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Roberto Floreanini

Roberto Percacci

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Ergin Sezgin

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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Roberto Floreanini
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste
Dipartimento di Fisica Teorica, Università di Trieste
Strada Costiera 11, 34014 Trieste, Italy

Roberto Percacci
International School for Advanced Studies, Trieste, Italy
and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Trieste, Italy

and

Ergin Sezgin
International Centre for Theoretical Physics, Trieste, Italy.

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ABSTRACT

We study the dynamics of sigma models in arbitrary dimensions with purely Wess–Zumino–Witten actions (i.e. without kinetic terms), both from the Lagrangian and Hamiltonian point of view. These models have nontrivial gauge groups which contain the *diffeomorphisms of space-time*, as well as *symmetry groups which in many cases turn out* to be infinite dimensional. We give examples in 1, 2, 3 and 4 space–time dimensions.

1. INTRODUCTION

In two recent papers Witten has considered Yang–Mills[1] and gravity[2] theories in 2+1 dimensions with purely Chern–Simons actions, leading to results in topology and string theory. The hallmark of these actions is that they do not depend on the space-time metric and therefore are topological in nature. These theories are exactly soluble and the (finite dimensional) space of all classical solutions of the Euler–Lagrange equations can be explicitly described.

In the case of the non-linear sigma models there also exists an action which is independent of the space-time metric, namely the Wess–Zumino–Witten (WZW) action

$$S = e\int_M \varphi^* B = \frac{e}{m!}\int d^m x\ \varepsilon^{i_1\ldots i_m}\partial_{i_1}\varphi^{\alpha_1}\ldots\partial_{i_m}\varphi^{\alpha_m} B_{\alpha_1\ldots\alpha_m}(\varphi(x)). \tag{1.1}$$

Here φ is a map from a manifold M to some manifold N, and B is an m-form on N, where $m = \dim M$. This action can be defined for every value of m. Usually, the term (1.1) is added to a kinetic term for φ. Here, instead, we shall take the total action to be given by (1.1). There are several motivations for this. From a mathematical point of view, it is conceivable that suitably defined path integrals based on (1.1) may lead to topological invariants, as happens in Refs. 1 and 2. Physically, one can interpret the fields φ as coordinates of an $(m-1)$–dimensional extended object propagating in the space-time N. In particular, for $m = 2$ actions like (1.1) could give rise to a new class of string theories. Furthermore, Faddeev and Shatashvili[3] have proposed that a gauged version of (1.1), without kinetic term for φ, be used in a program towards the consistent quantization of anomalous gauge theories.

In the present paper we discuss the properties of the purely WZW theory at the classical level, with particular emphasis on its symmetry properties. We begin in Sect. 2 by giving an account of the Lagrangian formulation of the theory, its gauge group and its symmetry group (the difference between the two being that the latter has Noether currents, the former not). The gauge group contains the diffeomorphisms of M and certain diffeomorphisms of N. It may come as a surprise that a theory can have gauge invariances without having covariant derivatives, but this is a consequence of its independence of the metric. The symmetry group consists of diffeomorphisms of N which preserve $H \equiv dB$ but at the same time are not in the gauge group (the diffeomorphisms of N which are gauge transformations also preserve H). In many cases this group is infinite dimensional. This is in contrast to what happens usually in theories with a kinetic term, and hence with a metric both on space-time and in the internal space. In these theories a diffeomorphism can be a symmetry only if it preserves the metric, and this restricts the symmetry group to be finite dimensional. In our case, due to independence from the metric, no such restriction arises. In the case when $n = \dim N = m+1$ we are able to describe all classical solutions of the Euler–Lagrange equations. Unlike in the theories considered in Refs. 1 and 2, the space of solutions modulo gauge transformations turns out to be infinite dimensional.

In Sect. 3 we discuss the constrained Hamiltonian formulation of the case $M = \mathbb{R}$; it corresponds to a charged particle moving in an external magnetic field, in the limit of slow motion, where the kinetic term can be neglected with respect to the interaction with the background.

In Sect. 4 we consider the case when $m = \dim M > 1$. The analysis is very similar to that of Sect. 3, but now in an infinite dimensional context. We find the reduced phase space of the theory and, at least formally, the Dirac brackets. Sect. 5 contains examples in dimensions $m = 1, 2, 3$ and 4. Some of these examples in 1 and 2 dimensions have been analyzed in a recent work[4], using a different formalism. Finally, in an Appendix we collect some properties of spaces of maps which are relevant to our treatment.

Throughout this paper we are going to make use of some terminology from differential geometry. If $v = v^\alpha\frac{\partial}{\partial x^\alpha}$ is a vector field and $\omega = \frac{1}{k!}\omega_{\beta_1\ldots\beta_k}dx^{\beta_1}\wedge\ldots\wedge dx^{\beta_k}$ is a k-form, $i_v\omega$ is a $(k-1)$-form obtained by contracting ω with v, namely $i_v\omega = \frac{1}{(k-1)!}v^\alpha\omega_{\alpha\beta_1\ldots\beta_{k-1}}dx^{\beta_1}\wedge\ldots\wedge dx^{\beta_{k-1}}$. As usual, d denotes the exterior differentiation, and the Lie derivative of ω along v is $(\mathcal{L}_v\omega)_{\beta_1\ldots\beta_k} = v^\alpha\partial_\alpha\omega_{\beta_1\ldots\beta_k} + \partial_{\beta_1}v^\alpha\omega_{\alpha\beta_2\ldots\beta_k} + \ldots + \partial_{\beta_k}v^\alpha\omega_{\beta_1\ldots\beta_{k-1}\alpha}$. We are going to make repeated use of the following formulae:

$$\mathcal{L}_v = i_v d + d i_v, \tag{1.2}$$

$$i_{[v,w]} = \mathcal{L}_v i_w - i_w\mathcal{L}_v. \tag{1.3}$$

2. LAGRANGIAN FORMALISM

We shall consider a non-linear sigma model with an action of the form (1.1), where φ is a map from the m–dimensional manifold M, with coordinates $\{x^i\}$, to the n–dimensional manifold N, with coordinates $\{y^\alpha\}$, and B is an m-form on N; e is an arbitrary dimensionless coupling constant and $\varepsilon^{1\ldots m} = 1$. An important feature of this action is that it is independent of any choice of metrics on M and N. Complete specification of a model in this class requires only a choice of the differentiable manifolds M and N, and of the form B.

If $dB = 0$, the action (1.1) is trivial, in the sense that the Euler–Lagrange equations (see Eq.(2.3) below) reduce to $0 = 0$. One has to distinguish two subcases. If $B = dC$ for some globally defined $(m-1)$-form C, the action vanishes identically for M compact and without boundary. If B represents a nontrivial cohomology class, i.e. one can write $B = dC$ only locally (or otherwise, C has singularities), then the action is a topological invariant of the map φ. In the following we shall consider exclusively the case when the curl of B is non-vanishing: $H \equiv dB \neq 0$. This requires $n \geq m+1$. Again $dH = 0$, so H defines a class in the cohomology group $H^{m+1}(N)$. If B is globally defined on N without singularities, the cohomology class of H is trivial. The most interesting situation occurs when the cohomology class of H is non-trivial. In this case the form B can be defined only locally on N, and the local results have to be patched together. We shall not discuss this point explicitly.

Under an arbitrary variation $\delta\varphi^\alpha = v^\alpha$ of the fields, the action (1.1) is changed by

$$\begin{aligned}\delta_v S &= e\int_M \varphi^*\mathcal{L}_v B = e\int_M d\varphi^* i_v B + e\int_M \varphi^* i_v H \\ &= \frac{e}{(m-1)!}\int d^m x\ \varepsilon^{i_1\ldots i_m}\partial_{i_1}\left[v^{\alpha_1}\partial_{i_2}\varphi^{\alpha_2}\ldots\partial_{i_m}\varphi^{\alpha_m}B_{\alpha_1\ldots\alpha_m}\right] \\ &\quad + \frac{e}{m!}\int d^m x\ \varepsilon^{i_1\ldots i_m}\partial_{i_1}\varphi^{\alpha_1}\ldots\partial_{i_m}\varphi^{\alpha_m}v^\beta H_{\beta\alpha_1\ldots\alpha_m}\ . \end{aligned} \tag{2.1}$$

where $H = dB$, or more explicitly

$$H_{\alpha_1\ldots\alpha_{m+1}} = (m+1)\ \partial_{[\alpha_1}B_{\alpha_2\ldots\alpha_{m+1}]}\ . \tag{2.2}$$

With appropriate boundary conditions, (2.1) leads to the Euler-Lagrange equations

$$\varepsilon^{i_1\ldots i_m}\partial_{i_1}\varphi^{\alpha_1}\ldots\partial_{i_m}\varphi^{\alpha_m}H_{\alpha_1\ldots\alpha_m\beta} = 0\ . \tag{2.3}$$

Let us observe that in the case $n = m+1$ the Euler-Lagrange equations have a simple geometrical meaning. In this case $H_{\alpha_1\ldots\alpha_n} = \varepsilon_{\alpha_1\ldots\alpha_n} f$, where f is a function which we assume to be nowhere vanishing. Thus H is a volume form on N and Eq.(2.3) is equivalent to

$$\varepsilon^{i_1\ldots i_m}\partial_{i_1}\varphi^{\alpha_1}\ldots\partial_{i_m}\varphi^{\alpha_m}\varepsilon_{\alpha_1\ldots\alpha_m\beta} = 0\ . \tag{2.4}$$

For fixed β, the quantity on the l.h.s. is just the determinant of the $m\times m$ minor which is obtained from the Jacobian matrix $J_i{}^\alpha = \partial_i\varphi^\alpha$ by deleting the β-th column. Thus, Eq.(2.4) implies that the Jacobian matrix has rank $m-1$ or less. The solutions of the Euler-Lagrange equations are all the maps φ whose image has dimension $\le m-1$ everywhere.

Let us now discuss the invariances of our model. Infinitesimally, these are given by transformations $\delta\varphi^\alpha = v^\alpha$ such that the integrand in the second term on the r.h.s. of the Eq.(2.1) is a total derivative, i.e.

$$\varphi^* i_v H = d\Theta_v\ . \tag{2.5}$$

Among these are the diffeomorphisms of M, for which

$$v^\alpha = \xi^i\partial_i\varphi^\alpha \tag{2.6}$$

(if M is non-compact or has a boundary one has to impose boundary conditions on ξ). The action is furthermore invariant under diffeomorphisms of N generated by vector fields v such that the Lie derivative of the m-form B is

$$\mathcal{L}_v B = d\Lambda_v\ , \tag{2.7}$$

for some $(m-1)$ form Λ_v. These vector fields satisfy Eq.(2.5) with $\Theta_v = \varphi^*(\Lambda_v - i_v B)$. For simplicity, we make the assumption that the cohomology group $H^m(N) = 0$. Then the form Λ_v is globally defined. Our conclusions however do not depend on this hypothesis. Equation (2.7) is equivalent to the invariance of H:

$$\mathcal{L}_v H = 0\ . \tag{2.8}$$

These vector fields v form a (possibly infinite dimensional) subalgebra of the algebra of all vector fields on N. The subgroup of $\mathcal{D}iff N$ generated by this subalgebra will be denoted $S(H)$. Since $dH = 0$, from (2.8) and (1.2) one finds that $i_v H = d\Omega_v$ for some $(m-1)$-form Ω_v; from this and Eq.(2.7) we must have,

$$\Omega_v = \Lambda_v - i_v B + \epsilon\ , \tag{2.9}$$

where ϵ is a closed $(m-1)$-form. As we shall see, it is important to distinguish the two cases

$$i_v H = 0 \tag{2.10}$$

and

$$i_v H = d\Omega_v \neq 0\ . \tag{2.11}$$

Using Eq.(1.3), one finds that if $i_v H = 0$ and $\mathcal{L}_w H = 0$, then $i_{[w,v]}H = 0$. Therefore, the vector fields satisfying (2.10) form an ideal (i.e. a normal subalgebra) of the algebra of vector fields satisfying (2.8). They generate a normal subgroup of $S(H)$, which we denote by $K(H)$. The vector fields satisfying (2.11) generate the quotient $S(H)/K(H)$. Note that if $K(H)$ is nontrivial, it is infinite dimensional, because if v satisfies (2.10), also fv does, where f is an arbitrary function on N.

In some dimensions, one can make general statements about the groups $S(H)$ and $K(H)$:

1) $n = m+1$. As mentioned above, in this case H is a volume form. Eq.(2.10) has no nontrivial solutions and Eq.(2.11) is solved by all vector fields that are divergence-free with respect to a metric $h_{\alpha\beta}$ such that: $H_{\alpha_1\ldots\alpha_n} = \sqrt{\det h}\ \varepsilon_{\alpha_1\ldots\alpha_n}$. Thus $K(H)$ is trivial and $S(H)$ is the group of volume-preserving diffeomorphisms of N.

2) $n = m+2$. In this case the group $K(H)$ is always nontrivial because the vector field v dual to H always satisfies (2.10).

The action of $\mathcal{D}iff M \times S(H)$ on the space $\Gamma(M,N)$ of all maps φ from M to N is defined as follows: if $f \in \mathcal{D}iff M$ and $h \in S(H)$,

$$\varphi \mapsto \varphi' = h^{-1}\circ\varphi\circ f\ . \tag{2.12}$$

In addition to $\mathcal{D}iff M \times S(H)$, one has to consider also field-dependent invariance transformations, for which v has to be regarded as a vector field on $\Gamma(M,N)$, but which are not field-dependent transformations of $\mathcal{D}iffM$ or $S(H)$ (see Appendix). One would ordinarily not consider these as invariances of the theory: usually one calls invariances only those infinitesimal transformations v for which $\int \varphi^* i_v H = 0$ for any φ. However, we shall see from the Hamiltonian treatment that these invariances play a role in the present models.

The Noether current corresponding to an invariance $\delta\varphi^\alpha = v^\alpha$ is

$$j_v{}^i = -\frac{e}{(m-1)!}\varepsilon^{ii_2\ldots i_m}(\Theta_v)_{i_2\ldots i_m}\ , \tag{2.13}$$

where Θ_v is the $(m-1)$-form defined by (2.5). In particular, the vector fields (2.6) and those satisfying (2.10) have identically vanishing Noether currents; they generate gauge invariances. On the other hand, for the transformations satisfying Eq.(2.11) we have

$$j_v{}^i = -\frac{e}{(m-1)!}\varepsilon^{ii_2\ldots i_m}\partial_{i_2}\varphi^{\alpha_2}\ldots\partial_{i_m}\varphi^{\alpha_m}(\Omega_v)_{\alpha_2\ldots\alpha_m}\ . \tag{2.14}$$

These are true symmetries of the theory.

The Noether charge corresponding to (2.14) is given by

$$Q_v = \int_\Sigma d^{m-1}x\ j_v^0 = -e\int_\Sigma \varphi^*\Omega_v\ , \tag{2.15}$$

where Σ denotes a surface of constant time, which we assume to be compact and without boundary. Note from Eq.(2.11) that Ω_v is determined up to the addition of a closed $(m-1)$-form ϵ. The addition of exact forms $\epsilon = d\eta$ does not affect the charge, but if the cohomology group $H^{m-1}(N)$ is nontrivial, Q_v is defined only up to a constant. In order to have a uniquely defined set of charges, one has to fix this ambiguity. We shall return to this point later.

3. THE CASE m = 1 (A CHARGED PARTICLE IN A MAGNETIC FIELD)

We consider first the case $M = \mathbb{R}$. In this case our model describes a particle with electric charge e moving in the n-dimensional space N in the presence of a "magnetic" field, in the limit when the kinetic term is negligible. We shall use here the more appropriate notation q^α instead of φ^α for the coordinates of the particle, $\mathcal{A}_\alpha(q)$ instead of $eB_\alpha(\varphi)$ for the magnetic potential, and $\mathcal{F}_{\alpha\beta}(q)$ instead of $eH_{\alpha\beta}(\varphi)$ for the field strength. The action (1.1) reduces to

$$S = \int dt\ \dot{q}^\alpha \mathcal{A}_\alpha(q)\ . \tag{3.1}$$

The Euler-Lagrange equations become

$$\dot{q}^\alpha \mathcal{F}_{\alpha\beta} = 0 \tag{3.2}$$

where $\mathcal{F}_{\alpha\beta} = \partial_\alpha\mathcal{A}_\beta - \partial_\beta\mathcal{A}_\alpha$ is the field strength. These equations demand that the velocity $\dot{q}^\alpha$ be a null eigenvector of $\mathcal{F}_{\alpha\beta}$. In particular, if $\mathcal{F}_{\alpha\beta}$ is nondegenerate, $\dot{q}^\alpha = 0$.

The action (3.1) is invariant under time-reparametrizations and the group $S(\mathcal{F})$ which is generated by the algebra of vector fields v such that $\mathcal{L}_v\mathcal{F} = 0$. The normal subgroup $K(\mathcal{F})$ generated by vector fields v such that $i_v\mathcal{F} = 0$ has vanishing Noether generator, while the quotient group $S(\mathcal{F})/K(\mathcal{F})$ is a true symmetry, with non-vanishing Noether generators.

We set up the Hamiltonian formalism on the phase space T^*N. The momentum conjugate to q^α is

$$p_\alpha = \mathcal{A}_\alpha(q). \tag{3.3}$$

Since the r.h.s. does not depend on the velocities $\dot{q}^\alpha$, this is really a constraint equation. We define the primary constraints C_α to be

$$C_\alpha = p_\alpha - \mathcal{A}_\alpha \approx 0. \tag{3.4}$$

The canonical Hamiltonian $H_C = p_\alpha\dot{q}^\alpha - \mathcal{L}$ is identically zero, therefore the primary Hamiltonian is just a linear combination of constraints:

$$H_P = \Lambda^\alpha C_\alpha\ , \tag{3.5}$$

with $\Lambda^\alpha(q,p)$ some as yet undetermined Lagrange multipliers. The Poisson brackets between the primary constraints are

$$\{C_\alpha, C_\beta\} = \mathcal{F}_{\alpha\beta}, \tag{3.6}$$

and the preservation in time of the primary constraints requires

$$0 = \dot{C}_\alpha|_\mathcal{P} = \{C_\alpha, H_P\}|_\mathcal{P} = \mathcal{F}_{\alpha\beta}\Lambda^\beta, \tag{3.7}$$

where $\mathcal{P}$ denotes the subspace of T^*N defined by the constraints (3.4). No secondary constraints arise.

From Eq.(3.6) we see that if v^α is a vector field on N which at each point is a null eigenvector of $\mathcal{F}_{\alpha\beta}$ (i.e. $i_v\mathcal{F} = 0$), then the linear combination $C_v = v^\alpha C_\alpha$ is a first class constraint. Therefore, the algebra of first class constraints is the Lie algebra of the group $K(\mathcal{F})$. If v is not a null eigenvector of $\mathcal{F}$, then C_v is a second class constraint. (In particular, if v is in the Lie algebra of $S(\mathcal{F})/K(\mathcal{F})$, then C_v is a second class constraint).

The first class constraints generate a gauge group, which in our case is $K(\mathcal{F})$. In order to understand the action of the gauge group in the present problem, it is necessary to study the geometry of the null eigenspaces of $\mathcal{F}$. At each point q, the null eigenspace of $\mathcal{F}$ is a certain subspace $ker_q\mathcal{F} \subset T_qN$ of dimension $(n-r)$, where r is the rank of the matrix

$\mathcal{F}_{\alpha\beta}(q)$. In general r could depend on q (and thus the number of first class constraints could vary from point to point in phase space), but we will assume in the following that r is constant. Thus, $ker\mathcal{F}$ defines a distribution of $(n-r)$–dimensional subspaces of TN. If v and w are vector fields on N which are everywhere in $ker\mathcal{F}$, i.e. $i_v\mathcal{F} = i_w\mathcal{F} = 0$, then using Eqs.(1.2), (1.3) and the fact that $d\mathcal{F} = 0$, it is easy to prove that also $i_{[v,w]}\mathcal{F} = 0$. It then follows from Fröbenius theorem that the distribution $ker\mathcal{F}$ is integrable, i.e. N is foliated with $(n-r)$–dimensional leaves, and for each q, $ker_q\mathcal{F}$ is the space tangent to the leaf at q. The Lie algebra of $K(\mathcal{F})$ consists of vector fields on N which are tangent to the leaves of the foliation. Therefore, the gauge group $K(\mathcal{F})$ consists of the diffeomorphisms of N which map each leaf into itself.

It is interesting to consider also the vector fields w such that $i_w\mathcal{F} = d\Omega_w \neq 0$, which generate the quotient group $S(\mathcal{F})/K(\mathcal{F})$. These vector fields are not tangent to the leaves. However, if v belongs to the Lie algebra of $K(\mathcal{F})$, then also $\mathcal{L}_w v$ does. So, w generates a transformation which maps vectors tangent to a leaf to vectors tangent to another leaf. From this one can conclude that $S(\mathcal{F})/K(\mathcal{F})$ consists of diffeomorphisms of N which map each leaf to another leaf.

We can now choose a local coordinate system $\{q'^\alpha\}$ which is adapted to the foliation, in the sense that $q'^1, \dots q'^r$ parametrize the different leaves, and $q'^{r+1} \dots q'^n$ are coordinates in the leaves (the action (3.1) is invariant under the coordinate transformation $q^\alpha \to q'^\alpha(q)$ provided the potential $\mathcal{A}_\alpha$ is also transformed into $\mathcal{A}'_\alpha(q') = \frac{\partial q^\beta}{\partial q'^\alpha}\mathcal{A}_\beta(q)$). In this coordinate system $\mathcal{F}'_{\alpha\beta} = 0$ whenever $\alpha > r$ or $\beta > r$. Therefore, the constraints C'_α are second class when $\alpha = 1, \dots, r$ and first class when $\alpha = r+1, \dots, n$. In the following we shall assume that the coordinates are adapted to the foliation and drop the primes for notational simplicity.

At this point, we can make contact with the Lagrangian formalism. Hamilton's equations read

$$\dot{q}_\alpha|_\mathcal{P} = \{q^\alpha, H_P\}|_\mathcal{P} = \Lambda^\alpha , \tag{3.8}$$

$$\dot{p}_\alpha|_\mathcal{P} = \{p_\alpha, H_P\}|_\mathcal{P} = \Lambda^\beta \partial_\alpha \mathcal{A}_\beta|_\mathcal{P} . \tag{3.9}$$

The first equation, when inserted in (3.7) gives back the Euler–Lagrange equation (3.2); furthermore, using (3.8), Eq.(3.9) is seen to be the time derivative of the constraint C_α, which we know we can set to zero by choosing the Lagrange multipliers appropriately. In fact, from Eq.(3.7), we have $\Lambda^\alpha = 0$ for $\alpha = 1, \dots, r$ whereas the Λ^α's for $\alpha = r+1, \dots, n$ remain arbitrary.

Following Dirac, in order to determine the dynamics completely, it is necessary to fix gauge conditions χ^α, with $\alpha = r+1, \dots, n$, in such a way that the matrix of Poisson brackets of all the $(2n-r)$ constraints C_α and χ_α is nondegenerate. Preservation in time of the gauge conditions will fix the as yet undetermined Lagrange multipliers. The r–dimensional subspace of T^*N which is defined by all the constraints (including

the gauge conditions) is called the reduced phase space; the time evolution of the system occurs entirely within this subspace, and is driven by a uniquely defined Hamiltonian. The disadvantage of this procedure is that it is purely local, and it does not yield any information on the global structure of the reduced phase space. We therefore follow a slightly different, more geometrical route which enables us to identify the reduced phase space, and only in the end will we go back to Dirac's method and fix the gauge.

Since $\mathcal{P}$ is obtained from T^*N by fixing the values of the momenta as functions of the coordinates, $\mathcal{P}$ may be regarded as (the image of) a global section $s : N \to T^*N$, defined by $s(q^\alpha) = (q^\alpha, \mathcal{A}_\alpha(q))$. s is a diffeomorphism of N onto $\mathcal{P}$, so in the following we can identify these two spaces and use $\{q^\alpha\}$ as coordinates on $\mathcal{P}$. A little calculation in local coordinates shows that the pullback of the canonical symplectic form $\omega = dq^\alpha \wedge dp_\alpha$ under the section s is just the field strength:

$$s^*\omega = -\mathcal{F} . \tag{3.10}$$

Therefore, the symplectic form induced on the constrained subspace $\mathcal{P}$ is degenerate: its kernel are precisely the tangent spaces to the leaves of the foliation defined by $\mathcal{F}$. The reduced phase space $\mathcal{Q}$ is defined to be the space of leaves of this foliation, i.e. the quotient of the constrained space $\mathcal{P}$ under the action of the gauge group $K(\mathcal{F})$.

In general, the reduced phase space will not turn out to be a manifold, but will have singularities. These corresponds to orbits of $K(\mathcal{F})$ in $\mathcal{P}$ which are not diffeomorphic to neighbouring orbits. These problems can be avoided by making some assumptions on N and $\mathcal{F}$. For instance, let $N = S^3$ and $B_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}\mathcal{F}_{\beta\gamma}$ be a nowhere vanishing magnetic field. The lines of force are the leaves of the foliation, because $ker\mathcal{F}$ is spanned by B. They are all diffeomorphic to S^1 and are the fibers of a bundle over $\mathcal{Q} = S^2$. Note that the group $S(\mathcal{F})/K(\mathcal{F})$ acts on $\mathcal{Q}$.

Returning now to Dirac's method, it is clear that choosing a gauge amounts to choosing a section of the natural projection from $\mathcal{P}$ to $\mathcal{Q}$. If the topology of $\mathcal{P}$ is that of a product $\mathcal{Q} \times F$, where F is a typical leaf, this section can be chosen globally and therefore it is legitimate to think of the reduced phase space $\mathcal{Q}$ as a subspace of $\mathcal{P}$ (and hence also of T^*N). However, the projection will in general not admit global sections and thus it will not be possible to choose the gauge globally. In this case it is not appropriate to think of $\mathcal{Q}$ as a subspace of $\mathcal{P}$.

Let $\{q^\alpha\}$ be a local coordinate system on $\mathcal{P} \equiv N$ adapted to the foliation. A local section can be defined by choosing gauge conditions

$$\chi_\alpha = q^\alpha \quad \text{for} \quad \alpha = r+1, \dots, n . \tag{3.11}$$

We have

$$\{C_\alpha, \chi_\beta\} = -\delta_{\alpha\beta} \quad \text{for} \quad \alpha, \beta = r+1, \dots n , \tag{3.12}$$

and therefore the set of all $2n - r$ constraints (3.4) and (3.11) is now of second class and can be imposed strongly. We can use the coordinates $\bar{q}^\alpha \equiv q^\alpha$ with $\alpha = 1, \dots, r$ as local coordinates on the reduced phase space. The Dirac bracket between these variables is

$$\{\bar{q}^\alpha, \bar{q}^\beta\}_* = (\overline{\mathcal{F}}^{-1})^{\alpha\beta} , \tag{3.13}$$

where $\overline{\mathcal{F}}$ is the nondegenerate symplectic form on $\mathcal{Q}$ defined by $\overline{\mathcal{F}}_{\alpha\beta} = \mathcal{F}_{\alpha\beta}$ for $\alpha, \beta \leq r$. By Darboux's theorem there exists another coordinate system on $\mathcal{Q}$ in which the matrix $\overline{\mathcal{F}}_{\alpha\beta}$ is zero except for

$$\overline{\mathcal{F}}_{2i-1,2i} = -\overline{\mathcal{F}}_{2i,2i-1} = 1, \quad \text{for} \quad i = 1, \dots, r/2 .$$

For instance, for $r = 2$, $\overline{\mathcal{F}}_{\alpha\beta} = \varepsilon_{\alpha\beta}$ and from (3.13) the only non-vanishing bracket is $\{\bar{q}^2, \bar{q}^1\}_* = 1$. Thus $\bar{q}^2$ can be interpreted as a local coordinate and $\bar{q}^1$ as a local momentum. In these coordinates $\overline{A}_\alpha = -\frac{1}{2}\varepsilon_{\alpha\beta}\bar{q}^\beta$, so the Lagrangian on the reduced phase space is just

$$\mathcal{L} = -\frac{1}{2}\dot{\bar{q}}^\alpha \varepsilon_{\alpha\beta}\bar{q}^\beta . \tag{3.14}$$

For larger r, the Lagrangian can be written locally as the sum of $r/2$ copies of (3.14).

On the reduced phase space $\mathcal{Q}$, the Hamiltonian is zero. Thus, Hamilton's equations reduce to $\dot{\bar{q}}^\alpha = 0$. Note that this is in agreement with Eq.(3.2), which implies that the gauge degrees of freedom have an arbitrary time evolution, whereas the physical degrees of freedom are time independent.

Recently, theories of the form (3.1) have been considered in Ref. 5, where it is shown that for such models it is not necessary to go through the constraint analysis, as we have done. Let us remark here that while the two procedures yield entirely equivalent results, the one we have followed here is more appropriate when we regard our model as the limit of a theory in which the kinetic term becomes negligible, and gives more information on the global structure of the reduced phase space.

Let us now discuss the realization of the group $S(\mathcal{F})/K(\mathcal{F})$ on the reduced phase space $\mathcal{Q}$. The group $S(\mathcal{F})$ acts on the space $\mathcal{P} = N$. Since $K(\mathcal{F})$ is a normal subgroup of $S(\mathcal{F})$, $S(\mathcal{F})/K(\mathcal{F})$ acts on $\mathcal{P}/K(\mathcal{F}) = \mathcal{Q}$. Every vector field v on N belonging to the Lie algebra of $S(\mathcal{F})/K(\mathcal{F})$ projects on a vector field $\bar{v}$ on $\mathcal{Q}$, whose components in our coordinates are $\bar{v}^\alpha = v^\alpha$, for $\alpha = 1, \dots, r$. This action is generated by the Noether charge

$$Q_v = -\Omega_v , \tag{3.15}$$

where

$$\bar{\partial}_\beta \Omega_v = \bar{v}^\alpha \overline{\mathcal{F}}_{\alpha\beta} . \tag{3.16}$$

Note that Ω_v is constant on the leaves, so Q_v can be regarded as a function on $\mathcal{Q}$. Clearly, the function Ω_v is defined up to a constant. Note that this is the ambiguity discussed in

the end of Sect. 2, since here $m = 1$ and the cohomology group $H^0(N)$ is nontrivial. In order to have a uniquely defined Noether charge, we shall now assume that the functions Ω_v are all normalized so that $\int_\mathcal{Q} \Omega_v d\mu = 0$, where $d\mu$ is some measure on $\mathcal{Q}$ satisfying $\int_\mathcal{Q} d\mu = 1$. The action of the charge (3.15) on the coordinates of $\mathcal{Q}$ is, as expected:

$$\delta\bar{q}^\alpha = \{\bar{q}^\alpha, Q_v\}_* = (\overline{\mathcal{F}}^{-1})^{\beta\gamma} \frac{\partial\bar{q}^\alpha}{\partial\bar{q}^\beta}\frac{\partial Q_v}{\partial\bar{q}^\gamma} = \bar{v}^\alpha . \tag{3.17}$$

The Dirac bracket of two charges is, using (3.16),

$$\{Q_u, Q_v\}_* = (\overline{\mathcal{F}}^{-1})^{\alpha\beta} \frac{\partial Q_u}{\partial\bar{q}^\alpha}\frac{\partial Q_v}{\partial\bar{q}^\beta} = -\overline{\mathcal{F}}(\bar{u}, \bar{v}) . \tag{3.18}$$

On the other hand $\Omega_{[u,v]}$ is defined by the equation $d\Omega_{[u,v]} = i_{[u,v]}\mathcal{F} = \mathcal{L}_u i_v \mathcal{F} = d i_u i_v \mathcal{F}$ and since $\mathcal{F}(u,v) = \overline{\mathcal{F}}(\bar{u}, \bar{v})$, we have

$$\Omega_{[u,v]} = -\overline{\mathcal{F}}(\bar{u}, \bar{v}) + \int_\mathcal{Q} \overline{\mathcal{F}}(\bar{u}, \bar{v}) d\mu . \tag{3.19}$$

Thus

$$\{Q_u, Q_v\}_* = -Q_{[u,v]} - \int_\mathcal{Q} \overline{\mathcal{F}}(\bar{u}, \bar{v}) d\mu , \tag{3.20}$$

and the algebra of the symmetry group is realized with a central extension. Whether the central term is removable or not has to be discussed case by case.

4. THE CASE $m \geq 2$

In order to set up the Hamiltonian formulation of the field theory model we have to assume that $M = \Sigma \times \mathbb{R}$, where Σ is a p-dimensional manifold (with $p = m - 1 \geq 1$) to be interpreted as space, and $\mathbb{R}$ is the time axis. This choice breaks the invariance of the action under $\mathcal{D}iff M$. The residual group is $\mathcal{D}iff\Sigma \times \mathcal{D}iff\mathbb{R}$.

The configuration space of the theory is the space $\Gamma(\Sigma, N)$ of smooth maps $\varphi : \Sigma \to N$, and the phase space is the cotangent bundle $T^*\Gamma(\Sigma, N)$. In describing geometric objects on $\Gamma(\Sigma, N)$ we are going to make use of a coordinate-independent notation, which is explained in the Appendix. We split the coordinates $\{x^i\}$ on M (with $i = 1, \dots, m$) into time t and coordinates $\{x^a\}$ on Σ (with $a = 1, \dots, p$). Then the action (1.1) can be rewritten in the form

$$S = \int dt \int d^p x \; \dot{\varphi}^\alpha(x,t) \mathcal{A}_\alpha(x,t) \tag{4.1}$$

where

$$\mathcal{A}_\alpha = \frac{c}{p!}\varepsilon^{a_1 \dots a_p}\partial_{a_1}\varphi^{\alpha_1} \dots \partial_{a_p}\varphi^{\alpha_p} B_{\alpha\alpha_1 \dots \alpha_p}(\varphi) \tag{4.2}$$

are the local components of a one-form on $\Gamma(\Sigma, N)$. At this point, two observations are in order. First of all, $\mathcal{A}$ is a local form on $\Gamma(\Sigma, N)$, in the sense that its components are functionals which depend only on a finite number of derivatives of φ. Secondly, note that the Lagrangian $L = \int d^px\ \dot{\varphi}^\alpha \mathcal{A}_\alpha = i_{\dot\varphi}\mathcal{A}$ is just the functional of φ obtained by contracting the one-form $\mathcal{A}$ with the vector $\dot\varphi$ tangent to $\Gamma(\Sigma, N)$, exactly as in Eq.(3.1) the Lagrangian $L = \dot q^\alpha \mathcal{A}_\alpha = i_{\dot q}\mathcal{A}$ was the function of q obtained by contracting the one-form $\mathcal{A}$ with the vector $\dot q$ tangent to N. It is therefore immediately clear that except for the replacement of the finite dimensional manifold N with the infinite dimensional manifold $\Gamma(\Sigma, N)$, the dynamical system considered here is identical to the one considered in the previous section.

The Euler–Lagrange equations (2.3) can be rewritten in a form analogous to (3.2):

$$\dot\varphi^\alpha \mathcal{F}_{\alpha\beta} = 0\ , \tag{4.3}$$

where $\mathcal{F} = d\mathcal{A}$ is the field strength, defined as in Eq.(A.8). Explicitly, we have for any pair of vectors v, w tangent to $\Gamma(\Sigma, N)$

$$\mathcal{F}(v,w) = \epsilon \int_\Sigma \varphi^* i_w i_v H = \int d^px \int d^py\ v^\alpha(x) w^\beta(y) \mathcal{F}_{\alpha\beta}(x,y)\ , \tag{4.4}$$

with

$$\mathcal{F}_{\alpha\beta}(x,y) = \mathcal{F}_{\alpha\beta}(x)\ \delta^{(p)}(x-y)\ , \tag{4.5a}$$

$$\mathcal{F}_{\alpha\beta}(x) = \frac{\epsilon}{p!}\ \varepsilon^{a_1\dots a_p}\partial_{a_1}\varphi^{\gamma_1}\dots\partial_{a_p}\varphi^{\gamma_p}\ H_{\alpha\beta\gamma_1\dots\gamma_p}(\varphi)\ . \tag{4.5b}$$

We will now give the Hamiltonian formulation of the model. This had been discussed previously in the presence of the kinetic term[6]. In order to emphasize the parallel with the charged particle case, we will follow closely the steps taken in the previous section. From (4.1), the momentum conjugate to $\varphi^\alpha(x)$ is

$$\pi_\alpha(x) = \mathcal{A}_\alpha(x)\ , \tag{4.6}$$

and therefore we have primary constraints

$$C_\alpha(x) = \pi_\alpha(x) - \mathcal{A}_\alpha(x) \approx 0\ . \tag{4.7}$$

It will be convenient to "smear" the constraints with test functions $v^\alpha(x)$ which are the components of vector fields tangent to $\Gamma(\Sigma, N)$

$$C_v = \int d^px\ v^\alpha(x) C_\alpha(x)\ . \tag{4.8}$$

Then, we find the following algebra

$$\{C_v, C_w\} = \mathcal{F}(v,w)\ . \tag{4.9}$$

The canonical Hamiltonian is identically zero, so the primary Hamiltonian is

$$H_P = \int d^px\ \Lambda^\alpha(x) C_\alpha(x)\ , \tag{4.10}$$

where $\Lambda^\alpha(x)$ are Lagrange multipliers. The preservation in time of the primary constraints yields

$$0 = \dot C_v|_{\mathcal{P}} = \{C_v, H_P\}|_{\mathcal{P}} = \mathcal{F}(v,\Lambda)\ , \tag{4.11}$$

where $\mathcal{P}$ is the subspace of $T^*\Gamma(\Sigma, N)$ defined by the constraints (4.7). Since v is arbitrary, this implies

$$\mathcal{F}_{\alpha\beta}(x)\Lambda^\beta(x) = 0\ . \tag{4.12}$$

One can use this to show that the Hamilton's equations which follow from (4.10) are equivalent to the Euler–Lagrange Eq.(4.3).

It appears from (4.9) that C_v is first class if and only if v belongs to the kernel of $\mathcal{F}$, i.e. $i_v\mathcal{F} = 0$. Note that since $\mathcal{F}$ is a functional of φ, the number of first class constraints will vary on $\Gamma(\Sigma, N)$. Unlike the particle case, where we could assume the rank of $\mathcal{F}$ to be constant, here there will always be φ's for which the rank of $\mathcal{F}$ is less than maximal. For example, if φ is independent from one of the coordinates of Σ, then $\mathcal{F}$ is identically zero. All one can do in this case is to restrict the discussion to a subspace of $\Gamma(\Sigma, N)$ where $\mathcal{F}$ has constant rank.

The first class constraints form a closed algebra and generate the gauge group of the theory, which we shall denote by $\mathcal{G}$. In order to determine the first class constraint, let us write explicitly:

$$(i_v\mathcal{F})_\beta = \frac{\epsilon}{p!}\varepsilon^{a_1\dots a_p} v^\alpha \partial_{a_1}\varphi^{\gamma_1}\dots\partial_{a_p}\varphi^{\gamma_p} H_{\alpha\beta\gamma_1\dots\gamma_p} = 0. \tag{4.13}$$

This equation is solved by vectors v for which $i_v H = 0$ and vectors v which are the images under φ of a vector field ξ on Σ (i.e. $v^\alpha = \xi^a\partial_a\varphi^\alpha$). The former generate the group $K(H)$ and were present already in the particle case, the latter generate the group $\mathcal{D}iff\Sigma$ and have no analog when $p = 0$. In addition there could be field-dependent solutions which do not correspond simply to field-dependent transformations of $K(H)$ or $\mathcal{D}iff\Sigma$. We shall see an example of this in Sect. 5.

In the case $m = n + 1$, $H_{\alpha_1\dots\alpha_n}(\varphi) = \varepsilon_{\alpha_1\dots\alpha_n} f(\varphi)$, with $f \neq 0$. Thus, from Eq.(4.5b) we see that $\mathcal{F}_{\alpha\beta}(x)$ is the determinant of the minor which is obtained from the Jacobian $J_a{}^\gamma(x) = \partial_a\varphi^\gamma(x)$ by deleting the columns α and β. If the rank of $J_a{}^\gamma(x)$ is $< p$, $\mathcal{F}_{\alpha\beta}(x) = 0$. If $J_a{}^\gamma(x)$ has maximal rank p, $\mathcal{F}_{\alpha\beta}(x)$ has exactly p linearly independent null eigenvectors, $v^\alpha \sim \partial_a\varphi^\alpha$, and its rank is $n - p = 2$. If φ is an embedding, the Jacobian J has maximal rank for all x and thus the kernel of $\mathcal{F}$ is spanned by the generators of $\mathcal{D}iff\Sigma$. Therefore, if we restrict our attention to the space $\mathcal{E}mb(\Sigma, N)$ of embeddings (which is dense in $\Gamma(\Sigma, N)$), the gauge group is $\mathcal{G} = \mathcal{D}iff\Sigma$.

Returning to the general case, one may wonder what happens of the vector fields v satisfying equation (2.11), which also generate invariances of the Lagrangian. For these vectors $i_v\mathcal{F} \neq 0$, so they do not belong to $ker\mathcal{F}$ and the corresponding constraints C_v are second class. This means that the group $S(H)/K(H)$ which is generated by these vector fields has to be regarded as a true symmetry of the system, and not as a gauge invariance. This concurs with the observation in Sect. 2 that the generators of $S(H)/K(H)$ have a non-vanishing Noether current, whereas the generators of $K(H)$ have vanishing Noether current.

The kernels of $\mathcal{F}$ define a distribution of (infinite dimensional) linear subspaces $ker\mathcal{F} \subset T\Gamma(\Sigma, N)$. As in the $p = 0$ case, it is easy to show using Fröbenius' theorem that this distribution is integrable. Thus $\Gamma(\Sigma, N)$ is foliated and the leaves of the foliation are just the orbits of the gauge group $\mathcal{G}$ on $\Gamma(\Sigma, N)$.

The constrained subspace $\mathcal{P}$ is the diffeomorphic image of $\Gamma(\Sigma, N)$ under a global section $s : \Gamma(\Sigma, N) \rightarrow T^*\Gamma(\Sigma, N)$, defined by $s(\varphi) = (\varphi, \mathcal{A}(\varphi))$. In the following we shall identify $\Gamma(\Sigma, N)$ with $\mathcal{P}$ using this map, and use $\{\varphi^\alpha(x)\}$ as coordinates on $\mathcal{P}$. The pullback under s of the canonical symplectic form of $T^*\Gamma(\Sigma, N)$, $\omega = \int dx\, d\varphi^\alpha(x) \wedge d\pi_\alpha(x)$, is

$$s^*\omega = -\mathcal{F} \; . \tag{4.14}$$

Therefore, the symplectic form induced on $\mathcal{P}$ is degenerate: its kernel are precisely the orbits of the gauge group $\mathcal{G}$. One defines the reduced phase space to be $\mathcal{Q} = \mathcal{P}/\mathcal{G}$. In general, $\mathcal{Q}$ will not be a manifold. However, note that in the particular case $n = m + 1$, $\mathcal{E}mb(\Sigma, N)/\mathcal{D}iff\Sigma$ is a manifold and is dense in $\mathcal{Q}$.

On $\mathcal{Q}$ there is a nondegenerate symplectic form, whose Poisson brackets are the Dirac brackets to be defined next. Since $\Gamma(\Sigma, N)$ is foliated, there exists locally a coordinate system adapted to the foliation. Let $\hat{\varphi}^\alpha(x)$ be the coordinates in the leaves, and $\bar{\varphi}^\alpha(x)$ be the coordinates which parametrize the different leaves. The space $\mathcal{Q}$ can be locally embedded in $\Gamma(\Sigma, N)$ by choosing gauge conditions analogous to (3.11): $\chi_\alpha(x) = \hat{\varphi}^\alpha(x) \approx 0$. The remaining coordinates $\bar{\varphi}^\alpha(x)$ can be taken as coordinates on $\mathcal{Q}$.

In adapted coordinates the two-form $\mathcal{F}$ has a block-diagonal matrix of components whose entries are zero whenever one of the indices is on the space tangent to the leaves. Let us denote $\overline{\mathcal{F}}_{\alpha\beta}(x, y)$ the invertible submatrix with all indices in the complementary subspace. $\overline{\mathcal{F}}$ is the symplectic form on $\mathcal{Q}$ and the Dirac bracket is

$$\{\bar{\varphi}^\alpha(x), \bar{\varphi}^\beta(y)\}_* = (\overline{\mathcal{F}}^{-1})^{\alpha\beta}(x, y) \; . \tag{4.15}$$

The reduced Hamiltonian vanishes, so the reduced system can be described on $\mathcal{Q}$ by the Lagrangian $\mathcal{L} = \int_\Sigma \bar{\pi}_\alpha(x)\dot{\bar{\varphi}}^\alpha(x) d^p x$. As in the particle case, Hamilton's equations give $\dot{\bar{\varphi}}$. It is important to observe that the variables $\bar{\varphi}$ are related to the original variables φ by a complicated non-local transformation, whose explicit form we do not know in general, but which can be found, in principle, in each specific case.

Finally, let us discuss the realization of the symmetry group $S(H)/K(H)$ on the reduced phase space $\mathcal{Q}$. As mentioned earlier, the vector fields v which generate

$S(H)/K(H)$ are not tangent to the leaves of the foliation. In adapted coordinates we can split $v = \hat{v} + \bar{v}$, where $\hat{v}$ is tangent to the leaves and $\bar{v}$ can be thought of as the projection of v onto $\mathcal{Q}$. An infinitesimal transformation of $S(H)/K(H)$ is of the form $\bar{\varphi}^\alpha(x) \rightarrow \bar{\varphi}^\alpha(x) + \bar{v}^\alpha(x)$. This transformation is generated by the Noether charge

$$Q_v = -c \int_\Sigma \varphi^*\Omega_v = -\frac{c}{p!} \int d^p x\, \varepsilon^{a_1 \dots a_p} \partial_{a_1}\varphi^{\alpha_1} \dots \partial_{a_p}\varphi^{\alpha_p} (\Omega_v)_{\alpha_1 \dots \alpha_p} \; . \tag{4.16}$$

To check this, we note that

$$\frac{\delta Q_v}{\delta\varphi^\gamma(x)} = -\frac{\epsilon}{p!} \epsilon^{a_1 \dots a_p} \partial_{a_1}\varphi^{\alpha_1} \dots \partial_{a_p}\varphi^{\alpha_p} v^\beta H_{\beta\gamma\alpha_1 \dots \alpha_p} = -(i_v\mathcal{F})_\gamma(x) \; ; \tag{4.17}$$

in particular $\frac{\delta Q_v}{\delta\hat{\varphi}^\gamma(x)} = 0$, showing that Q_v can be thought of as a function on $\mathcal{Q}$, and $\frac{\delta Q_v}{\delta\bar{\varphi}^\gamma(x)} = -(i_{\bar{v}}\overline{\mathcal{F}})_\gamma(x)$. Thus

$$\{\bar{\varphi}^\alpha(x), Q_v\}_* = \int d^p y \int d^p z\, (\overline{\mathcal{F}}^{-1})^{\beta\gamma}(y, z) \frac{\delta\bar{\varphi}^\alpha(x)}{\delta\bar{\varphi}^\beta(y)} \frac{\delta Q_v}{\delta\bar{\varphi}^\gamma(z)} = \bar{v}^\alpha(x). \tag{4.18}$$

Using (4.17) one also proves easily that

$$\{Q_u, Q_v\}_* = \bar{\mathcal{F}}(u, \bar{v}). \tag{4.19}$$

On the other hand

$$d\Omega_{[u,v]} = i_{[u,v]}H = \mathcal{L}_u i_v H = d i_u i_v H \tag{4.20}$$

and thus, up to a closed form, $\Omega_{[u,v]} = i_u i_v H$. As observed in Sect. 2, if $H^p(N)$, the p-th cohomology group of N, is trivial, this arbitrariness in Ω_v does not affect the charge Q_v and therefore from (4.20) we conclude that $Q_{[u,v]} = -c \int_\Sigma \varphi^*\Omega_{[u,v]} = -c \int_\Sigma \varphi^* i_u i_v H = \mathcal{F}(u, v) = \overline{\mathcal{F}}(\bar{u}, \bar{v})$. In this case,

$$\{Q_u, Q_v\}_* = \cdot\; Q_{[u,v]}. \tag{4.21}$$

If the p-th cohomology group of N is non-trivial, *one has to choose a normalization for the charges. In analogy to what happens in the particle case, this can lead in general to an extension of the algebra (4.21).*

5. EXAMPLES

In this section we discuss examples of purely WZW actions in 1, 2, 3 and 4 spacetime dimensions. In all examples the form H defines a nontrivial cohomology class and thus the models have interesting topological features.

0+1 dimensions

We consider a point particle moving on $N = \mathbb{R}^3 \backslash \{0\} = S^2 \times \mathbb{R}^+$ in the background of the magnetic field generated by a monopole at the origin. In polar coordinates r, θ, φ the field strength is $\mathcal{F}_{\theta\varphi} = g \sin\theta$, where g is the monopole charge, and the magnetic potential is $\mathcal{A}_r = \mathcal{A}_\theta = 0$, $\mathcal{A}_\varphi = g(1 - \cos\theta)$, in a gauge in which the Dirac string points along the negative z axis. The equation $i_v \mathcal{F} = 0$ is solved by all radial vector fields ($v^\theta = v^\varphi = 0$). Therefore, the leaves of the foliation $\ker\mathcal{F}$ are the radii and the gauge group consists of transformations of the form $(r, \theta, \varphi) \mapsto (r', \theta, \varphi)$, with $r' = r'(r, \theta, \varphi)$. Thus $K(\mathcal{F}) = \Gamma(S^2, \mathcal{D}iff\,\mathbb{R}^+)$. The equation $\mathcal{L}_v \mathcal{F} = 0$, when written out explicitly, yields

$$\partial_r v^\theta = 0 \ , \tag{5.1a}$$

$$\partial_r v^\varphi = 0 \ , \tag{5.1b}$$

$$\partial_\theta(\sin\theta\ v^\theta) + \sin\theta\ \partial_\varphi v^\varphi = 0 \ . \tag{5.1c}$$

The radial component of v remains undetermined, and the projection of v onto S^2 must satisfy (5.1c). We note that the solutions to (5.1c) are the divergence-free vector fields (with respect to the standard metric on S^2). These generate the group $\mathcal{SD}iff S^2$ of volume-preserving diffeomorphisms of S^2. It follows that the group $S(\mathcal{F})$ is the semidirect product of $K(\mathcal{F})$ and $S(\mathcal{F})/K(\mathcal{F}) = \mathcal{SD}iff S^2$.

The Euler–Lagrange equations (3.2) imply $\dot\theta = 0$, $\dot\varphi = 0$, while $\dot r$ remains undetermined. However, r is the gauge degree of freedom in this model, so the equations imply that all physical degrees of freedom remain constant in time.

From the Hamiltonian point of view, there are two second class constraints C_θ, C_φ and one first class constraint C_r. The reduced phase space is $\mathcal{Q} = N/K(\mathcal{F}) = S^2$. Note that since $N = S^2 \times \mathbb{R}^+$, a global choice of gauge is possible and therefore one can think of $\mathcal{Q}$ as a submanifold of N. In fact, we can choose $\chi_r = r$ as a gauge condition to match C_r. The reduced Hamiltonian on $\mathcal{Q} = S^2$ vanishes, and the Dirac bracket is

$$\{\theta, \varphi\}_* = -\frac{1}{g \sin\theta} \ . \tag{5.2}$$

The symmetry group $S(\mathcal{F})/K(\mathcal{F}) = \mathcal{SD}iff S^2$ acts on the reduced phase space S^2. The Noether charge is

$$Q_v = -\Omega_v \ , \tag{5.3}$$

where

$$v^\theta = \frac{\partial_\varphi \Omega_v}{g \sin\theta} \ , \qquad v^\varphi = -\frac{\partial_\theta \Omega_v}{g \sin\theta} \ , \tag{5.4}$$

and Ω_v is an arbitrary function on S^2 normalized so that $\frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta\ \Omega_v = 0$. The algebra of the Noether charges is given by Eq.(3.20). However, using Eq.(5.4) it can be seen that in this particular case the central term vanishes. This agrees with the results of Ref. 7.

1 + 1 dimensions

Our next example is a non-linear sigma model with values in $N = SU(2)$ on $M = S^1 \times \mathbb{R}$. Let L_a^α be the components of a set of left-invariant vector fields on $SU(2)$ satisfying the algebra

$$[L_a, L_b] = \varepsilon_{abc} L_c \tag{5.6}$$

and L_α^a be the inverse matrix, i.e. the components of the left-invariant Maurer–Cartan form. We take

$$H_{\alpha\beta\gamma} = -\frac{1}{4}\varepsilon_{abc} L_\alpha^a L_\beta^b L_\gamma^c \ . \tag{5.7}$$

In this case Eq.(4.13), which determines the first class constraints, becomes

$$[v, W_1] = 0 \ , \tag{5.8}$$

where $v^a = v^\alpha L_\alpha^a$, $W_1^a = \partial_1 \varphi^\alpha L_\alpha^a$ and the bracket is the commutator in the Lie algebra of $SU(2)$. If $W_1(x) \neq 0$ for all x (which is the case when φ is an embedding), the only solutions to this equation are vectors v proportional to W_1. These generate the gauge group $\mathcal{G} = \mathcal{D}iff S^1$. (In particular the group $K(H)$ is trivial.) The symmetry group is $S(H) = \mathcal{SD}iff SU(2)$, the volume-preserving diffeomorphisms of $SU(2)$, and is generated by all vector fields which are divergence-free with respect to the invariant metric $h_{\alpha\beta} = L_\alpha^a L_\beta^b \delta_{ab}$. According to the discussion in Sect. 4, the reduced phase space of the model contains $\mathcal{E}mb(S^1, SU(2))/\mathcal{D}iff S^1$ as a dense subspace.

The Noether charge is $Q_v = -c \int dx^1 \partial_1 \varphi^\alpha (\Omega_v)_\alpha$, where

$$v^\alpha = 16\ H^{\alpha\beta\gamma} \partial_{[\beta} (\Omega_v)_{\gamma]} \ ; \tag{5.9}$$

the indices on H have been raised using $h_{\alpha\beta}$. Note that the divergence-free vector fields on $SU(2)$ are in bijective correspondence with one–forms modulo exact forms. Since the cohomology group $H^1(SU(2)) = 0$, the algebra of the generators of the symmetry group $\mathcal{SD}iff S^2$ is as in (4.21), without extensions.

A more complicated example occurs when N is an arbitrary semisimple Lie group G with structure constants f_{abc}, and

$$H_{\alpha\beta\gamma} = \frac{1}{4} f_{abc} L_\alpha^a L_\beta^b L_\gamma^c \ . \tag{5.10}$$

In this case equation (4.13) can be written again in the form (5.8). For generic φ, W_1 will be a regular element of the Lie algebra, and therefore belongs to a Cartan subalgebra. The solutions of (5.8) are given by all v's belonging to the Cartan subalgebra containing W_1; if G has rank r, the space of solutions is r-dimensional. Solutions proportional to W_1 correspond again to diffeomorphisms of S^1. Note that again the group $K(H)$ is trivial, since no v commutes with all vectors in the algebra. In the neighborhood of a generic point, the reduced phase space will be parametrized by $n-r$ functions. It is interesting to observe that this is also the number of degrees of freedom of the gauge field in an anomalous gauge theory in two dimensions[8].

The group $S(H)$ is generated by those vector fields which satisfy Eq.(2.11): $v^\alpha H_{\alpha\beta\gamma} = 2\,\partial_{[\beta}(\Omega_v)_{\gamma]}$. Since $f_{abc}f_{dbc} \propto \delta_{ad}$, this can be inverted giving a relation analogous to Eq.(5.9). Therefore also here the generators of $S(H)$ are in bijective correspondence with all one-forms on G modulo exact forms. If the cohomology group $H^1(G)=0$, the algebra of the charges Q_v is as in (4.21), without extensions.

2 + 1 dimensions

Let us consider first the case when $M = S^2 \times \mathbb{R}$, $N = S^4$ and H is a volume form. For generic φ, the gauge group is $\mathcal{D}iff S^2$, and the reduced phase space Q contains $\mathcal{E}mb(S^2,S^4)/\mathcal{D}iff S^2$ as a dense subspace. The symmetry group is $S(H) = \mathcal{SD}iff S^4$. This model is related to the $SU(2)$–Yang–Mills theory considered in Ref. 1. In fact, let $A_\alpha(y)$ be the fundamental, $O(5)$–invariant instanton solution on S^4. Then given the map $\varphi : M \to S^4$ one can construct the pullback $A_i = \partial_i\varphi^\alpha A_\alpha(\varphi)$ and take the following action for φ:

$$S = \frac{\epsilon}{3!}\int d^3x\, \varepsilon^{ijk}\,\mathrm{Tr}(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k). \qquad (5.11)$$

This is equal to the action (1.1), when B is the Chern–Simons form constructed with the instanton of S^4

$$B_{\alpha\beta\gamma} = \mathrm{Tr}(A_{[\alpha}\partial_\beta A_{\gamma]} + \frac{2}{3}A_{[\alpha}A_\beta A_{\gamma]}) \;. \qquad (5.12)$$

The curl of this form is $\mathrm{Tr}F_{[\alpha\beta}F_{\gamma\delta]}$ and for the instanton it is equal to the $O(5)$–invariant volume element on S^4. Denoting by $g_{\alpha\beta}$ the $O(5)$–invariant metric, we have

$$H_{\alpha\beta\gamma\delta} = \mathrm{Tr}\ F_{[\alpha\beta}F_{\gamma\delta]} = \sqrt{g}\,\epsilon_{\alpha\beta\gamma\delta} \;. \qquad (5.13)$$

The theory discussed in Ref. 1 has exactly the action (5.11), but with A_i an independent dynamical variable, not a composite of the scalar φ. The equations of motion that one obtains from (5.11) are in this case:

$$F_{ij} = 0 \;. \qquad (5.14)$$

This should be compared with the equation (2.3), with $H_{\alpha\beta\gamma\delta} = \mathrm{Tr}\ F_{[\alpha\beta}F_{\gamma\delta]}$, which is obtained by treating A_i as a composite field. Clearly, if $F_{ij}=0$, also (2.3) is satisfied.

This was to be expected, since every variation of φ gives rise to a variation of A_i, but not every variation of A_i can be obtained from a variation of φ, and thus the equations of motion (5.14) considered in Ref. 1 are stronger than those of the sigma model.

One can construct a more complicated sigma model whose equations of motion are equivalent to Eq.(5.4). Namely, replace $N = S^4 = H\mathbb{P}^1$ by $N = H\mathbb{P}^m$, the quaternionic projective m–space. By the theorem of Narasimhan and Ramanan[9], every $Sp(1) = SU(2)$ gauge field on M can be obtained by pulling back the canonical $Sp(1)$-connection A_α on $H\mathbb{P}^m$ when m is sufficiently large. Therefore, in such a model, the equation of motion (2.3) becomes equivalent to the equation

$$F_{ij} \equiv \partial_i\varphi^\alpha\partial_j\varphi^\beta F_{\alpha\beta}(\varphi) = 0 \;. \qquad (5.14)$$

3+1 dimensions

Our last example is an $SU(3)$-valued non-linear sigma model on $M = \mathbb{R}\times S^3$. In the same notation of Eq. (5.10), we take

$$H_{\alpha\beta\gamma\delta\epsilon} = iL^a_\alpha L^b_\beta L^c_\gamma L^d_\delta L^e_\epsilon B_{[abcde]}, \qquad (5.15)$$

where $B_{abcde} = \mathrm{Tr}(T_aT_bT_cT_dT_e)$. The form $B_{[abcde]}$ on the Lie algebra of $SU(3)$ is dual to the form f_{abc} defined by the structure constants. It satisfies the condition $B_{[abcde]}B_{[fbcde]} = c\delta_{af}$, with c some constant. Therefore, the equation $v^\alpha H_{\alpha\beta\gamma\delta\epsilon} = 4\partial_{[\beta}(\Omega_v)_{\gamma\delta\epsilon]}$ can be inverted:

$$v^\alpha = -\frac{4}{c}\,H^{\alpha\beta\gamma\delta\epsilon}\partial_{[\beta}(\Omega_v)_{\gamma\delta\epsilon]}\;; \qquad (5.16)$$

the indices on H have been raised by means of the invariant metric $h_{\alpha\beta} = L^a_\alpha L^b_\beta\delta_{ab}$. The group $K(H)$ is trivial and the group $S(H)$ is generated by vector fields of the form (5.16). These are in bijective correspondence with the three forms modulo closed three forms. Since the cohomology group $H^3(SU(3)) = \mathbb{R}$, the Noether charge (4.16) is defined up to an arbitrary constant. As noted in general at the end of Sect. 4, this may lead to an extension of the algebra (4.21).

APPENDIX

Differential geometry of spaces of maps

A vector tangent to $\Gamma(M,N)$ at a point φ is a map v from M to TN which projects onto φ, i.e. for each x in M, $v(x)$ is a vector tangent to N at $\varphi(x)$. This conforms to the intuitive idea of v being an infinitesimal deformation of φ. Given local coordinates on N, one can write

$$v = \int dx\ v^\alpha(x)\frac{\delta}{\delta\varphi^\alpha(x)}\ , \tag{A.1}$$

where $\frac{\delta}{\delta\varphi^\alpha(x)}$ are the basis vectors. A one-form on $\Gamma(M,N)$ at φ is a linear map from the tangent space $T_\varphi\Gamma(M,N)$ to the reals. In local coordinates,

$$\omega = \int dx\ \omega_\alpha(x)\delta\varphi^\alpha(x) \tag{A.2}$$

and the value of the form ω on the vector v is

$$\omega(v) \equiv i_v\omega = \int dx\ \omega_\alpha(x)v^\alpha(x). \tag{A.3}$$

Similarly, a k-form can be written as

$$\omega = \frac{1}{k!}\int dx^1\dots\int dx^k\ \omega(x^1,\dots,x^k)_{\alpha_1\dots\alpha_k}\delta\varphi^{\alpha_1}(x^1)\wedge\dots\wedge\delta\varphi^{\alpha_k}(x^k) \tag{A.4}$$

and the contraction of the k-form ω with a vector v is given by

$$i_v\omega = \frac{1}{(k-1)!}\int dx^1\dots\int dx^k\ v^{\alpha_1}(x^1)\omega_{\alpha_1\dots\alpha_k}(x^1,\dots,x^k)\delta\varphi^{\alpha_2}(x^2)\wedge\dots\wedge\delta\varphi^{\alpha_k}(x^k). \tag{A.5}$$

A vector field v on $\Gamma(M,N)$ is the assignment of a tangent vector $v(\varphi)$ to each point φ in $\Gamma(M,N)$. A tangent vector field can be written again as in (A.1), but now with the components $v^\alpha(x)$ being also functionals of φ: $v^\alpha(x) = (v(\varphi))^\alpha(x)$. Similarly, we can define a field of forms. The Lie bracket of two vector fields v and w is given by

$$[v,w] = \int dx\int dy\left(v^\alpha(x)\frac{\delta w^\beta(y)}{\delta\varphi^\alpha(x)} - w^\alpha(x)\frac{\delta v^\beta(y)}{\delta\varphi^\alpha(x)}\right)\frac{\delta}{\delta\varphi^\beta(y)} \tag{A.6}$$

and the exterior derivative of a k-form ω is defined by

$$\begin{aligned} d\omega(v_1,\dots v_{k+1}) = & \sum_i(-)^{i+1}v_i(\omega(v_1,\dots,\hat{v}_i,\dots v_{k+1})) \\ & -\sum_{i<j}(-)^{i+j}\omega([v_i,v_j],v_1,\dots\hat{v}_i\dots,\hat{v}_j,\dots v_{k+1}), \end{aligned} \tag{A.7}$$

where v_i on the r.h.s. has to be regarded as a directional derivative acting on the function in brackets, and the hat denotes that one argument is missing. In particular, if ω is a one-form,

$$\begin{aligned} d\omega(v,w) &= v(\omega(w)) - w(\omega(v)) - \omega([v,w]) \\ &= \int dx\int dy\ v^\alpha(x)w^\beta(y)\left(\frac{\delta\omega_\beta(y)}{\delta\varphi^\alpha(x)} - \frac{\delta\omega_\alpha(x)}{\delta\varphi^\beta(y)}\right)\ . \end{aligned} \tag{A.8}$$

Most of the standard formulae of differential geometry can be generalized to infinite dimensions. In particular, Eqs.(1.2) and (1.3) hold also in $\Gamma(M,N)$.

The groups $\mathcal{D}iffM$ and $\mathcal{D}iffN$ act on $\Gamma(M,N)$ as in Eq.(2.12):

$$\varphi\mapsto\varphi' = h^{-1}\circ\varphi\circ f, \tag{A.9}$$

where $f\in\mathcal{D}iffM$ and $h\in\mathcal{D}iffN$. Infinitesimally, vector fields on M and N give rise to vector fields on $\Gamma(M,N)$. If $\xi = \xi^i\partial_i$ is a vector field on M, we define a vector field $\tilde{\xi}$ on $\Gamma(M,N)$ by

$$(\tilde{\xi}(\varphi))(x) = (T\varphi(\xi))(x). \tag{A.10}$$

In local coordinates the components of $\tilde{\xi}$ are $(\tilde{\xi}(\varphi))^\alpha(x) = \xi^i(x)\partial_i\varphi^\alpha$. If $v = v^\alpha\partial_\alpha$ is a vector field on N, we define a vector field $\bar{v}$ on $\Gamma(M,N)$ by

$$(\bar{v}(\varphi))(x) = v(\varphi(x)). \tag{A.11}$$

In local coordinates, the components of $\bar{v}$ are $(\bar{v}(\varphi))^\alpha(x) = v^\alpha(\varphi(x))$. It is easy to show that

$$[\tilde{\xi}_1,\tilde{\xi}_2] = \widetilde{[\xi_1,\xi_2]}\ , \tag{A.12a}$$

$$[\tilde{\xi},\bar{v}] = 0\ , \tag{A.12b}$$

$$[\bar{v}_1,\bar{v}_2] = \overline{[v_1,v_2]}\ . \tag{A.12c}$$

Thus the maps $\xi\mapsto\tilde{\xi}$ and $v\mapsto\bar{v}$ are homomorphisms from the algebras of vector fields on M and N to the algebra of vector fields on $\Gamma(M,N)$.

REFERENCES

1. E. Witten, "Quantum field theory and the Jones polynomial", in the Proceedings of the IAMP Congress, Swansea, 1988, Princeton-preprint (IASSNS-HEP-88/83), 1988.

2. E. Witten, "2+1 dimensional gravity as an exactly soluble model", Princeton preprint, 1988.

3. L.D. Faddeev and S.L. Shatashvili, *Phys. Lett.* **167 B**, 225, (1986).

4. A. Stern, "Current algebra from pure Wess-Zumino actions", University of Alabama-preprint (UAHEP 884), 1988.

5. L.D. Faddeev and R. Jackiw, *Phys. Rev. Lett.* **60**, 1692, (1988).

6. T.R. Ramadas, *Comm. Math. Phys.* **93**, 355, (1984).
   J. Wudka, *Nucl. Phys.* **B 288**, 649, (1987).

7. I. Bars, C. Pope and E. Sezgin, *Phys. Lett.* **B210**, 85 (1988).

8. J. Lott and R. Rajaraman, *Phys. Lett.* **165B**, 321, (1985).

9. M.S. Narasimhan and S. Ramanan, *Am. J. of Math.* **83**, 563, (1961).

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