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INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

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# TWO EXAMPLES OF ESCAPING HARMONIC MAPS *

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## Introduction

The subject of this communication is part of the study of the existence of special harmonic maps on complete non-compact Riemannian manifolds. In fact we pretend to generalize the notion of escaping geodesic (see [1]), and prove some results on the existence of escaping harmonic maps.
We will first introduce the notion of end of a manifold and of escaping map and will relate one with the other. We will also see how a proper map maps the ends of a non-compact manifold. Since neither these definitions nor the propositions proved in the first part need the structure of a Riemannian manifold, we will always assume X and Y are Hausdorff spaces.
In the second part we will explain two examples of escaping harmonic maps, the first from the 2-dimensional hyperbolic disc into the 3-dimensional hyperbolic disc and the second from a manifold with an infinite set of ends in another also with an infinite set of ends. The first example can be considered as the analog one dimension higher, both on the domain and on the range, of the geodesic on the 2-dimensional hyperbolic disc whose limit set for any point is the sphere at infinity $S^1_\infty$. These maps are in fact proper, therefore as we will see all ends of the first manifold are mapped into ends of the second manifold.

## 1-Ends and escaping properties

In this paragraph all definitions and propositions are going to be presented in a more general case, in fact we will only require two Hausdorff spaces.

**Definition 1.1**: ([2], [4], [11]) Let X be a Hausdorff space. An end of X is a function $\varepsilon$ that assigns to each compact subset $K \subset X$, a connected component $\varepsilon(K)$ of X-K and that verifies condition (1)

(1) if K and L are two compact subsets of X such that $K \subset L$ then $\varepsilon(L) \subset \varepsilon(K)$

**Remark 1.1**

1- Let X be a Hausdorff space,the set of ends of X is empty if and only if X is compact.

2- If X is a Hausdorff space, we say X has an infinite number of ends if the inverse limit of $\Pi_0(X\text{-}K)$, with $K \subset X$ compact , has an infinite number of components.

3- ([11]) Let End(X) be the set of all ends of X. We can define a topology on $X \cup \mathrm{End}(X)$ by choosing as neighbourhoods $N_c(\varepsilon_0)$ of an end $\varepsilon_0$ the sets

$$X\text{-}C \cup \{\text{ends } \varepsilon: \varepsilon(C) = \varepsilon_0(C)\} \text{ for all compact } C \subset X.$$

With this topology $X \cup End(X)$ is compact.

4- Any group determines up to homotopy the Eilenberg-MacLane space $K(\Gamma,1)$, and by definition, the ends of the group are the ends of this space.

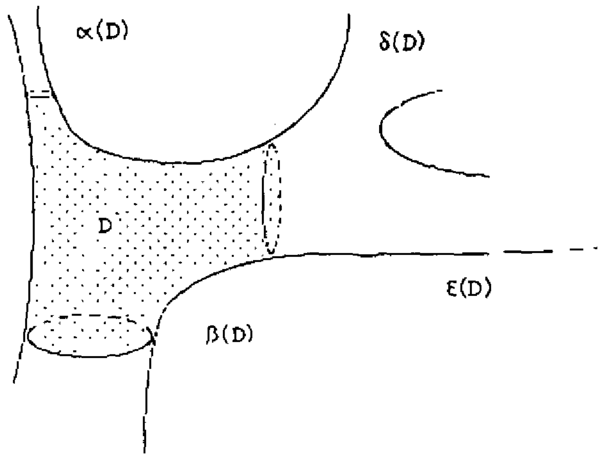


figure 1

**Example 1.1**

Let $\Gamma$ be a free group with more than two generators acting freely, properly and discontinuously by isometries on the hyperbolic plane $\mathbb{H}^2$. The quotient space $\mathbb{H}^2/\Gamma$ has as many ends as $\Gamma$ (see remark 1.1.4).

**Definition 1.2**: Let X and Y be Hausdorff spaces and $h: X \to Y$ be a map. Let $\varepsilon$ be an end of X and $\delta$ be an end of Y. We say h maps $\varepsilon$ into $\delta$ if for each compact set $D \subset Y$, there exists a compact $C \subset X$ such that $h(\varepsilon(C)) \subset \delta(D)$.

We say h respects ends if all ends of X are mapped into ends of Y.

**Example 1.2**

1-Let X be the surface drawn in figure 2.1 , Y be the surface drawn in figure 2.2 and h(X) be the shadowed surface in Y. h does not respect ends. Only two of the three ends of X are mapped into ends of Y.

2- Let X be once again the surface in figure 2.1, Y be the surface in figure 2.3 , and h(X) be the



shadowed surface on Y (h is surjective). h respects ends ("two ends of X are mapped into the same end of Y").

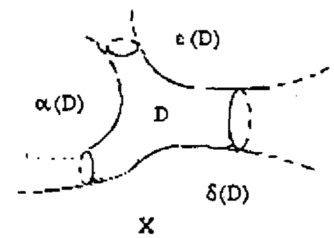


figure 2.1

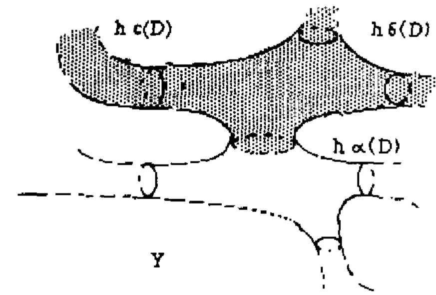


figure 2.2

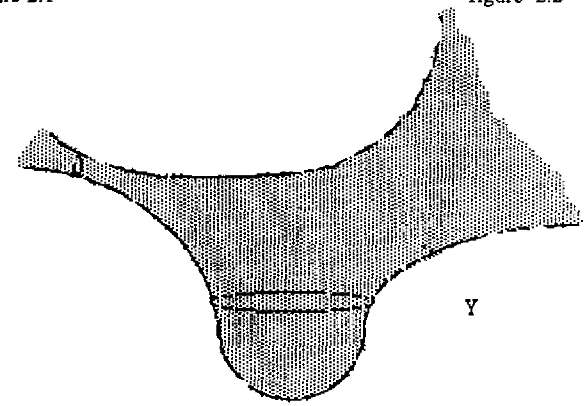


figure 2.3

**Definition 1.4**: Let X and Y be non-compact Hausdorff spaces and $h: X \to Y$ be a map such that h(X) is not bounded. h is escaping if there exists a compact set $K \subset Y$ such that h is proper outside K (i.e. for each compact set $D \subset (Y\text{-}intK)$, $h^{-1}(D)$ is compact in X).

**Example 1.3**:

-Let (M,g) be a Riemannian manifold. Any escaping geodesic in M is an escaping harmonic map from $\mathbb{R}$ into M.

-Obviously, any proper map between non-compact Hausdorff spaces escapes

In the following proposition we are going to relate the ends of the Hausdorff spaces with the escaping property.

Proposition 1.1

Let X and Y be Hausdorff spaces and $h:X\rightarrow Y$ be a map.

1-If h is proper, then h respects ends.

2-If h is escaping, then at least an end of X is mapped into an end of Y.

**proof**:

1-Let h be proper and $\varepsilon$ be an end of X. Let D be any compact of Y.

There must exist an end $\delta$ of Y, such that for some $C_\delta \subset X$ compact

$$h(\varepsilon(C_\delta)) \cup \delta(D) \neq \emptyset.$$

Otherwise, there would exist a compact set, namely $C=\bigcap_\delta C_\delta$, such that $h(\varepsilon(C))\subset D$ and the map would be bounded, hence not proper.

Let us suppose $h(\varepsilon(C_\delta))$ is not contained in $\delta(D)$, then $h(\varepsilon(C_\delta)) \cap D\neq\emptyset$ and since D is compact $h^{-1}(D)$ is compact and $\varepsilon(C_\delta) \cap h^{-1}(D)$ is bounded. Then there exists a compact set $A\subset X$ such that

$$[C_\delta \cup (\varepsilon(C_\delta) \cap h^{-1}(D))] \subset A.$$

Since $C_\delta \subset A$, $\varepsilon(A)\subset\varepsilon(C_\delta)$, and $h(\varepsilon(A)) \cap D=\emptyset$, $h(\varepsilon(A))\subset\delta(D)$.

2- Let us suppose for all ends $\varepsilon$ of X and all ends $\delta$ of Y there exists a compact set $D_\delta \subset Y$, such that

(3) for all compact sets $C\subset X$, $h(\varepsilon(C))$ is not contained in $\delta(D_\delta)$.

For some end $\delta$ of Y , we must have $h(\varepsilon(C)) \cap D_\delta\neq\emptyset$ and $h(\varepsilon(C))\cap\delta(D_\delta)\neq\emptyset$, otherwise h is bounded.

a) Let us suppose $D_\delta\text{-int}K \neq\emptyset$, then since it is compact and contained in Y-intK, $h^{-1}(D_\delta\text{-int}K)$ is also a compact set and once again there must exist a set $A \subset X$ such that

$$C \cup [\varepsilon(C) \cap h^{-1}(D_\delta\text{-int}K)] \subset A$$

hence $h(\varepsilon(A)) \cap D_\delta=\emptyset$ and $h(\varepsilon(A)) \subset\delta(D_\delta)$, which contradicts (3).

b) Let us now suppose $D_\delta\text{-int}K=\emptyset$.

$X\text{-}h^{-1}(K)$ cannot be bounded, (otherwise the map would be bounded), and there exists an end $\varepsilon_1$ of X such that for any compact set $C\subset X$

$$( X\text{-}h^{-1}(K)) \cap \varepsilon_1(C) \neq\emptyset$$

and $h(\varepsilon_1(C))$ is not contained in K. (If there were no such end the map would have to be bounded).



Let $D_1$ be any compact set such that $D_1\subset Y\text{-int}K$, and $h(\varepsilon_1(C)) \cap D_1\neq\emptyset$, then for some $\delta_1$, we must have $h(\varepsilon_1(C))\cap \delta_1(D_1)\neq\emptyset$ otherwise the map would be bounded. We can now follow the usual reasoning: $h^{-1}(D_1)$ is compact therefore there exists one compact set $A\subset X$ with

$$C \cup [\varepsilon_1(C) \cap h^{-1}(D_1)] \subset A$$

hence $\varepsilon_1(A) \cap D_1=\emptyset$ and $h(\varepsilon_1(A)) \subset \delta_1(A)$.◊◊

If a map is proper we know how ends are mapped, but if **h** is escaping we only have the information that at least an end of Y is occupied (i.e. for any compact set $D\subset Y$, $(h(X)\cap\delta(D))\neq\emptyset$) but we know nothing about the way the other ends are mapped. In figure 3 we can see an example were all ends of Y are "occupied" and there is no end of X which is "mapped into one end" of Y.

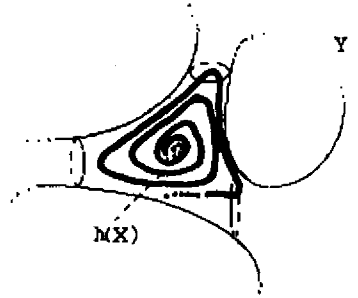


figure 3

**Lemma 1.1**

Let X be a Hausdorff spaces, Y a metric space such that all closed bounded sets are compact, let $h:X\rightarrow Y$ be an escaping map and $\varepsilon$ be an end of X which is not mapped into any end of Y, then there exists compact sets $C\subset X$ and $D\subset Y$ such that $h(\varepsilon(C))\subset D$.

**proof**

Since $\varepsilon$ is not mapped into any end of X, we know that for any end $\delta$ of Y there exists a compact set $D_\delta\subset Y$ such that for all compact sets $C\subset X$, $h(\varepsilon(C))$ is not contained in $\delta(D_\delta)$.

If $h(\varepsilon(C))\subset D_\delta$ for some compact set $C\subset X$, there is nothing to prove.

Let us suppose there exists an end $\delta_1$ of Y such that for all compact set $C\subset X$

$$h(\varepsilon(C)) \text{ is not contained in } \delta_1(D_1) \text{ and } h(\varepsilon(C))\cap\delta_1(D_{\delta 1})\neq\emptyset$$

If this were not possible then $h(\varepsilon(C))$ would be contained in a compact set and the lemma would be proved.

Let $K \subset Y$ be the compact set from the definition of escaping map.

Let $D = D_{\delta 1} \cup K$. Since $h(\varepsilon(C)) \cap D_{\delta 1} \neq \emptyset$, we know that $h(\varepsilon(C)) \cap D \neq \emptyset$.

Let $A^a = \{ p \in Y\text{-int}D : d(p, \partial D) \leq a \}$, $a \in \mathbb{R}$ and $a > 0$.

$A^a$ is compact and $A^a \subset Y\text{-int}K$, therefore $h^{-1}(A^a)$ is compact.

Let $C^a = h^{-1}(A^a) \cup C$.

$$\varepsilon(C^a) \cap C^a = \emptyset \text{ and } h(\varepsilon(C^a)) \cap A^a = \emptyset.$$

Then, since $\varepsilon(C^a)$ is connected and so $h(\varepsilon(C^a))$ has to be connected, we conclude that either

$h(\varepsilon(C^a)) \subset D$ and the lemma is proved,

or

$h(\varepsilon(C^a)) \subset \delta(A^a \cup D)$ for some end $\delta$ of Y. But then $h(\varepsilon(C^a)) \subset \delta(A^a \cup D) \subset \delta(D)$ and the end $\varepsilon$ is mapped in the end $\delta$ which is impossible.◊◊

**Corollary 1.1**

Let X, Y and $h: X \to Y$ be as in lemma 1.1. Let $\{\varepsilon_i\}_{i \in I}$ be the set of ends of X which are not mapped into any end of Y. Let $K \subset Y$ be the compact set from the definition of escaping map, then for each $i \in I$ there is a compact set $C_i \subset X$ such that $h(\varepsilon_i(C_i)) \subset K$.

**proof**

Let $\varepsilon_i$ be an end of X as in the hypothesis, and $C_i$ and D be two compacts given by lemma 1.1. If D is not contained in K, just consider $h^{-1}(D\text{-int}K)$ which is compact, and let $C \subset X$ be a compact set such that

$$C_i \cup [\varepsilon_i(C_i) \cap h^{-1}(D\text{-int}K)] \subset C.$$

Then $h(\varepsilon_i(C)) \subset D$ and $h(\varepsilon_i(C)) \cap (D\text{-int}K) = \emptyset$, therefore $h(\varepsilon_i(C)) \subset K$.◊◊

**Example 1.4**

Let Y be the surface represented in figure 4 and X a space with an infinite number of ends. Let us suppose h(X) is the set marked in darker ink in Y. h(X) is not bounded but h is not escaping. The ends are mapped "further and further away".



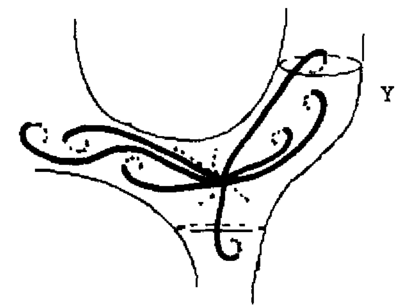


figure 4

In the following proposition we will give a necessary and sufficient condition for a map to be escaping. In fact we must have at least an end of X mapped into an end of Y and must be able to "catch" in a compact set all ends which are not mapped into ends.

**Proposition 1.2**

Let X and Y be as in lemma 1.1, and $h: X \to Y$ a map. h is escaping if and only if both conditions 1 and 2 are verified.

1. There exists at least an end of X which is mapped into an end of Y.

2. Let $\{\varepsilon_i\}_{i \in I}$ is the set of the ends of X which are not mapped into any end of Y. There exists a compact set $K \subset Y$ such that for each i there exists a compact set $C_i \subset X$, such that $h(\varepsilon_i(C_i)) \subset K$.

**proof**

$\Rightarrow$

is an immediate consequence of propositions 1.1 and corollary 1.1

$\Leftarrow$

Since h verifies condition 1, h is not bounded.

Let D be any compact in Y such that $D \subset Y\text{-}K$ and $h^{-1}(D) \neq \emptyset$. Is $h^{-1}(D)$ compact in X? Let us suppose not. In this case it cannot be bounded and there exists an end $\varepsilon$ of X such that for any compact set $C \subset X$,

$$\varepsilon(C) \cap h^{-1}(D) \neq \emptyset.$$

Hence $\varepsilon$ is not mapped into any end $\delta$ of Y (in fact if for all compact sets $C \subset X$, $h(\varepsilon(C)) \cap D \neq \emptyset$, $h(\varepsilon(C))$ cannot be contained in $\delta(D)$ for any $\delta$).

But condition 2 implies that there exists a compact set $C_i \subset X$ such that $h(\varepsilon(C_i)) \subset K$. Since $h(\varepsilon(C_i)) \cap D \neq \emptyset$ and $[h(\varepsilon(C_i)) \cap D] \subset Y\text{-}K$, there is a contradiction, therefore $h^{-1}(D)$ is compact and the map is escaping.◊◊

## 2-Two examples of escaping harmonic maps

For all notation and results on harmonic maps see [5] and [6] , and on hyperbolic manifolds see [9].

**Example 2.1**

Let (M,g) be a compact surface with genus strictly greater than 1.

Let $f:M \to M$ be a diffeomorphism which is isotopic to a pseudo-Anosov diffeomorphism (for the definition see [9]).

Let $F: M \times \mathbb{R} \to M \times \mathbb{R}$ be given by $F(x,t) = (f(x),t+1)$. We have a free, proper and totally discontinuous action of $\mathbb{Z}$ on $M \times \mathbb{R}$ given by

$$G(n,(x,t)) = F^n(x,t), \quad (F^n = F \circ \ldots \circ F).$$

The quotient space $N_f$, given by the orbits of the action is the mapping torus and it can also be described as the quotient

$$M \times \mathbb{R}/(x,0)\sim(f(x),1)$$ (see figures 5.1 to 5.3)

The manifold $N_f$ fibers differentiably over $S^1$ with fibre M. Let $p:N_f \to S^1$ denote this fibration.

**Definition 2.1** ([7], [9]) A Jørgensen-Thurston manifold is a compact hyperbolic 3-dimensional manifold, which fibers differentiably over $S^1$, and whose fiber is a smooth orientable surface of genus greater than 1.

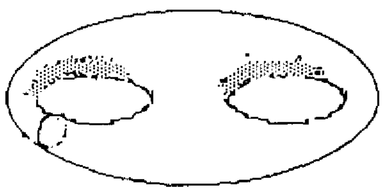

figure 5.1



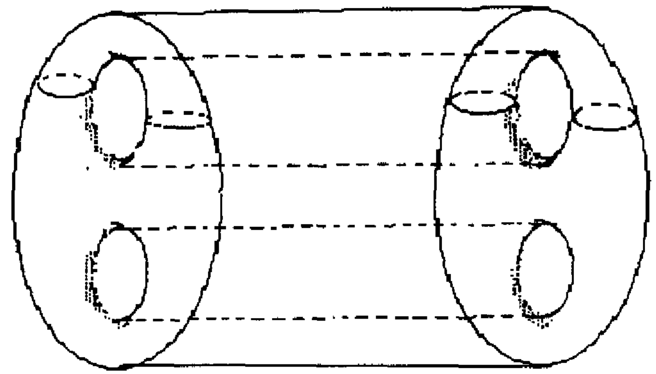

figure 5.2

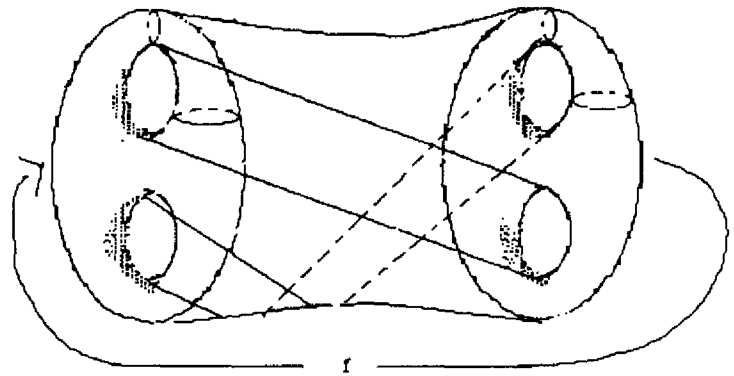


figure 5.3

Thurston has shown [10] that the mapping torus $N_f$ of any pseudo-Anosov diffeomorphism is a complete hyperbolic manifold, hence a Jørgensen-Thurston manifold. Furthermore this hyperbolic metric is unique because of Mostow's rigidity theorem [8]. Let h be this hyperbolic metric.

**Remark 2.1**

Let $f:M \to M$ be a diffeomorphism that satisfies the following condition:

A) The map $f_*:H_1(M,\mathbb{R}) \to H_1(M,\mathbb{R})$ does not have an eigenvalue with norm equal to 1.

If f satisfies A), f is isotopic to a pseudo-Anosov diffeomorphism and the manifold $N_f$ associated to f is a Jørgensen-Thurston manifold.

From now on we will always assume that f is a diffeomorphism that satisfies the condition A) in remark 2.1.

Using Wang's sequence we can see that $H_1(N,\mathbb{R})=\mathbb{R}$.

Let g be a hyperbolic metric on M and let $A=\{\alpha_1,\alpha_2,\ldots,\alpha_\gamma,\beta_1,\beta_2,\ldots,\beta_\gamma\}$ ($\gamma$ is the genus of M), be a set of generators of $\Gamma=\Pi_1(M)$. Then (M,g,A) as a marked hyperbolic surface represents a point on the Teichmüller space $T(\Gamma)$.

Let $A'=\{\alpha'_1,\ldots,\alpha'_\gamma,\beta'_1,\ldots,\beta'_\gamma\}$, be a basis of $\Pi_1(M\times\{0\})$, where $M\times\{0\}$ is a fibre of $N_f$.

Let $\boldsymbol{H}: T(\Gamma)\to C^\infty(M,N_f)$ be given as follows.

$$\boldsymbol{H}\ ((M,g,A)) := \varphi_g\ ,$$

where $\varphi_g : (M,g)\to(N_f,h)$ is the unique harmonic map given by Eells and Sampson's theorem, (see[5] and [6]), such that

$$(\varphi_g)_* : \Pi_1(M)\to\Pi_1(N_f)$$

has the following property

$$(\varphi_g)_*(\alpha_i) = \alpha'_i \qquad \text{and} \qquad (\varphi_g)_*(\beta_i) = \beta'_i.$$

By regularity theorems for harmonic maps (see [5],[6]), the mapping $\boldsymbol{H}: T(\Gamma)\to C^\infty(M,N_f)$ is a $C^\omega$ mapping from the Teichmüller space $T(\Gamma)$ into the Fréchet manifold $C^\infty(M,N_f)$.

**Remark 2.2**

If there exists an element of $T(\Gamma)$, (M,g,A) such that $\varphi_g$ is harmonic and conformal then it follows that $\varphi_g$ is a branched immersion ( see [6]).

Let $\pi_g : \mathbb{H}^2\to M$ be the universal covering projection from the hyperbolic plane $\mathbb{H}^2$, considered as the unit disc in the complex plane with the hyperbolic metric, to (M,g), such that $\pi_g$ is a local isometry and gives the chosen marking A of M. Namely one fixes the group of deck transformations $\Gamma'\subset PSL(2,\mathbb{R})$, ($PSL(2,\mathbb{R})$ is the group of orientation-preserving isometries of $\mathbb{H}^2$), such that $\Gamma'\equiv\Gamma$.

Let $\pi: \mathbb{H}^3\to N_f$ be the local isometric universal covering projection from the hyperbolic space $\mathbb{H}^3$, identified with the open unit ball in $\mathbb{R}^3$, with the Poincaré metric. The group of orientation-preserving isometries of $\mathbb{H}^3$ is $PSL(2,\mathbb{C})$.



From the exact sequence of homotopy groups associated with the fibration p, one obtains:

$$0\to\Gamma=\Pi_1(M)\to\Pi_1(N)\to\mathbf{Z}\to 0.$$

Therefore as a group $\Pi_1(N_f)$ is a semi-direct product of $\Gamma$ with the integers, and we have a natural monomorphism from $\Gamma$ onto a normal subgroup of $\Pi_1(N_f)$,

$$(\varphi_g)_*(\Gamma) \lhd \Pi_1(N_f).$$

$PSL(2,\mathbb{C})$ has a natural extension to the closed disc $D^3$, and acts by Möbius transformations on the sphere at infinity $S^2_\infty$.

Let $\tilde{\Gamma}=(\varphi_g)_*(\Gamma)$ and $\Delta=\Pi_1(N_f)$. We will identify $\Delta$ with the group of deck transformations of $\pi$ and consider $\Delta\subset PSL(2,\mathbb{C})$. Then since $\tilde{\Gamma}$ is a normal subgroup of $\Delta$, it follows by lemma 8.1.3 in [9] that $\tilde{\Gamma}$ acts also minimally on $S^2_\infty$, hence the limit set of $\tilde{\Gamma}$ is $S^2_\infty$.

As a consequence of the previous statements we have the following commuting diagram consisting of local isometries and harmonic maps.

$$\begin{array}{ccc} & \tilde{\varphi}_g & \\ \mathbb{H}^2 & \to & \mathbb{H}^3 \\ \pi_g \downarrow & & \downarrow \pi \\ M & \to & N_f \\ & \varphi_g & \end{array}$$

More precisely we have the following theorem.

**Theorem 2.1**

$\tilde{\varphi}_g: \mathbb{H}^2\to\mathbb{H}^3$ is a harmonic map such that, given any point $x\in S^2_\infty=\partial\,\mathbb{H}^3$, there exists a sequence $\{x_i\}_{i\in I}$ in $\mathbb{H}^2$, such that $x_i$ converges to $y\in S^1_\infty$ and

$$\lim_{i\to\infty}\tilde{\varphi}_g(x_i) = x.$$

In fact one has the following:

**Theorem 2.2** ([3])

$\tilde{\varphi}_g$ has a continuous extension to the closed disc $D^2=\mathbb{H}^2\cup S^1_\infty$, and the restriction to $S^1_\infty$,

$$\tilde{\varphi}_g|_{S^1_\infty} : S^1_\infty \to S^2_\infty$$

is a Peano curve, which means its image is all of $S^2_\infty$. Furthermore this map only depends on the homotopy data, namely if $\psi : M \to N_f$ is homotopic to $\varphi_g$ and if $\tilde{\psi} : \mathbb{H}^2 \to \mathbb{H}^3$ is its lifting, then $\tilde{\psi}$ also has a continuous extension to all of $\mathbb{H}^2 \cup S^1_\infty$, and its restriction to $S^1_\infty$ is equal to $\Phi_g$.

**Example 2.2**

Let $\bar{M}$ be a covering of M by a non-compact surface with a Cantor set of ends (see figure 6).

Let $\bar{\pi}_g : \bar{M} \to M$ be the corresponding covering projection considered as a local isometry , and let $\Gamma_C$ be the group of deck transformations of $\pi_g$.

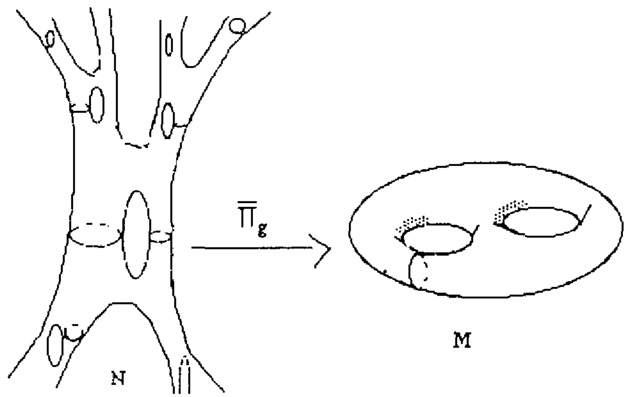


figure 6

Let $\bar{N}_f$ be the covering of $N_f$ corresponding to $(\varphi_g)_*(\Gamma_C)$ and $\pi_C : \bar{N}_f \to N_f$ the corresponding local isometry. Then we have the following theorem.

**Theorem 2.3**

$\bar{\varphi} : \bar{M} \to \bar{N}_f$ is a harmonic map that sends the ends of $\bar{M}$ surjectively to the ends of $\bar{N}$, hence $\bar{\varphi}$ respects ends.




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