



THE LIGHT-CONE GAUGE AND THE PRINCIPAL VALUE
PRESCRIPTION

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ABSTRACT

The principal value prescription is used to treat the unphysical pole $(K.n)^{-1}$ in the basic one-loop light-cone integral. It is shown that the prescription is well suited to such a task, contrary to what has been previously thought till now.

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1. INTRODUCTION

The first time modern theoretical physicists were introduced to the light-cone formulation for quantum field theories, goes as far back as 1949, when Dirac^[1] discussed the front form of relativistic dynamics. For the next two decades or so since its debut, interest in this subject lay dormant, with only few sporadic appearances in the literature². These focused not on the gauge *per se* but rather on investigating the structure of relativistic theories in the light-cone frame^[3]. It was only during the seventies that a handful of researchers started venturing to take a fresh look at the gauge proper^[4].

In the beginning of this decade, the light-cone gauge was shown to be especially appropriate to the computation of quantum effects contributing to the leading logarithm approximation in deep inelastic processes^[5]. However, as much as this feature was attractive and desirable, hand in hand with it there emerged an ominous one: Feynman amplitudes at the one-loop level exhibited double-pole singularities^[6]. This idiosyncrasy has caused many to consider the light-cone gauge with reserve, if not suspicion.

In tracing back such a pathological behaviour, this has always been ascribed to the Principal Value (PV) prescription employed to treat the unphysical poles $(k \cdot n)^{-1}$ of the gauge boson propagator. (Direct application of dimensional regularization to one-loop light-cone integrals do also yield double poles^[7].)

Therefore, alternative prescriptions to handle the

² For a very nice historical survey we refer the reader to a well written review article on non-covariant gauges by G. Leiberman^[2].

$(k \cdot n)^{-1}$ singularities in the light-cone gauge were searched for. Mandelstam⁽⁸⁾, and Leibbrandt⁽⁹⁾ authored two seemingly different prescriptions independently; later, Lee and Milgram⁽¹⁰⁾ showed that they were, in fact, equivalent. These new prescriptions were devised in such a way so as to ensure that the location of the $(k \cdot n)^{-1}$ poles in the k^0 -complex plane would not hinder Wick rotation nor spoil power counting thereby: qualities which, as is often claimed, were missing in the PV prescription.

Moreover, Basselo et al⁽¹¹⁾ demonstrated that the Leibbrandt prescription could be recovered via canonical quantization, and more recently, within the framework of functional path integrals, Slavnov and Frolov⁽¹²⁾ reproduced a similar result.

Successful one- and two-loop calculations employing either of those new prescriptions have already come out of the press⁽¹³⁾, enhancing their credibility as well as their suitability to handle the troublesome $(k \cdot n)^{-1}$ factors in the light-cone gauge.

So, with all this, the role of the PV prescription in this particular gauge waned, while the prescription itself was left to sink into oblivion. To add insult to injury, it went carrying an unwelcome reputation of being the prime culprit for all the unwieldy pathologies manifested at its implementation.

However, the fascinating story of such an ugly duckling in the realm of light-cone gauge is not over yet, we believe. The starting point for its full-fledged come back we trace to the above quoted paper by Slavnov and Frolov. In it the authors came up with another distinct set of propagator for the Yang-Mills

field in the light-cone gauge, which, though not quite the same, exhibits a structure that resembles the propagator in the PV prescription. Novel and noteworthy, this result strongly suggested us to take a second look at the ill-favoured prescription. The subsequent discovery, detailed in the following section, is simple, but a fascinating one nonetheless. It clearly shows that the deficiency is not in the prescription *per se*, but rather in its mistaken implementation, which used to overlook basic first principles. In other words, we will understand why the unwieldy higher order poles that plagued earlier calculations of quantum effects in this gauge are physically unacceptable.

2. PRELIMINARIES

At this point we introduce our notation and convention. Throughout this article, the signature for the Minkowski (flat) space-time metric is $(+,-,-,-)$, with x^0 being the time-component of a general four-vector x^μ . This is written in light-cone components as

$$x^\mu = (x^+, x^-, x^1) \quad ; \quad i = 1, 2, \quad (2.1)$$

where

$$x^\pm = \frac{x^0 \pm x^3}{\sqrt{2}} = \frac{x_0 \mp x_3}{\sqrt{2}} = x_\mp, \quad (2.2)$$

and

$$x^i = (\hat{Q})^i = (x^1, x^2). \quad (2.3)$$

It is convenient to define the light-like vectors n_μ and \bar{n}_μ with cartesian components

$$n_\mu = \frac{1}{\sqrt{2}} (1, 0, 0, 1) \quad , \quad (2.4)$$

and

$$\bar{n}_\mu = \frac{1}{\sqrt{2}} (1, 0, 0, -1) \quad . \quad (2.5)$$

These therefore satisfy

$$n^2 = \bar{n}^2 = 0 \quad , \quad (2.6)$$

and

$$n^\mu \bar{n}_\mu = 1 \quad , \quad (2.7)$$

while the "plus" and "minus" components are conveniently reproduced now by

$$x^+ = x^\mu n_\mu \quad , \quad (2.8)$$

and

$$x^- = x^\mu \bar{n}_\mu \quad . \quad (2.9)$$

The scalar product between two vectors becomes

$$x^\mu y_\mu (\equiv x \cdot y) = x^+ y^- + x^- y^+ - x^i y^i = x^+ y^- + x^- y^+ - \hat{x} \cdot \hat{y} \quad . (2.10)$$

In particular,

$$x^2 = 2x^+ x^- - \hat{x}^2 \quad . \quad (2.11)$$

The volume element is written

$$d^4 x = d^2 \hat{x} dx^- dx^+ \quad , \quad (2.12)$$

and dimensional regularization is to be implemented by continuing the $\mu = 1, 2$ components to $D-2$ dimensions, i.e.,

$$d^D x = d^{D-2} \hat{x} dx^- dx^+ \quad . \quad (2.13)$$

We also employ, according to convenience, the definition

$$D-2 = 2(w-1) \quad . \quad (2.14)$$

For a vector gauge field A_μ , the light-cone gauge is defined by

$$n^\mu A_\mu (\equiv n \cdot A \equiv A^\nu) = 0 \quad ; \quad n^2 = 0 \quad . \quad (2.15)$$

where n^μ is an arbitrary light-like constant vector.

Formally inverting the "regularized" quadratic form in the Lagrangian density, the propagator in momentum space reads

$$G_{\mu\nu}(k) = \frac{-1}{(k^2 + i\epsilon)} \left\{ g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{(k \cdot n)} + \frac{(k^2 + n^2) k_\mu k_\nu}{(k \cdot n)^2} \right\} .$$

Letting the gauge parameter α to vanish and using (2.6), the effective propagator in the light-cone gauge becomes

$$G_{\mu\nu}(k) = \frac{-1}{(k^2 + i\epsilon)} \left\{ g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{(k \cdot n)} \right\} \quad ; \quad \epsilon > 0 \quad . \quad (2.16)$$

where the unphysical singularity $(k \cdot n)^{-1} = (k^+)^{-1}$ has yet to be defined.

In what follows, our chief concern will be the evaluation of the basic one-loop light-cone integral ($dk \equiv d^D k$)

$$\frac{(\mu)^{4-2\omega}}{(2\pi)^{2\omega}} \int \frac{dk}{(k^2 + i\epsilon) [(k-p)^2 + i\epsilon] k^+} ,$$

where μ is the mass scale produced by the renormalization procedure and p is the external momentum. For convenience, we leave the factor $\mu^4 (2\pi\mu)^{-2\omega}$ out and focus our attention on the

Integral

$$I = \int \frac{dk}{(k^2 + i\epsilon) [(k-p)^2 + i\epsilon] k^{\omega}} \quad (2.17)$$

Until 1982, the only known way to treat the pole at $k^{\omega} = 0$ was to do it in the sense of Cauchy principal-value, i.e.,

$$\frac{1}{k^{\omega}} \rightarrow P \frac{1}{k^{\omega}} = \lim_{\zeta \rightarrow 0} \frac{1}{2} \left\{ \frac{1}{k^{\omega} + i\zeta} + \frac{1}{k^{\omega} - i\zeta} \right\} ; \quad \zeta > 0 \quad (2.18)$$

Using (2.18), Capper et al.^[14] arrived at the following result for (2.17):

$$I = i (-\pi)^{\omega} \frac{(p^2)^{\omega-2}}{p^{\omega}} \frac{1}{(\omega-2)} \frac{\Gamma(2-\omega) \Gamma(\omega-1) \Gamma(\omega-1)}{\Gamma(2\omega-3)} \quad (2.19)$$

where the double pole for $\omega \rightarrow 2$ is clearly seen. This result is physically unacceptable.

On the other hand, by using the Mandelstam prescription, that is, making the substitution

$$\frac{1}{k^{\omega}} \rightarrow \lim_{\zeta \rightarrow 0} \frac{1}{k^{\omega} + i\zeta k^{-\omega}} ; \quad \zeta > 0 \quad (2.20)$$

the same authors quoted last showed that

$$1 = 1 (-\pi)^{\omega} \frac{(p^{\pm})^{\omega-2}}{p^{\pm}} \Gamma(2-\omega) \left\{ \frac{\Gamma(\omega-2) \Gamma(\omega-1)}{\Gamma(2\omega-3)} - \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(2-\omega+l)}{\Gamma(\omega-2+l) \Gamma(2-\omega)} \left(\frac{p_{\pm}^2}{p^{\pm}} \right)^l \right\} \quad (2.21)$$

which, for $\omega \rightarrow 2$ yields

$$1 = \frac{1}{p^{\pm}} \left\{ \frac{\pi^2}{6} - S(\lambda) \right\} + O(2-\omega) \quad (2.22)$$

where $S(\lambda)$ is the Spence (or "dilogarithm") integral

$$S(\lambda) = - \int_0^1 dx x^{-1} \ln(1-\lambda x) \quad (2.23)$$

with

$$\lambda = - \frac{p_{\pm}^2}{p^{\pm}} \quad (2.24)$$

Note that (2.22) is finite, so that it is in harmony with the naive power counting assessment done on (2.17).

From the work of Lee and Milgram^[10], we know that the Leibbrandt prescription, namely,

$$\frac{1}{k^{\pm}} \rightarrow \lim_{\zeta \rightarrow 0} \frac{k^{\pm}}{k^{\pm} k^{\mp} + 1\zeta} \quad ; \quad \zeta > 0 \quad (2.25)$$

when implemented in (2.17), does also yield the same result as

that obtained through the Mandelstam prescription.

The status quo for the main properties of both prescriptions (2.20) and (2.25) can be summarized as follows^[15],

- (a) Poles at $k^{\dagger}=0$ do not hinder Wick rotation;
- (b) Degree of divergence of momentum integrals can be assessed by naive power counting;
- (c) Evaluated momentum integrals exhibit divergent parts that are local functions of the external momentum;
- (d) There are no higher order divergences at the one loop level;
- (e) Ward and BRS identities are satisfied;
- (f) Integrals are internally consistent.

3. THE PV PRESCRIPTION REVISITED

Employing the standard trick of exponentiating propagators, (2.17) becomes

$$1 = - \int_0^\infty d\alpha \int_0^\infty d\beta e^{i\beta p^2} M_{LC}(\alpha, \beta) \quad (3.1)$$

where $M_{LC}(\alpha, \beta)$ stands for the light-cone momentum integral

$$M_{LC}(\alpha, \beta) \equiv \int \frac{dk}{k^+} e^{i((\alpha+\beta)k^2 - 2\beta p \cdot k)} \quad (3.2)$$

For convenience, we define

$$x \equiv (\alpha + \beta) \quad (3.3a)$$

$$\beta p \equiv -xQ \quad (3.3b)$$

$$\beta p^2 \equiv xQ \quad (3.3c)$$

With these definitions, and using (2.10)-(2.12), the momentum integral (3.2) becomes

$$\begin{aligned} M_{LC}(\alpha, \beta) &= \int \frac{dk}{k^+} e^{ix(k^2 + 2k \cdot Q)} \\ &= \int d^{D-2}k e^{-ix(k^2 + 2k \cdot Q)} J_{LC} \\ &= \left[-\frac{i\pi}{x} \right]^{D-1} e^{ixQ^2} J_{LC} \quad (3.4) \end{aligned}$$

where the standard Gaussian integral in $D/2$ ($D=31, 19, 20$) dimensional Euclidean space has been performed and J_{LC} is the integral over k^+ and k^- :

$$J_{LC} \equiv \int_{-\infty}^{+\infty} dk^- \cdot 2ixQ^+ k^- \int_{-\infty}^{+\infty} dk^+ (k^+)^{-1} \cdot 2ixk^+(k^-+Q^-) \quad (3.5)$$

Let us consider the integral over k^+ :

$$K_{LC} \equiv \int_{-\infty}^{+\infty} dk^+ (k^+)^{-1} \cdot 2ixk^+(k^-+Q^-) \quad , \quad (3.6)$$

which, in the PV sense is understood as (see (2.1b))

$$K_{LC} = \frac{1}{2} \lim_{\zeta \rightarrow 0} \left\{ K^+(\zeta) + K^-(\zeta) \right\} \quad ; \quad \zeta > 0 \quad , \quad (3.7)$$

with

$$K^+(\zeta) \equiv \int_{-\infty}^{+\infty} \frac{dk^+}{(k^+ + i\zeta)} \cdot 2ixk^+(k^-+Q^-) \quad , \quad (3.8)$$

and

$$K^-(\zeta) \equiv \int_{-\infty}^{+\infty} \frac{dk^+}{(k^+ - i\zeta)} \cdot 2ixk^+(k^-+Q^-) \quad . \quad (3.9)$$

We note from (3.1) and (3.3a) that x is strictly positive ($x > 0$) while the factor $(k^- + Q^-)$ can either be positive or negative. Therefore, in applying the residue theorem to evaluate (3.6) and (3.9), we have the following cases to consider.

(i) $(k^- + Q^-) < 0$ (or $k^- < -Q^-$): In this case, contour C in the complex k^+ -plane must close in the lower half plane to ensure convergence at infinity. This contour encloses the simple pole $k^+ = -1\zeta$ of (3.8), whose residue is

$$\text{Res}_C(-1\zeta) = e^{-2\pi(k^- + Q^-)\zeta} \quad , \quad (3.10)$$

so that

$$K_C^+(\zeta) = -2\pi i e^{-2\pi(k^- + Q^-)\zeta} \quad ; \quad k^- < -Q^- \quad , \quad (3.11)$$

while the pole at $k^+ = 1\zeta$ of (3.9) is outside C and its residue is zero. So

$$K_C^-(\zeta) = 0 \quad . \quad (3.12)$$

(ii) $(k^- + Q^-) > 0$ (or $k^- > -Q^-$): For this case, contour C' must close in the upper half of the complex k^+ -plane. Now the pole at $k^+ = -1\zeta$ of (3.8) is outside C' and the corresponding residue vanishes. Therefore,

$$K_{C'}^+(\zeta) = 0 \quad , \quad (3.13)$$

while the pole at $k^+ = 1\zeta$ of (3.9) contributes a residue of

$$\text{Res}_{C'}(1\zeta) = e^{-2\pi(k^- + Q^-)\zeta} \quad , \quad (3.14)$$

so that

$$K_{C'}^-(\zeta) = 2\pi i e^{-2\pi(k^- + Q^-)\zeta} \quad ; \quad k^- > -Q^- \quad . \quad (3.15)$$

Substituting (3.11) and (3.15) into (3.7) and inserting the result into (3.5) we obtain

$$J_{LC} = -i\pi \left\{ \int_{-\infty}^{-Q^-} dk^- \cdot 2ix(Q^- - i\epsilon)k^- - \int_{-Q^-}^{+\infty} dk^- \cdot 2ix(Q^- + i\epsilon)k^- \right\} \quad (3.16)$$

Note that the upper (lower) limit of integration over k^- in the first (second) integral of (3.16) is a direct consequence of cases (i) and (ii) considered above. Straightforward integration over k^- (with $\epsilon=0$) and further integrations over x and y according to the standard change of variables

$$\alpha = xy \quad , \quad (3.17a)$$

$$\beta = x(1-y) \quad , \quad (3.17b)$$

(with (3.3a) satisfied), will lead us to the physically unacceptable result (2.10).

Here comes our crucial argument: We cannot and must not overlook the pole generated by the covariant piece of the propagator, namely, $k^2 + i\epsilon$, $\epsilon > 0$, which, using (2.11) reads

$$k^2 + i\epsilon = 2k^+k^- - k^{\perp 2} + i\epsilon \quad ; \quad \epsilon > 0 \quad . \quad (3.18)$$

This shows that there is a simple pole at

$$k^+ = \frac{k^2}{2k^-} - \frac{1\epsilon}{2k^-} \quad ; \quad \epsilon > 0 \quad . \quad (3.19)$$

Whatever the value of k^2 (in particular, $k^2 = 0$), the sign of the imaginary part of (3.19) must agree with the sign of the poles in prescription (2.18), i.e., we must have: in (3.8):

$$\zeta = \frac{\epsilon}{2k^-} \quad ; \quad \zeta > 0 \quad , \quad \epsilon > 0 \quad , \quad (3.20)$$

while in (3.9):

$$\zeta = -\frac{\epsilon}{2k^-} \quad ; \quad \zeta > 0 \quad , \quad \epsilon > 0 \quad . \quad (3.21)$$

Since ζ and ϵ are both strictly positive, (3.20) obliges us to take only $k^- > 0$; conversely, (3.21) entails us to take only $k^- < 0$. As they stand in (3.16), both integrals over k^- inadvertently allows for forbidden regions of k^- . They include those regions of k^- which forces a change in the sign of ϵ in (3.20) and in (3.21), thus mixing positive-frequency (or -energy) radiation propagating forward in time, with negative-frequency one. This violates causality, a basic physical principle.

Therefore, physically acceptable regions of integration over k^- are confined to the following intervals: in (3.11),

$$0 < k^- < -Q^- \quad , \quad (3.22)$$

while in (3.15),

$$-Q^- < k^- < 0 \quad , \quad (3.23)$$

so that

$$J_{LC} = -i\pi \left\{ \int_0^{-Q^-} dk^- \bullet^{2ix(Q^+ - i\zeta)k^-} - \int_{-Q^-}^0 dk^- \bullet^{2ix(Q^+ + i\zeta)k^-} \right\} \quad . \quad (3.24)$$

Carrying out the k^- -integration we obtain

$$J_{LC} = \frac{-nQ^+}{x[(Q^+)^2 + \zeta^2]} \left\{ \bullet^{-2ixQ^+Q^-} - \cosh(2x\zeta Q^-) - \frac{i\zeta}{Q^+} \sinh(2x\zeta Q^-) \right\} \quad . \quad (3.25)$$

Taking the limit $\zeta \rightarrow 0$, (3.25) reduces to

$$J_{LC} = \frac{-n}{xQ^+} \left\{ \bullet^{-2ixQ^+Q^-} - 1 \right\} \quad , \quad (3.26)$$

which, inserted into (3.4) results in

$$M_{LC} = (-1)^{\omega-1} \frac{\pi^\omega}{x^\omega Q^+} \left\{ \bullet^{-ixQ^2} - \bullet^{ixQ^2} \right\} \quad . \quad (3.27)$$

Finally, changing variables according to (3.17a) and (3.17b) and integrating over x ($0 < x < \omega$) and y ($0 < y < 1$), we arrive at the same result as the one quoted in (2.21). Therefore,

the PV prescription, when correctly implemented, leads to the same result as the one obtained through the Mandelstam prescription. Consequently, all the properties (a)-(f) given at the end of section 2 resulting from the use of "good" prescriptions authored by Mandelstam and Leibbrandt, do also result from a careful implementation of the PV prescription.

4. COMMENTS AND CONCLUSIONS

By looking at the poles in the k^+ -complex plane generated by the covariant piece of the gauge boson propagator and noting that the signs of their imaginary parts must be concordant with the signs of the corresponding poles at $k^+=0$ in the sense of Cauchy PV, we observed that:

(i) The constraint defines a finite range of integration over the k^- -integrals^[16], and

(ii) The basis for this restriction has to do with ensuring that positive energy quanta do not become mixed up with quanta of negative energy propagating forward in time.

One should note that the poles of prescription (2.18) in the k^0 -complex plane, when analysed under observation (ii) above, are located in the second and fourth quadrants. This ensures legitimacy to Wick rotation and thus for power counting.

We want to emphasize that careful implementation of the PV prescription in the light-cone gauge reproduces the well-behaved properties ensuing from treating the $(k^0)^{\pm 1}$ poles via either the Mandelstam or the Leibbrandt prescriptions. It should be stressed that earlier attempts at applying the PV technique in the light-cone gauge computations failed because they overlooked the fact that there is a strong constraint over the range of k^- component of momentum. Overlooking this amounts to violating causality, and so, no wonder there appeared unwieldy double-poles in one-loop calculations.

Of course, we have only evaluated the basic one-loop

integral in the light-cone gauge, (2.17). However, in computing, e.g., one-loop gluon self-energy, one encounters tadpole-like integrals as well as integrals with tensorial structure in the integrand. These can be evaluated following the same steps as the ones used for evaluating (2.17), or by employing techniques already developed for the Mandelstam and/or Leibbrandt prescriptions to compute those kind of integrals in terms of the basic one (2.17).

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