 4\pi\delta Q}{2}}\right) .$$

For $\delta Q = 0$, the stopband width vanishes, but the values of $\Delta Q$ are exactly twice the values of the dashed line shown in Fig. 2.

The threshold of instability, starting at $Q = 1$ and $\Delta Q = 0$, is given by the relation

$$\Delta Q = \frac{\sin 2\pi Q + \sqrt{(1 - \cos 2\pi Q)(1 + \cos 4\pi\delta Q)}}{4\pi(\cos 4\pi\delta Q - \cos 2\pi Q)} \qquad (6)$$

The limiting value of this expression for $\delta Q = 0$ is

$$\Delta Q = \frac{\cos \pi Q + 1}{4\pi \sin \pi Q} \qquad (7)$$

which agrees with Piwinski's expression for the full line in Fig. 2.

## 4. Tracking with non-linear forces

The linear model described so far is a reasonable approximation only in the case where the oscillation amplitudes are a small fraction of the beam size. On the other hand, such amplitudes have a negligible effect on the overlap of the counter-rotating bunches and hence on the luminosity. In order to study this question further, we have introduced non-linear kicks at the crossing points and followed the coherent bunch oscillations for a large number of turns.

### 4.1 Non-linear forces

In the linear model, the kicks are simply proportional to the difference $\Delta z$ in the positions of the colliding bunches, as can be seen from (1). This is now generalized by making the kicks proportional to a function of this difference. Two specific kicks have been included in the analysis, for elliptic and round beams with a Gaussian density distribution.

For an elliptic beam, the kick is obtained by following the calculation of Montague[4]. If the vertical rms beam radius is normalized to unity, i.e. if all amplitudes are measured in units of the rms beam radius, and if the horizontal rms beam radius is $\sigma_x > 1$, the kick is given by the following expression:

$$\phi_b = 4\pi \, \Delta Q_y (\sigma_x + 1) \, \phi(\sigma_x, \Delta z) \quad . \qquad (8)$$

The function $\phi(\sigma, \Delta)$ is defined as follows:

$$\phi(\sigma, \Delta) = \sqrt{\frac{\pi}{2(\sigma^2-1)}} \left\{ w\left(\frac{i\Delta}{\sqrt{2(\sigma^2-1)}}\right) - \exp\left(-\frac{\Delta^2}{2}\right) w\left(\frac{i\sigma\Delta}{\sqrt{2(\sigma^2-1)}}\right) \right\} \qquad (9)$$

where w is the complex error function. For computations it is useful to remember that

$$w(ix) = (1 - \text{erf } x) \exp(x^2) \quad .$$

For a round beam, the kick may either be obtained from (9) by letting $\sigma_x \to 1$, or more directly from the fields. The result is:

$$\phi_b = 4\pi \Delta Q_y \frac{2}{\Delta z} \left[ 1 - \exp\left( - \frac{(\Delta z)^2}{2} \right) \right]$$

## 4.2 Results of tracking

The coherent oscillations of the bunches were launched by displacing the first bunch in each beam by equal amounts but with opposite sign, while the remaining bunches were not displaced. In the results presented later, the displacement was 0.1 beam radii, but this value is not critical. The resulting coherent oscillations were followed for 500 turns. After this number of turns the coherent oscillations are in full swing as is demonstrated by recording the number of the turn where the maximum amplitude occurred. Hence, 500 turns is considered to be sufficiently high to obtain significant results.

The important parameter for the luminosity is the average separation between the centres of the colliding bunches. It is drawn in Figs. 8 to 13 as a function of the tune Q, for several beam-beam tune shifts ΔQ and number of bunches k.

It is instructive to compare the ranges of Q where the linear theory yields stability and where tracking shows no growth. The stable regions from linear theory are shown in Table II.

TABLE II. Stable ranges of Q from linear theory

| k | ΔQ | Q | Q | Q | Q |
|---|---|---|---|---|---|
| 1 | 0.02 | 0 - 0.8432 | | | |
| | 0.04 | 0 - 0.7035 | | | |
| | 0.06 | 0 - 0.5887 | | | |
| 2 | 0.02 | 0 - 0.8432 | 1 - 1.6865 | | |
| | 0.04 | 0 - 0.7035 | 1 - 1.4070 | | |
| | 0.06 | 0 - 0.5887 | 1 - 1.1774 | | |
| 3 | 0.02 | 0 - 0.7814 | 1 - 1.7483 | 2 - 2.5297 | |
| | 0.04 | 0 - 0.6130 | 1 - 1.4975 | 2 - 2.1104 | |
| | 0.06 | 0 - 0.4909 | 1 - 1.2752 | - | |
| 4 | 0.02 | 0 - 0.7243 | 1 - 1.6865 | 2 - 2.6486 | 3 - 3.3730 |
| | 0.04 | 0 - 0.5363 | 1 - 1.4070 | 2 - 2.2777 | - |
| | 0.06 | 0 - 0.4135 | 1 - 1.1774 | - | - |

Comparing these ranges to those found by tracking shows very good agreement. Hence, the linear theory gives a very good indication over what ranges of Q the coherent oscillations are stable, even when realistic non-linear forces are used in the tracking. Conversely, the realistic forces are not non-linear enough to stabilize the coherent motion at separations small compared to a vertical rms beam radius, over much larger ranges of the tune Q than those given by the linear theory. Hence, the choice of the working point must avoid regions of coherent instabilities in order not to lose luminosity significantly.

## References


1) D. Ritson and J. Rees, SLAC-TN-65-49 (Revised) (1965).
2) A. Piwinski, 8th Internat. Conf. on High Energy Accelerators, CERN 1971, 357 (1971).
3) E.D. Courant and H.S. Snyder, Annals of Phys. 3, 1 (1958).
4) B.W. Montague, CERN/ISR-GS/75-36 (1975).


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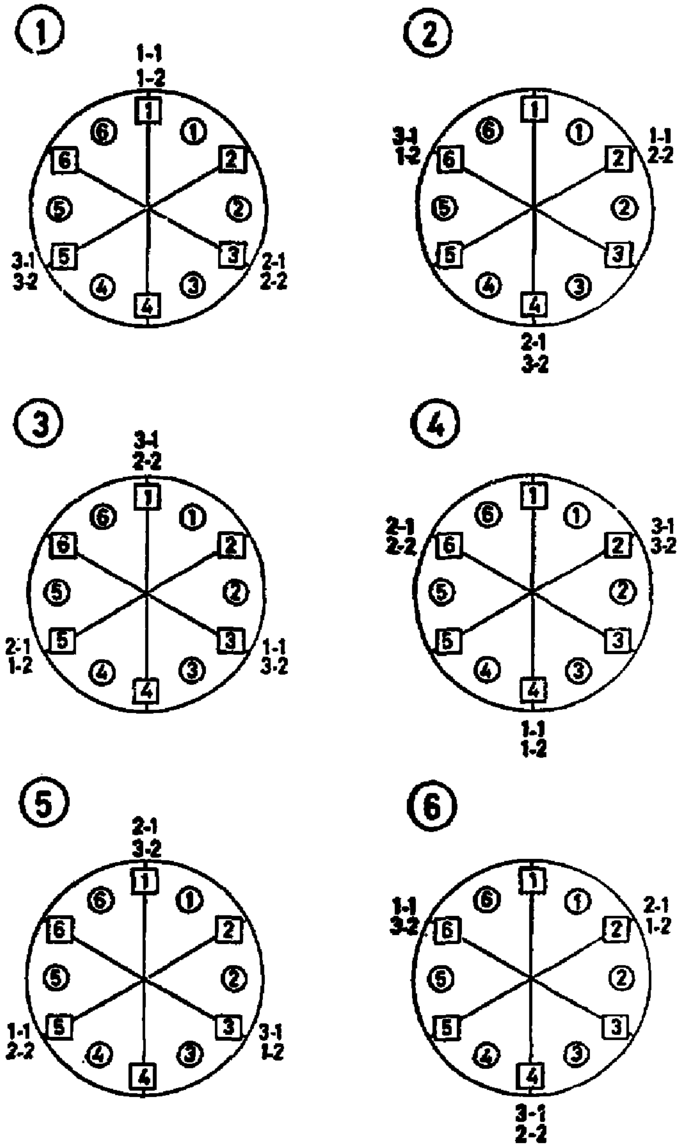



Fig. 1. The numbers in circles (squares) label machine sectors (crossing points); the bunches are labelled n(m) is the bunch (beam) number.

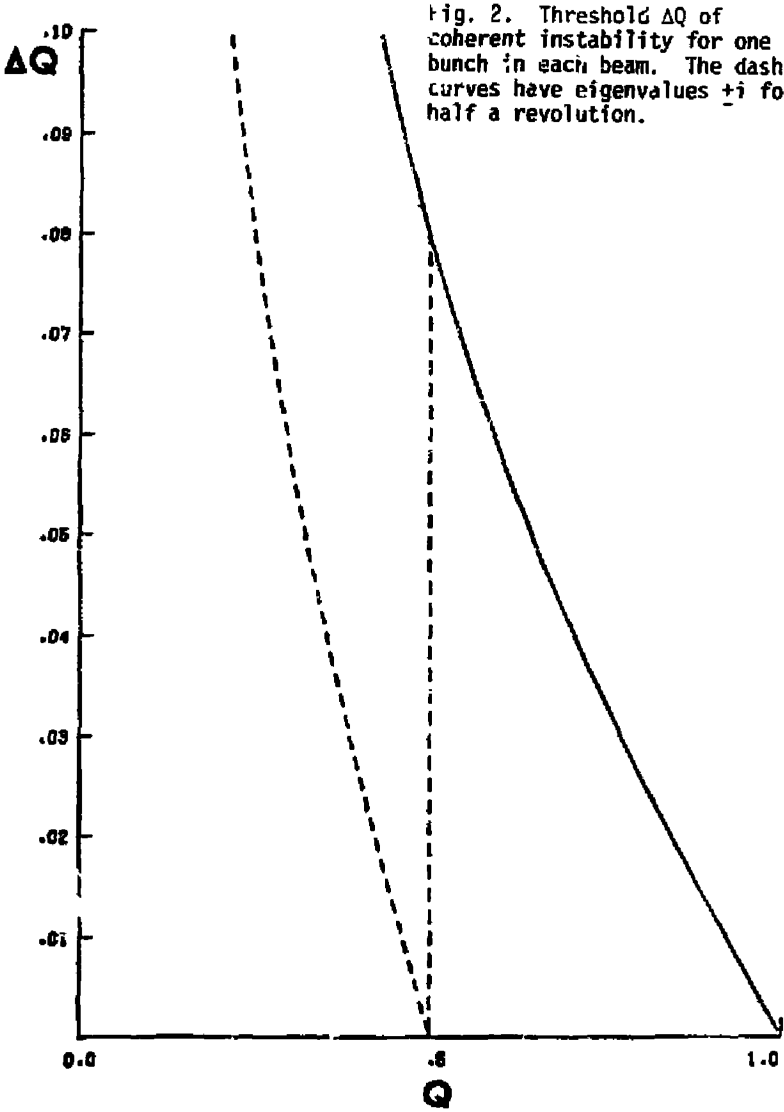
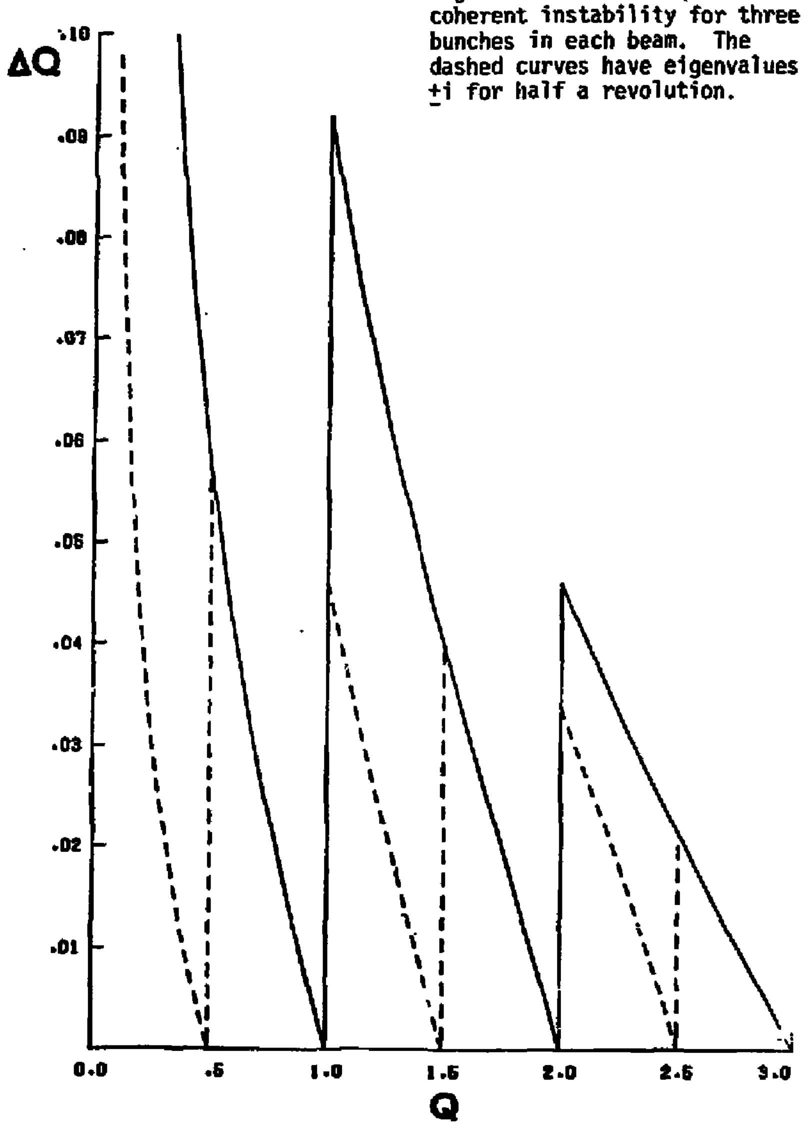



Fig. 2. Threshold ΔQ of coherent instability for one bunch in each beam. The dashed curves have eigenvalues $\pm i$ for half a revolution.

Fig. 3. Threshold ΔQ of coherent instability for three bunches in each beam. The dashed curves have eigenvalues ±i for half a revolution.



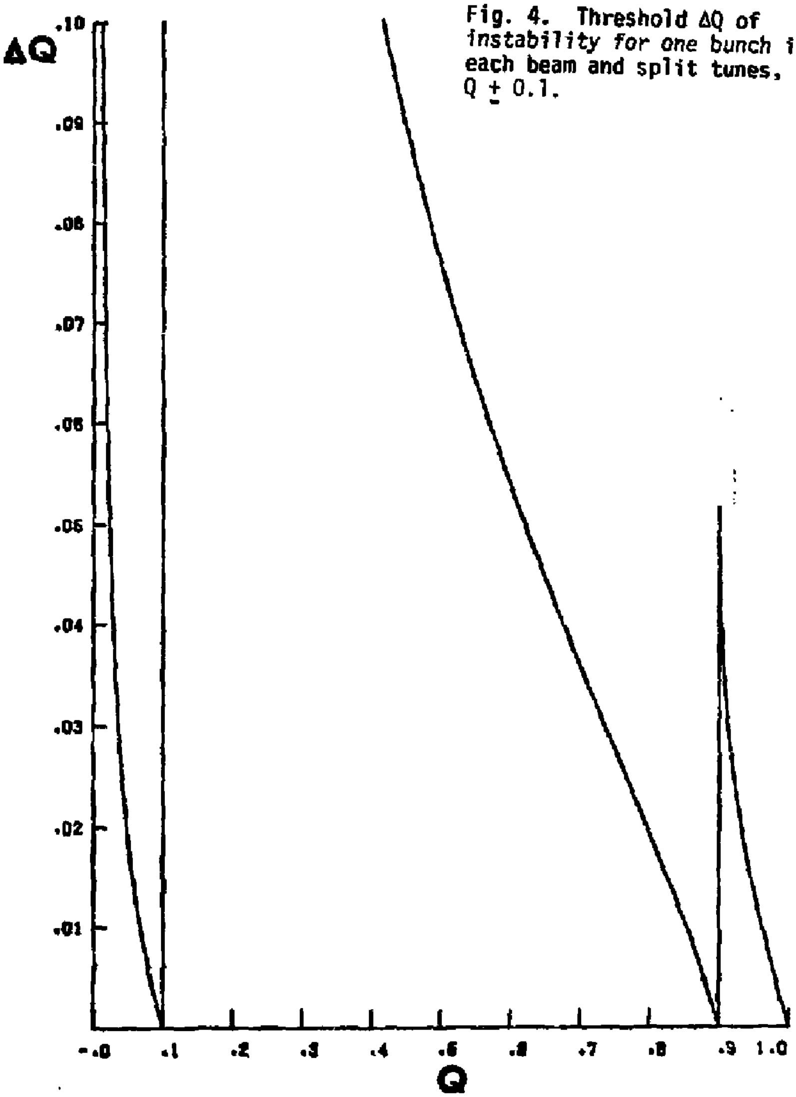
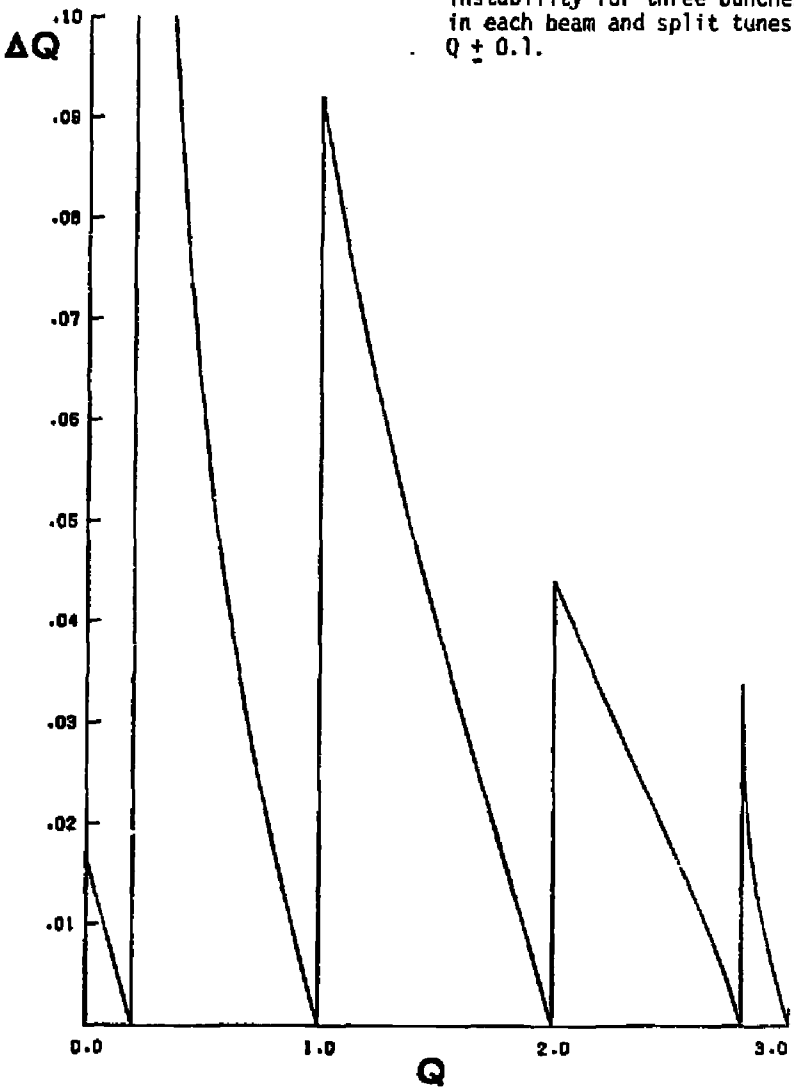



Fig. 4. Threshold ΔQ of *instability for one bunch in* each beam and split tunes, Q ± 0.1.

Fig. 5. Threshold $\Delta Q$ of instability for three bunches in each beam and split tunes, $Q \pm 0.1$.



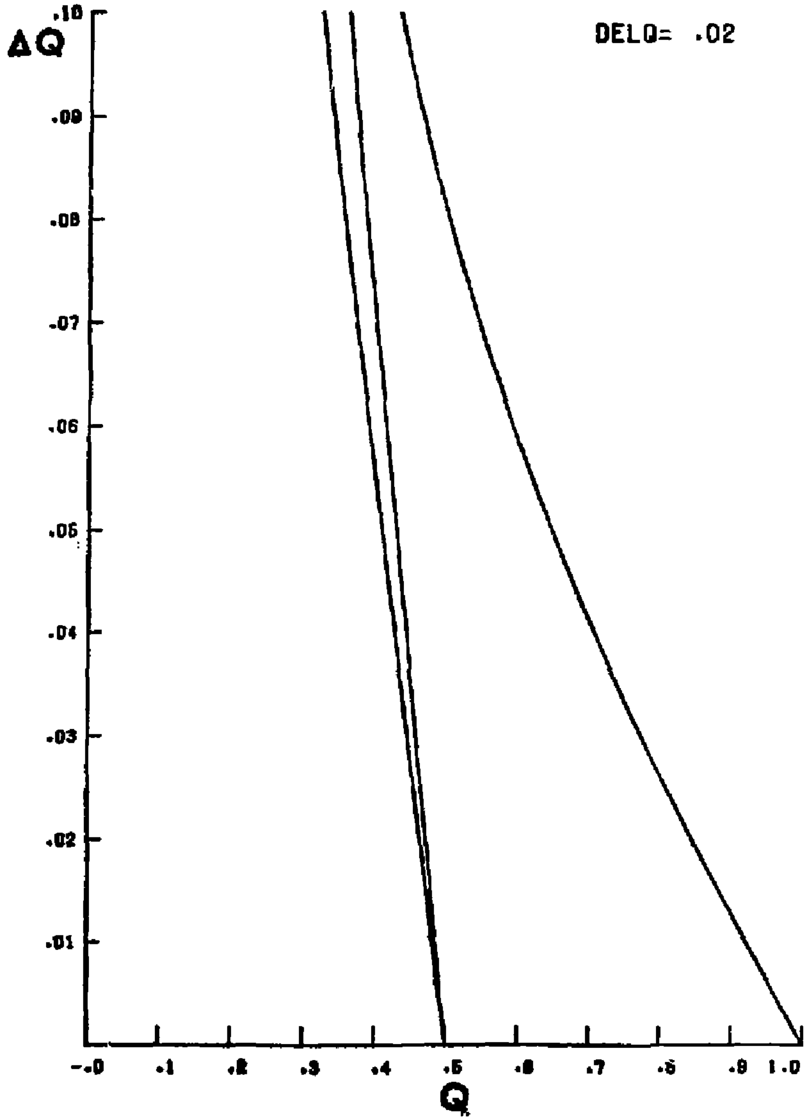



Fig. 6. Threshold and stopband for one bunch in each beam, and different phase advances in the two arcs, $2\pi(\frac{1}{2}Q \pm 0.02)$.

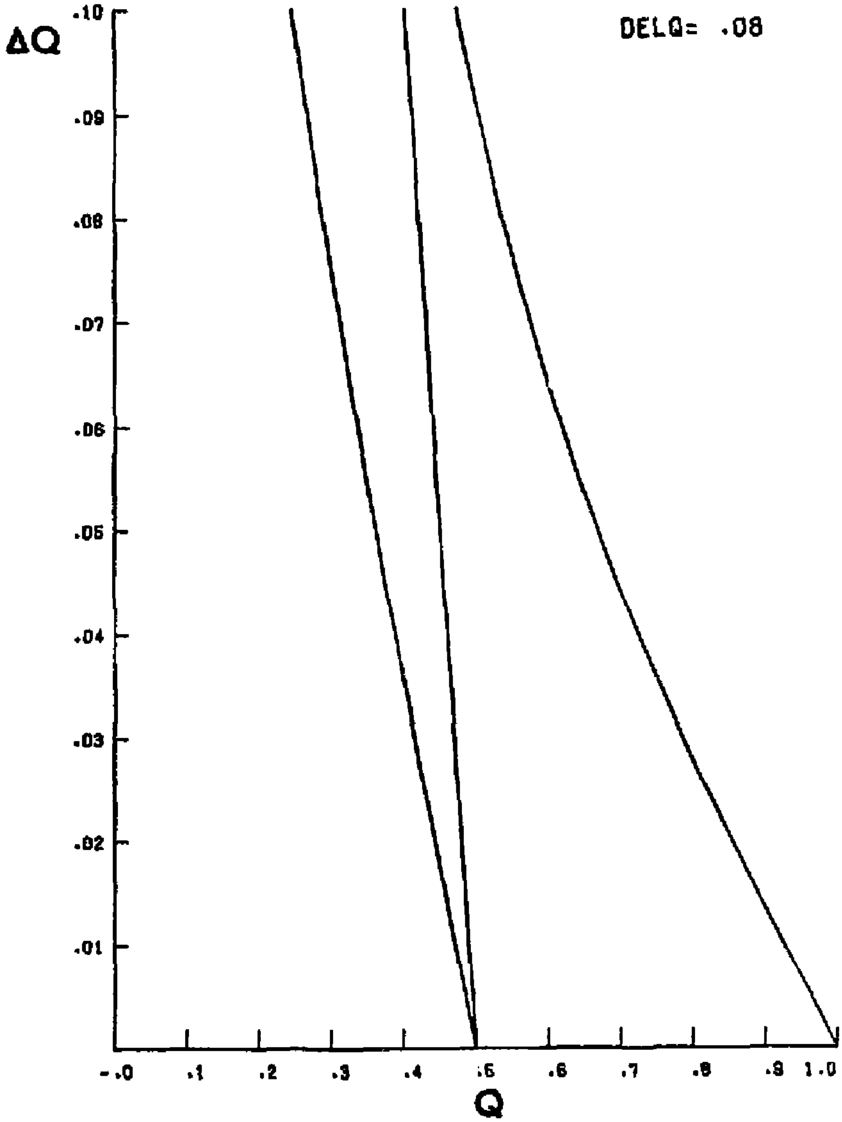



Fig. 7. Threshold and stopband for one bunch in each beam, and different phase advances in the two arcs, $2\pi(\frac{1}{2}Q \pm 0.08)$.

Fig. 8. Average separation S between colliding bunches for one bunch in each beam and ΔQ = 0.02.

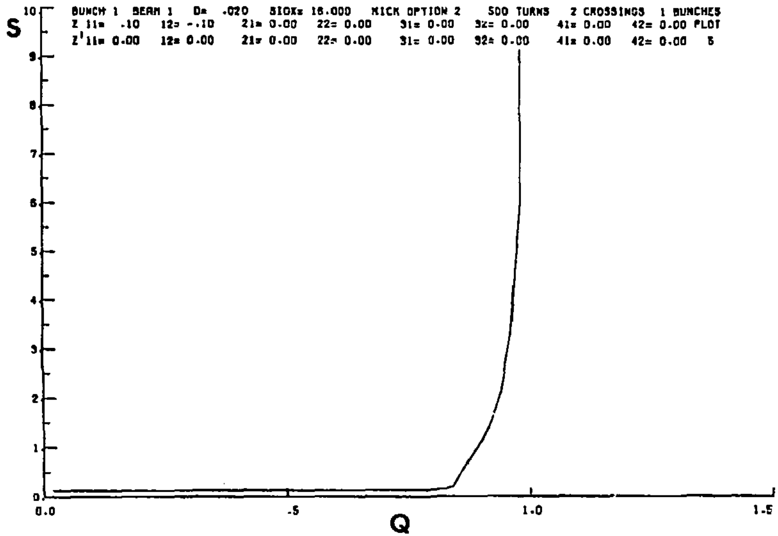


Fig. 9. Average separation S between colliding bunches for one bunch in each beam and ΔQ = 0.04.

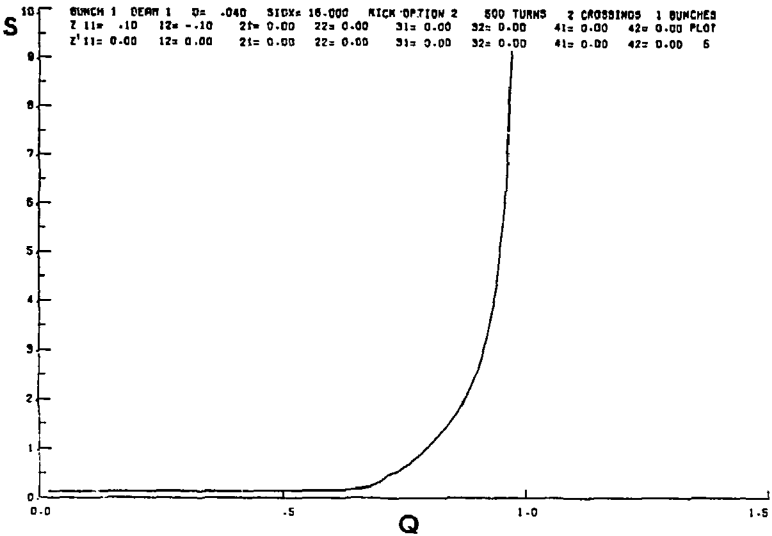


Fig. 10. Average separation S between colliding bunches for one bunch in each beam and $\Delta Q = 0.06$.

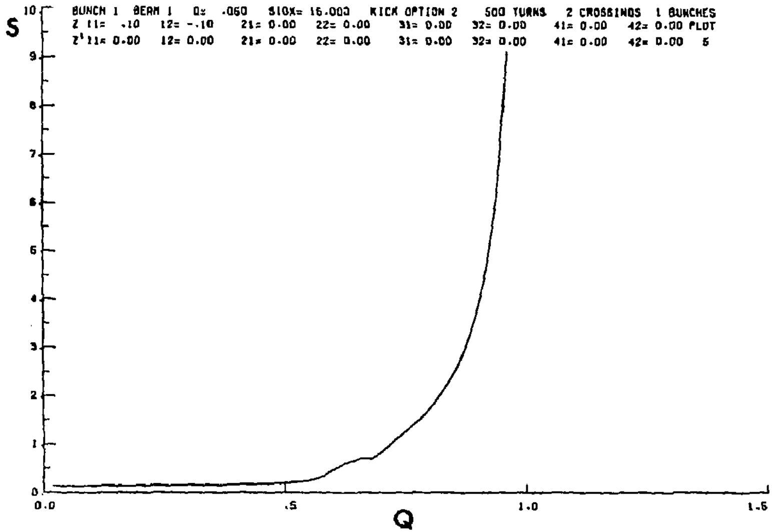


Fig. 11. Average separation S between colliding bunches for three bunches in each beam and $\Delta Q = 0.02$.

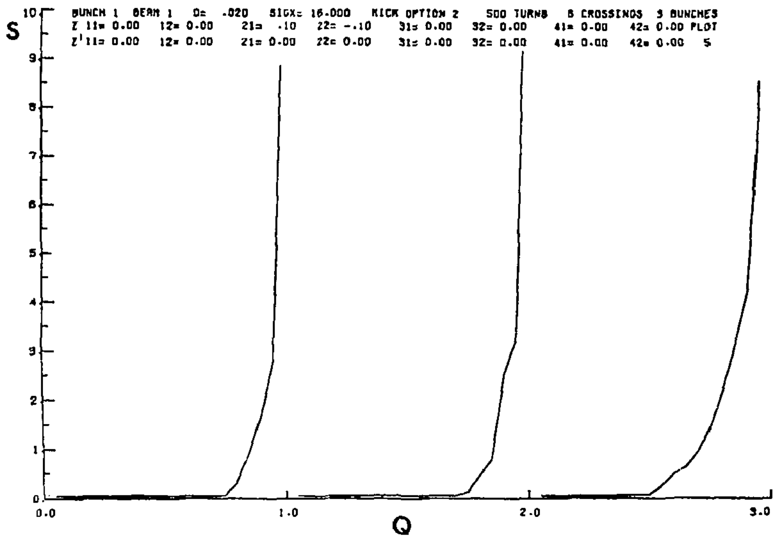


Fig. 12. Average separation S between colliding bunches for three bunches in each beam and $\Delta Q = 0.04$.

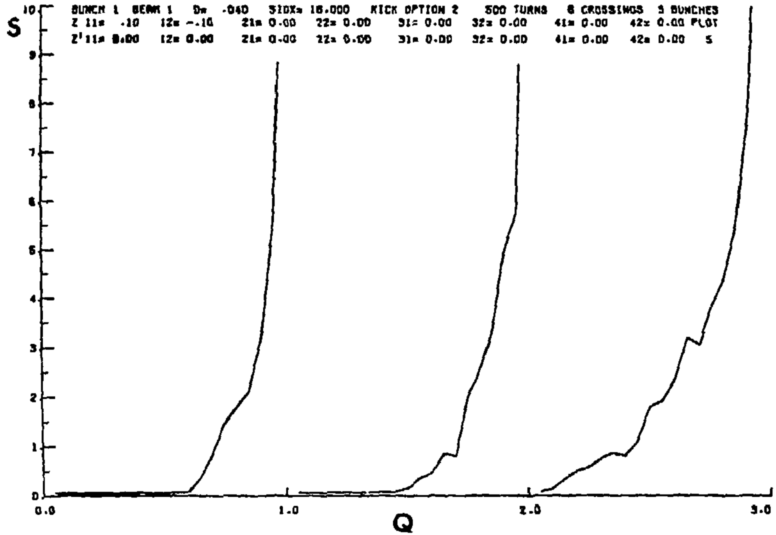


Fig. 13. Average separation S between colliding bunches for three bunches in each beam and Q = 0.06.