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ABSTRACT

A semigroup S which has a left regular band of groups as a semigroup of left quotients is shown to be the semigroup which is a left regular band of right reversible cancellative semigroups. An alternative characterization is provided by using spinned products. These results are applied to the case where S is superabundant whose set of idempotents forms a left normal band.

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INTRODUCTION

In this paper we characterize semigroups S which have a semigroup Q of left quotients, where Q is an $\mathbb{R}$-unipotent semigroup which is a band of groups. Recall that an $\mathbb{R}$-unipotent (or right inverse) semigroup S is a regular semigroup whose set of idempotents is a left regular band in that $efe = ef$ for any idempotents e, f in S. $\mathbb{R}$-unipotent semigroups were studied by several authors, see for example [1] and [12]. Bailes [1] characterized $\mathbb{R}$-unipotent semigroups which are bands of groups. This characterization extended the structure of inverse semigroups which are semilattices of groups. Recently, Gould studied in [7] the semigroup S which has a semigroup Q of left quotients where Q is an inverse semigroup which is a semilattice of groups. However, many definitions of semigroups of quotients have been proposed and studied. For a survey, the reader may consult Weinert's paper [13]. These definitions have been motivated by corresponding definitions in ring theory. In this paper we are concerned with a concept of semigroups of left quotients adopted by Fountain and Petrich [5]. The definition proposed there is restricted to completely 0-simple semigroups of left quotients. The idea is that for a completely 0-simple semigroup Q containing a subsemigroup S to be a semigroup of left quotients of S if every element q in Q can be written as $q = a^{-1}b$ for some elements a, b in S with $a^2 \neq 0$ and a^{-1} is the inverse of a in the group $\mathcal{H}$-class H_a of Q. In this case S is called a left order in Q. This definition and its dual were used in [5] to characterize semigroups S which have a completely 0-simple semigroup of quotients. An extension of this definition was used in [6] to obtain necessary and sufficient conditions for a semigroup S to have bisimple inverse ω-semigroup of left quotients. This extended definition was also used in [7] to characterize semigroups S which have a semigroup Q of left quotients where Q is an inverse semigroup which is a semilattice of groups. In this paper we consider the corresponding problem for $\mathbb{R}$-unipotent semigroups which are bands of groups.

After preliminary results, we obtain in Sec. 2, the necessary and sufficient conditions for a semigroup S to have a semigroup Q of left quotients where Q is an $\mathbb{R}$-unipotent semigroup which is a band of groups. This result will be used in Sec. 3 together with the characterization of $\mathbb{R}$-unipotent semigroups which are bands of groups in terms of spined products to obtain an alternative structure for a semigroup S to have a left regular band of groups as a semigroup of left quotients. Sec. 4 is devoted to the case where the left orders are in a class of $\mathbb{R}^*$-unipotent semigroups.

We use the notation and terminology of Howie [8]. Other undefined terms can be found in Fountain's paper [4].

1. PRELIMINARIES

Recall that for a semigroup S, any two elements a,b in S are $\mathbb{R}^*$-related if they are related by Green's relation $\mathbb{R}$ in some oversemigroup of S. The following Lemma from [9] and [10] gives an alternative definition of $\mathbb{R}^*$.

LEMMA 1.1 For any two elements a, b in a semigroup S, the following statements are equivalent:

i) $a\mathbb{R}^*b$ in S

ii) $sa = ta \iff sb = tb;\ \forall\ s, t \in S^1$.

The following Corollary is a consequence of Lemma 1.1:

COROLLARY 1.2 If a is an element of a semigroup S and e is an idempotent in S, then $a\mathbb{R}^*e$ in S if and only if;

i) $ea = a$

ii) $sa = ta \Rightarrow se = te;\ \forall\ s, t \in S^1$.

The dual relation of $\mathbb{R}^*$ is $\mathcal{L}^*$. It is easy to see that $\mathbb{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence. Thus the intersection of $\mathbb{R}^*$ and $\mathcal{L}^*$ is an equivalence relation denoted by $\mathcal{H}^*$. It is evident for $K = \mathbb{R}$, $\mathcal{L}$ or $\mathcal{H}$ that $K \subseteq K^*$ and for any regular elements a, b in a semigroup S; aKb follows from aK^*b. Therefore $K = K^*$ on regular semigroups.

As a consequence of Corollary 1.2 and its dual or from [4] we have the following Lemma:

LEMMA 1.3 If e is an idempotent of a semigroup S, then H^*_e is a cancellative monoid.

Moreover, we have:

LEMMA 1.4 If e, f are $\mathcal{L}$-related idempotents in a semigroup S, then $H^*_e = H^*_f$.

PROOF Define $\psi : H^*_e \to H^*_f$ by $x\psi = fx$ for any x in H^*_e. If x is in H^*_e, then for any $s, t \in S$;

$$sfx = fxt \Rightarrow sfe = tfe \Rightarrow sf = tf \Rightarrow sfx = tfx$$

and

$$fxs = fxt \Rightarrow efxs = efxt \Rightarrow exs = ext \Rightarrow xs = xt \Rightarrow fxs = fxt.$$

Hence, $fx\mathcal{H}^*f$. Thus, ψ is a well-defined map. Clearly, it is one-to-one and for any $y \in H^*_f$, $ey \in H^*_e$ and $(ey)\psi = y$ and so ψ is a bijection.

Now let x,y be in H^*_e. Then $xy \in H^*_e$ (Lemma 1.3) and

$$(xy)\psi = fxy = fxfy = (x\psi)(y\psi)$$

Therefore, ψ is an isomorphism.

It is known that if Q is an $\mathbb{R}$-unipotent semigroup which is a band of groups, then Q can be written as a disjoint union of groups G_α, $\alpha \in Y$, that is, $Q = \bigcup_{\alpha\in Y} G_\alpha$, where Y is a band isomorphic to the band of idempotents of Q. In particular Y is left regular; so we may call Q in this case; a left regular band of groups. From [1] we have the following result:

THEOREM 1.5 Let Q be an $\mathbb{R}$-unipotent semigroup. Then:

a) Q is a union of groups if and only if $\mathbb{R} = \mathcal{H}$ in Q.

b) Q is a band of groups if and only if $\mathbb{R}$ is a congruence on Q.

The following Corollary is a consequence of Theorem 1.5.

COROLLARY 1.6 Let Q be a left regular band of groups and E be its band of idempotents. Then Q is the left regular band E of the $\mathcal{H}$-classes H_e; $e \in E$ of Q.

The central concept in our work is the concept of semigroups of left quotients. We say an oversemigroup Q of a semigroup S is a semigroup of left quotients of S if for any element q of Q, there exist a,b in S such that $q = a^{-1}b$ where a^{-1} is the inverse of a in a subgroup of Q. Similar definitions were adopted in [5], [6] and [7]. If Q is a semigroup of left quotients of a semigroup S, then S is said to be a left order in Q. Our investigation will be on the light of left regular bands of right reversible, cancellative semigroups. We recall from [2] that a semigroup S is right reversible if $Sa \cap Sb \neq \Box$ for all a, b in S, that is, for any a, b in S, there exist x, y in S with $xa = yb$.

For cancellativity, we conclude from [7] the following lemma:

LEMMA 1.7 If S is a semilattice of cancellative semigroups, then $\mathcal{H}^*$ is the greatest band congruence on S all of whose classes are cancellative.

2. LEFT ORDERS IN A BAND OF GROUPS

Let Q be an $\mathbb{R}$-unipotent semigroup with set of idempotents E. Recall that E is a left regular band and so every $\mathbb{R}$-class in Q contains a unique idempotent. Consider Q to be a band Y of groups G_α; $\alpha \in Y$, where for any α, $\beta \in Y$, $G_\alpha \cap G_\beta = \Box$ if $\alpha \neq \beta$ and $Q = \bigcup_{\alpha\in Y} G_\alpha$; $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ such that $E \cong Y$, that is Y is a left regular band. Now let S be a semigroup which is a left order in Q. Put $S_\alpha = S \cap G_\alpha$ for any $\alpha \in Y$. It follows that for any $\alpha \in Y$, $a \in G_\alpha$, there exist x, $y \in S$ with $a = x^{-1}y$, where $x \in S_\beta$, $y \in S_\gamma$ for some β, $\gamma \in Y$. Since $x^{-1} \in G_\beta$, $y \in G_\gamma$, then $\alpha = \beta\gamma$ and $xy \in S_\beta S_\gamma \in S_{\beta\gamma} = S_\alpha$ so that S_α is not empty for any α in Y. Clearly, for any α in Y; S_α is a subsemigroup of S. Since S_α is a subsemigroup of the group G_α, then S_α is cancellative. Now it is evident that S is the band Y of the semigroups S_α; $\alpha \in Y$. Moreover we have

PROPOSITION 2.1 S is a left regular band Y of right reversible, cancellative semigroups S_α ; $\alpha \in Y$.

PROOF It remains to prove that S_α is right reversible for any $\alpha \in Y$. Let $\alpha \in Y$ and $a, b \in S_\alpha$. Choose $s \in S_\alpha$. Since $b^{-1} \in G_\alpha$, then $sab^{-1} \in G_\alpha$. By the ordering of S in Q, there exist $x \in S_\beta$, $y \in S_\gamma$ for some β, γ in Y such that $sab^{-1} = x^{-1}y$. It follows that $\alpha = \beta\gamma$ and $xsae_\alpha = e_\beta yb$ where e_α and e_β are the identity elements of G_α and G_β respectively. Therefore $\beta\alpha = \alpha$ and $xsa \in G_\beta G_\alpha \subseteq G_{\beta\alpha} = G_\alpha$. Hence $xsa = e_\beta yb$. Let z be in S_β. Then $xsa = z^{-1}zyb$ and $zxsa = zyb$. It is clear that $zxs \in S$, $zxs \in G_\beta G_\alpha \subseteq G_{\beta\alpha} = G_\alpha$ and $zxs \in S_\alpha$. Similarly $zy \in S$, $zy \in G_\beta G_\gamma \subseteq G_{\beta\gamma} = G_\alpha$ and $zy \in S_\alpha$. Hence S_α is right reversible.

COROLLARY 2.2 For any $\alpha \in Y$; G_α is a group of left quotients of S_α.

PROOF For any $\alpha \in Y$, let $g \in G_\alpha$ and choose $a \in S_\alpha$. Since $ag \in G_\alpha$, then there exists $x \in S_\beta$, $y \in S_\gamma$ for some β, γ in Y such that $ag = x^{-1}y$. Notice that $x^{-1} \in G_\beta$, $\beta\gamma = \alpha$ and we have $xag = e_\beta y$. Let $b \in S_\beta$ and write $bxag = be_\beta y = by$. It follows that $\beta\alpha = \alpha$, $by \in S_\beta S_\gamma \subseteq S_{\beta\gamma} = S_\alpha$ and $bxa \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_\alpha$. Therefore $g = e_\alpha g = (bxa)^{-1}by$.

COROLLARY 2.3 If $q \in Q$, then there exist a, b in S with $a\mathbb{R}b$ in Q and $q = a^{-1}b$.

PROOF This follows directly from Corollary 2.2 and from the fact that every two elements in G_α are $\mathcal{H}$-related.

The objective now towards a converse of Proposition 2.1. Let S be a semigroup which is a left regular band Y of right reversible, cancellative semigroups S_α; $\alpha \in Y$. By (5, Proposition 6.4), for each $\alpha \in Y$, S_α has a group G_α of left quotients. We may assume $G_\alpha \cap G_\beta = \square$ for any $\alpha, \beta \in Y$, $\alpha \neq \beta$. Put $Q = \bigcup_{\alpha\in Y} G_\alpha$. Define a product $\cdot$ on Q by

$$a^{-1}b \cdot c^{-1}d = (xa)^{-1}yd$$

where, if $a, b \in S_\alpha$; $c, d \in S_\beta$, then $x, y \in S_{\alpha\beta}$ are chosen so that $xb = yc$.

Since Y is a left regular band, then

$$xa \in S_{\alpha\beta}S_\alpha \subseteq S_{\alpha\beta\alpha} = S_{\alpha\beta} \quad \text{and} \quad (xa)^{-1} \in G_{\alpha\beta}$$

also

$$yd \in S_{\alpha\beta}S_\beta \subseteq S_{\alpha\beta\beta} = S_{\alpha\beta} \quad \text{and} \quad yd \in G_{\alpha\beta}$$

Therefore, the product $(xa)^{-1}yd$ is taken as the product in $G_{\alpha\beta}$. We notice that the property of left regularity of the band Y together with right reversibility and cancellativity of S_α; $\alpha \in Y$ are sufficient to carry out the proof of [7] for Q to be a groupoid under the given product, so we omit it here. To prove that product is associative we need the following lemma

LEMMA 2.4 If $\alpha \in Y$ and a,b are elements of S_α, then $a\mathbb{R}^*b$ in S

PROOF Let $\alpha \in Y$; $a,b \in S_\alpha$ and $s \in S_\lambda$, $t \in S_\mu$ for some λ, μ in Y with $sa = ta$. Then $S_{\lambda\alpha} = S_{\mu\alpha}$. Put $\beta = \lambda\alpha = \mu\alpha$. Since sa, ta in S_β, then by right reversibility of S_β, there exist m, n in S_β such that $msa = nta$. Notice that

$$ms \in S_{\lambda\alpha}S_\lambda \subseteq S_{\lambda\alpha\lambda} = S_{\lambda\alpha} = S_\beta; \quad nt \in S_{\mu\alpha}S_\mu \subseteq S_{\mu\alpha\mu} = S_{\mu\alpha} = S_\beta$$

and again by right reversibility of S_β, there exist u, v in S_β with $ums = vnt$ and $umsa = vnta$. Since um, sa, vn and ta are in S_β, $sa = ta$ and S_β is cancellative, then $um = vn$ ($=k$, say). It follows that $ks = kt$ and $ksb = ktb$. Since k, sb and tb are in S_β. Then, by cancellativity in S_β we obtain $sb = tb$.

Similarly, $sb = tb$ implies $sa = ta$. Therefore, by Lemma 1.1, $a\mathbb{R}^*b$ in S.

COROLLARY 2.5 $a\mathcal{H}^*a^2$ for any element a in S.

PROOF Let $a \in S_\alpha$, $s \in S_\lambda$, $t \in S_\mu$ with $a^2s = a^2t$. It is clear that $a^2 \in S_\alpha$ and $\alpha\lambda = \alpha\mu$ ($= \gamma$, say). Choose $k \in S_\gamma$ and write $ka\cdot as = ka\cdot at$. Notice that

$$ka \in S_{\alpha\lambda}S_\alpha \in S_{\alpha\lambda\alpha} = S_{\alpha\lambda} = S_\gamma; \quad as, at \in S_\gamma$$

and S_γ is cancellative. Hence $as = at$. It is obvious that $as = at$ implies $a^2s = a^2t$. Therefore by the dual of Lemma 1.1; $a\mathcal{L}^*a^2$ in S. But $a\mathbb{R}^*a^2$ in S by Lemma 2.4. Hence $a\mathcal{H}^*a^2$ in S.

Returning now to the product of Q, the associativity of the product of Q in the inverse semigroup case was proved in [7] by using only the property that $a\mathbb{R}^*b$ in S for any $a,b \in S_\alpha$; $\alpha \in Y$. Therefore, we can see by Lemma 2.4 that the product of Q is associative by a similar proof as that in [7]. Moreover, it can be seen that the product of Q is an extension of that in S. It is immediate from the definition of the product of Q, that $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ for any $\alpha, \beta \in Y$. Therefore Q is a left regular band of groups G_α; $\alpha \in Y$. From its construction, Q is a semigroup of left quotients of S. In conclusion we have established the following result.

THEOREM 2.6 A semigroup S has a left regular band of groups as a semigroup of left quotients if and only if S is a left regular band of right reversible cancellative semigroups.

Theorem 2.6 shows that, if S is a left regular band of right reversible, cancellative semigroups, then for any decomposition of S as a left regular band of right reversible, cancellative semigroups, we can construct a semigroup Q of left quotients of S where Q is a left regular band of groups. Neither the decomposition of S nor the construction of Q is unique (see [7]). In order to overcome this problem of uniqueness in the inverse semigroup case, the notion of stratified semigroup of left quotients was introduced in [7]. Similarly, we say that an oversemigroup of a semigroup S is a stratified semigroup of left quotients of S if it is a semigroup of left quotients of S and for any a, b of S

$$a\mathcal{R}^*b \text{ in } S \Rightarrow a\mathcal{R}b \text{ in } Q \text{ and that } a\mathcal{L}^*b \text{ in } S \Rightarrow a\mathcal{L}b \text{ in } Q.$$

Let Q be an oversemigroup of a semigroup S. Then for any element a in S, we have

$$a\mathcal{H}a^2 \text{ in } Q \Leftrightarrow a \text{ is in a subgroup of } Q$$

and by definition

$$a\mathcal{H}a^2 \text{ in } Q \Rightarrow a\mathcal{H}^*a^2 \text{ in } S$$

If Q is a stratified semigroup of left quotients of S, then the reverse implication of the last statement holds. If Q is a left regular band of groups, then for any $q \in Q$, there exist $a, b \in S$ such that $q = a^{-1}b$ with $a\mathcal{R}b$ in Q (Corollary 2.3). In this case, by [7, Theorem 4.1], Q is unique up to isomorphism.

3. PUNCHED SPINED PRODUCTS

In this section we provide an alternative characterization of a semigroup S which has a semigroup Q of left quotients where Q is a left regular band of groups. This characterization will be in terms of spined products. Recall that, if E is a band and M is a semigroup with a semilattice congruence τ and a semigroup isomorphism $\phi : E/\epsilon \to M/\tau$ where ϵ is the minimum semilattice congruence on E, then the subdirect product

$$P = \left\{(e,x) \in E\times M : e\epsilon^{\natural}\phi = x\tau^{\natural}\right\}$$

is called a spined product of E and M. We call a subdirect product S of $E \times M$ is a punched spined product of E and M if S is a subset of a spined product of E and M such that for any $e \in E$, there

exists $x \in M$ with $(e, x) \in S$ and for any $y \in M$, there exists $f \in E$ with $(f, y) \in S$. The aim of this section is to prove that the left orders characterized in Sec. 2 are in fact punched spined products.

Let Q be an $\mathbb{R}$-unipotent semigroup and E be its band of idempotents. Let ϵ be the minimum semilattice congruence on E. For any $e \in E$, denote the ϵ-class containing e by $\bar{e}$ or $E(e)$. Write $Y = \{E(e) : e \in E\}$. Since E is left regular, then $E(e)$ is a left zero semigroup. Let $\gamma = \{(x,y) \in Q \times Q : V(x) = V(y)\}$. It is well known that γ is the minimum inverse semigroup conguence on Q and $\gamma|E = \epsilon$. Suppose that Q is a band of groups, then Q/γ is a semilattice of groups and we can write $Q/\gamma = \bigcup_{\bar{e}\in Y} H_{\bar{e}}$ where $H_{\bar{e}}$ is the group $\mathcal{H}$-class in Q/γ containing $\bar{e}$. Moreover, Q is a spined product P of E and Q/γ, that is,

$$Q = P = \bigcup_{\bar{e}\in Y} (E(e) \times H_{\bar{e}}) = \{(x^{-1}x, x\gamma) : x \in Q\}$$

We emphasize that P is a semilattice of the direct products $E(e)\times H_{\bar{e}}$; $\bar{e} \in Y$ and the product of P is reduced from the Cartesian product $E \times Q/\gamma$. Moreover, $(f, x^{-1}\gamma)$ is an inverse of $(e,x\gamma)$ for any $f \in E(e)$. In particular, for any $(f, y\gamma)$ in $E(e)\times H_{\bar{e}}$; $(f, y\gamma)\,\mathcal{H}\,(f, \bar{e})$ in P and the inverse of $(f,y\gamma)$ in $H_{(f,\bar{e})}$ is $(f,y^{-1}\gamma)$. We refer the reader to [1] and [11] for further details.

Let S be a semigroup which has P as a semigroup of left quotients. For any $\bar{e} \in Y$, define a subset $M_{\bar{e}}$ of Q/γ by the rule; $m \in M_{\bar{e}}$ if and only if $m \in Q/\gamma$ and $(f,m) \in S$ for some $f \in E(e)$.

LEMMA 3.1 For any $\bar{e}$ in Y; $M_{\bar{e}}$ is a cancellative semigroup.

PROOF Let $e \in E$ and $(e, a\gamma) \in E(e)\times H_{\bar{e}}$. Since P is a semigroup of left quotients of S, then there exist $(k, x\gamma)$, $(g, y\gamma)$ in S and $(f, x^{-1}\gamma)$ the inverse of $(k, x\gamma)$ in a subgroup of P, that is, $f \in E(k)$ such that

$$(e, a\gamma) = (f, x^{-1}\gamma)(g, y\gamma)$$

It follows that $e = fg$, $fe = e$, $ke = kg$, $ke \in E(f)E(e) \subseteq E(fe) \subseteq E(e)$ and $x\gamma y\gamma \in H_{\bar{f}\bar{g}} = H_{\bar{e}}$. Therefore,

$$(k,x\gamma)(g, y\gamma) = (kg, x\gamma y\gamma) \in S \cap (E(e) \times H_{\bar{e}})$$

Hence $xy\gamma \in M_{\bar{e}}$, and so $M_{\bar{e}}$ is not empty.

Clearly, $M_{\bar{e}}$ is a subsemigroup of $H_{\bar{e}}$ where $H_{\bar{e}}$ is a group and $M_{\bar{e}}$ is cancellative.

LEMMA 3.2 For any $\bar{e}$ in Y; $M_{\bar{e}}$ is right reversible.

PROOF Let $a\gamma$, $b\gamma$ be in $M_{\bar e}$ and g, h in $E(e)$ so that $(g, a\gamma)$, $(h, b\gamma)$ are in S. Choose $c\gamma \in M_{\bar e}$ and take $(k, c\gamma) \in S$ for some $k \in E(e)$. Let $(n, b^{-1}\gamma)$ be the inverse of $(h, b\gamma)$ in a subgroup of P, that is, $n \in E(h)$ and

$$(k, c\gamma)(g, a\gamma)(n, b^{-1}\gamma) \in E(e) \times H_{\bar e}$$

By left ordering of S in P, there exist $(f, q\gamma)$, $(i, d\gamma)$ in S and $(t, q^{-1}\gamma)$ the inverse of $(f, q\gamma)$ in a subgroup of P, that is, $t \in E(f)$ such that

$$(k, c\gamma)(g, a\gamma)(n, b^{-1}\gamma) = (t, g^{-1}\gamma)(i, d\gamma)$$

Thus

$$(f, q\gamma)(k, c\gamma)(g, a\gamma)(n, b^{-1}\gamma)(h, b\gamma) = (f, q\gamma)(t, q^{-1}\gamma)(i, d\gamma)(h, b\gamma)$$

and

$$(fk, qc\gamma)(g, a\gamma)(n, \bar e) = (fi, \bar f(d\gamma))(h, b\gamma)$$

choose an element $v\gamma$ in $M_{\bar e}$ and j in $E(e)$ so that $(j, v\gamma) \in S$ to get

$$(j, v\gamma)(fk, qc\gamma)(g, a\gamma) = (j, v\gamma)(fi, \bar f(d\gamma))(h, b\gamma)$$

that is

$$(jfk, vqc\gamma)(g, a\gamma) = (jfi, (v\gamma)\bar f(d\gamma))(h, b\gamma)$$

Recall that $k = ti$ and notice $fk = fi$, $tk = k$ so that $E(f)E(e) \subseteq E(e)$ and $jfi = jfk$, $ef = efe$ are in $E(e)$. Moreover, $(v\gamma)\bar f(d\gamma) = (v\gamma)\bar e \bar f(d\gamma) = (v\gamma)\bar e(d\gamma) = v\gamma d\gamma$. Therefore $vqc\gamma$, $vd\gamma$ are in $M_{\bar e}$ and $vqc\gamma = vd\gamma\, b\gamma$.

Now we put $M = \bigcup_{\bar e \in Y} M_{\bar e}$, M is a semilattice Y of right reversible, cancellative semigroups $M_{\bar e}$, $\bar e \in Y$. It is easy to see that $\bigcup_{\bar e \in Y}(E(e) \times M_{\bar e})$ is a spined product containing S. Moreover, we have

LEMMA 3.3 (i) For any $e \in E$, there exists $x\gamma \in H_{\bar e}$ with $(e, x\gamma) \in S$.

(ii) For any $f \in E$, $y\gamma \in M_{\bar f}$, there exists $g \in E(f)$ with $(g, y\gamma) \in S$

PROOF (i) Let $e \in E$ and $(e, a\gamma) \in E(e) \times M_{\bar e}$. Then

$$(e, a\gamma) = (f, x^{-1}\gamma)(g, y\gamma)$$

where $(f, x\gamma)$, $(g, y\gamma)$ are in S and $(f, x^{-1}\gamma)$ is the inverse of $(f, x\gamma)$ in $H_{(f, \bar f)}$ of P. Therefore $e = fg$, and

$$(f, x\gamma)(g, y\gamma) = (fg, x\gamma y\gamma) = (e, xy\gamma) \text{ in } S$$

(ii) Straightforward.

Now it follows that S is a punched spined product and the following result is established.

PROPOSITION 3.4 Let P be a left regular band of groups and S be a semigroup. If P is a semigroup of left quotients of S, then S is a punched spined product of left regular band and a semilattice of right reversible, cancellative semigroups.

For a converse of Proposition 3.4, let S be a punched spined product of a left regular band E and a semilattice Y of right reversible, cancellative semigroups M_α, $\alpha \in Y$. By [5, Proposition 6.4], there is a group of left quotients G_α of M_α for any α in Y. We may assume that $G_\alpha \cap G_\beta = \square$ for all $\alpha, \beta \in Y$, $\alpha \neq \beta$. Let $T = \bigcup_{\alpha \in Y} G_\alpha$. Define a product $\cdot$ on T by

$$a^{-1}b \cdot c^{-1}d = (xa)^{-1}\, yd$$

where, if $a, b \in M_\alpha$; $c, d \in M_\beta$, then $x, y \in M_{\alpha\beta}$ are chosen such that $xb = yc$. From the proof of [7, Theorem 3.1] we conclude that T is a semilattice of groups. Moreover, T is a semigroup of left quotients of M where $M = \bigcup_{\alpha \in Y} M_\alpha$. Put $P = \bigcup_{\alpha \in Y}(E_\alpha \times G_\alpha)$. By [11, Theorems 3.2 and 4.1], P is a regular semigroup which is a band of groups and whose set of idempotents is a subsemigroup isomorphic to E. Therefore P is a left regular band of groups. In fact we have

LEMMA 3.5 P is a semigroup of left quotients of S.

PROOF Let $\alpha \in Y$ and $(e, m) \in E_\alpha \times G_\alpha$. Recall that S is a punched spined product of E and M. Since $e \in E_\alpha$, there exist an element z in M_α such that (e, z) in S. From $m \in G_\alpha$ and M_α is a left order in G_α, there exist $x, y \in M_\alpha$ such that $m = x^{-1}y$, and thus there exist $f, g \in E_\alpha$ with (f, x) and (g, y) in S. Notice that $x^{-1} \in G_\alpha$ and there exist u, v in M_α with $x^{-1} = u^{-1}v$, that is, $ux^{-1} = v$ and $zux^{-1} = zv$. Let i, j be in E_α so that (i, u) and (j, v) are in S. Clearly $(ei, zu) = (e, zu)$ in S and $(e, (zu)^{-1})$ is the inverse of (e, zu) in $H_{(e, \bar e)}$ of P. Also $(ejg, zvy) = (e, zvy)$ in S. Moreover, $(e, m) = (e, (zu)^{-1}zvy) = (e, (zu)^{-1})(e, zvy)$.

Now a converse of Proposition 4 is evident. In conclusion we have the following result.

THEOREM 3.6 A semigroup S has a left regular band of groups as a semigroup of left quotients if and only if S is a punched spined product of a left regular band and a semilattice of right reversible, cancellative semigroups. The following Corollary is an immediate consequence of Theorem 3.6.

COROLLARY 3.7 If S is a spined product of a left regular band and a semilattice of right reversible, cancellative semigroups, then S has a left regular band of groups as a semigroup of left quotients.

For the rest of the section, let S be a spined product of a left regular band E and a semilattice Y of cancellative semigroups M_α; $\alpha \in Y$. Put $E = \bigcup_{\alpha\in Y} E_\alpha$, $M = \bigcup_{\alpha\in Y} M_\alpha$ and $S = \bigcup_{\alpha\in Y} (E_\alpha \times M_\alpha)$.

LEMMA 3.8 The relation $\mathcal{H}^*$ is the greatest semilattice congruence on M all of whose classes are cancellative.

PROOF by Lemma 1.7, $\mathcal{H}^*$ is the greatest band congruence on M all of whose classes are cancellative. The relation γ defined on M by the rule

$(a, b) \in \gamma$ if and only if $a, b \in M_\alpha$ for some $\alpha \in Y$

is a band congruence on M all of whose classes are cancellative. Therefore, $\gamma \subseteq \mathcal{H}^*$. Now for any elements a, b in M, we have $(ab, ba)\in\gamma$. Hence $ab\mathcal{H}^*ba$ and $M/\mathcal{H}^*$ is a semilattice.

Identify the semilattice $M/\mathcal{H}^*$ by J, that is, M is a semilattice J of H_j^*; $j \in J$. For each $j \in J$, let $Z_j = \{\alpha \in Y;\ M_\alpha \subseteq H_j^*\}$. Readily, Z_j is a subsemilattice of Y. For any $j \in J$. Put $F_j = \bigcup_{\alpha\in Z_j} E_\alpha$ and $S_j = \bigcup_{\alpha\in Z_j} (E_\alpha \times M_\alpha)$.

We are now in a position to prove the final result of this section.

PROPOSITION 3.9 The following statements concerning the semigroup S are equivalent:

(1) each $\mathcal{H}^*$-class of M is right reversible.

(2) for any a, b in M, there exist x, y in M with $xa = yb$ and $x\mathcal{H}^*y\ \mathcal{H}^*ab$.

(3) S_j is right reversible for any j in J.

(4) There is an oversemigroup T of S which is a left regular band χ of right reversible, cancellative semigroups T_α; $\alpha \in \chi$ where for any $j \in J$; H_j^* is isomorphic to T_α for some $\alpha \in \chi$.

PROOF *Recall that* $\mathcal{H}^*$ *is a semilattice congruence on* M.

(1) $\Longleftrightarrow$ (2) If (1) holds and a, b in M, then ab, ba are in H_{ab}^* and there exist $c, d \in H_{ab}^*$ with $cab = dba$ where $ca \in H_{ab}^* \cdot H_a^* \subseteq H_{ab}^*$ and $db \in H_{ab}^* \cdot H_b^* \subseteq H_{ab}^*$. Put $x = ca$, $y = db$ to get $xa = yb$ and $x\mathcal{H}^*y\ H_{ab}^*$. Hence (2) holds.

If (2) holds, $z \in M$ and $a, b \in H_z^*$, then in particular, there exist x, y in M with $xa = yb$ and $x\mathcal{H}^*y\ \mathcal{H}^*ab$. Since $a^2\mathcal{H}^*ab$, then $a\mathcal{H}^*ab$ and x, y are in H_z^* so (1) holds.

(1) $\Longleftrightarrow$ (3) If (1) holds and $j \in J$, (e, a), (f, b) in S_j, that $(e, a)\in E_\alpha \times M_\alpha$, $(f, b) \in E_\beta \times M_\beta$, say, where M_α and M_β are subsets of H_j^*. Then $a, b \in H_j^*$ with $xa = yb$, where $x \in M_\lambda$, $y \in M_\mu$ for some $\lambda, \mu \in Z_j$. It follows that $\lambda\alpha = \mu\beta$. Let $g \in E_\lambda$, $h \in E_\mu$ and $s \in M_{\lambda\alpha} = M_{\mu\beta}$. Then

$$gehf \in E_\lambda E_\alpha E_\mu E_\beta \subseteq E_{\lambda\alpha},\quad sx \in M_{\lambda\alpha}M_\lambda \subseteq M_{\lambda\alpha},\quad sy \in M_{\mu\beta}M_\beta \subseteq M_{\mu\beta}$$

whence, $sxa = syb$. The elements

$(gehf, sx)$, $(gehf, sy)$ are in $(E_{\lambda\alpha} \times M_{\lambda\alpha})$

so that, they are in S_j. Moreover,

$(gehf, sxa) = (gehf, syb)$, that is, $(gehfe, sxa) = (gehff, syb)$

and $(gehf, sx)(e, a) = (gehf, sy)(f, b)$. Hence (3) holds.

If (3) holds and a, b in H_j^*, then for some $\alpha, \beta \in Z_j$, $a \in M_\alpha$, $b \in M_\beta$. Let $e \in E_\alpha$, $f \in E_\beta$ so that (e, a), (f, b) are in S_j. Then there exist (g, x), (h, y) in S_j with $(g, x)(e, a) = (h, y)(f, b)$. In particular; $x, y \in H_j^*$, $xa = yb$ and (1) holds.

(1) $\Longleftrightarrow$ (4) If (1) holds, then by Lemma 3.8, H_j^* and hence $\{e\} \times H_j^*$ is a right reversible, cancellative *semigroup* for any $e \in E_\alpha$, $\alpha \in Z_j$. For any $j \in J$, $\alpha \in Z_j$, put

$$N_\alpha = \bigcup_{e\in E_\alpha} (\{e\}\times H_j^*) \quad \text{so that} \quad F_j \times H_j^* = \bigcup_{\alpha\in Z_j} N_\alpha$$

and

$$T = \bigcup_{j\in J} (F_j \times H_j^*) = \bigcup_{j\in J} \Big(\bigcup_{\alpha\in Z_j} N_\alpha \Big)$$

$$= \bigcup_{j\in J} \Big(\bigcup_{\alpha\in Z_j} \Big(\bigcup_{e\in E_\alpha} (\{e\}\times H_j^*) \Big)\Big)$$

is a left regular band of right reversible, cancellative semigroups. Clearly, for any $j\in J$, $\alpha\in Z_j$, $e \in E_\alpha$; $\{e\} \times H_j^* \cong H_j^*$ and S is a subsemigroup of T. Hence (4) holds.

If (4) holds, then trivially (1) holds.

4. $\mathbb{R}^*$-UNIPOTENT SEMIGROUPS

Recall from (4) that a semigroup S is abundant if each $\mathbb{R}^*$-class and each $\mathcal{L}^*$-class of S contains an idempotent. If a is an element of S, then a^+ and a^* denote typical idempotents in $\mathbb{R}_a^*$ and $\mathcal{L}_a^*$ respectively. A semigroup S is superabundant if each $\mathcal{H}^*$-class contains an idempotent. In this section we consider the class of abundant semigroups S in which the set of idempotents forms a left regular band. In this case every $\mathbb{R}^*$-class of S contains a unique idempotent, thus S is called *$\mathbb{R}^*$-unipotent*. The objective is to characterize a class of $\mathbb{R}^*$-unipotent semigroups which have a semigroup Q of left quotients, where Q is a left regular band of groups. This is a special case of the subject discussed in the previous sections. The dual of $\mathbb{R}^*$-unipotent semigroups was studied in [3] from which we conclude the following result.

LEMMA 4.1 Let S be an $\mathbb{R}^*$-unipotent semigroup. Then

(1) S is superabundant if and only if $\mathbb{R}^* = \mathcal{H}^*$ on S.

(2) S is a band of cancellative monoids if and only if S is superabundant and $\mathcal{H}^*$ is a congruence on S.

Throughout this section, let S be an $\mathbb{R}^*$-unipotent semigroup.

PROPOSITION 4.2 If S is a left regular band Y of right reversible, cancellative semigroups S_α, $\alpha \in Y$, then the following statements are equivalent

(1) S is superabundant

(2) for any $\alpha \in Y$, $a \in S_\alpha$, there exists an idempotent e_γ in S_γ for some $\gamma \in Y$ with $e_\gamma \mathcal{L}^* a$ and $S_\gamma S_\alpha \subseteq S_\alpha$.

PROOF

(1) $\Rightarrow$ (2) Let $\alpha \in Y$, $a \in S_\alpha$ and $a\mathbb{R}^* e_\gamma$ where e_γ is an idempotent in S_γ. Since $\mathbb{R}^* = \mathcal{H}^*$ on S (Lemma 4.1) then $a \mathcal{L}^* e_\gamma$ and $e_\gamma a = a$, that is $S_\gamma S_\alpha \subseteq S_\alpha$.

(2) $\Rightarrow$ (1) Let $a \in S_\alpha$ where $a\mathbb{R}^* e_\delta$, e_δ is an idempotent in S_δ. Then $e_\delta a = a$, that is, $\delta\alpha = \alpha$. It follows that $\alpha\delta\alpha = \alpha$ and $\alpha\delta = \alpha$. In particular $ae_\delta \in S_\alpha$. By right reversibility of S_α; $xa = yae_\delta$ for some $x, y \in S_\alpha$, that is $xae_\delta = yae_\delta$. The cancellation in S_α implies $x = y$ and $xa = xae_\delta$. Thus $a = ae_\delta$. Now let e_γ be an idempotent in S_γ with $e_\gamma \mathcal{L}^* a$ and $S_\gamma S_\alpha \subseteq S_\alpha$. Since $ae_\gamma = a = a \cdot e_\delta$ and $e_\gamma \mathcal{L}^* a$, then $e_\gamma = e_\gamma e_\delta$. Recall that $e_\gamma a \in S_\gamma S_\alpha \subseteq S_\alpha$, $a \in S_\alpha$, S_α is right reversible we have $ue_\gamma a = va$ for some $u, v \in S_\alpha$ and $ue_\gamma = v$. Therefore, $ue_\gamma e_\gamma a = va$ and $e_\gamma a = a$. Since $e_\gamma a = a = e_\delta a$ and $e_\delta \mathbb{R}^* a$, then $e_\gamma e_\delta = e_\delta$. Hence $e_\gamma = e_\delta$ and $a \mathcal{L}^* e_\delta$, that is, $a \mathcal{H}^* e_\delta$ and (1) holds.

LEMMA 4.3 If S is superabundant in which for any elements a, b in S there exist x, y in S with $xa = yb$ and $x \mathcal{H}^* y \mathcal{H}^* ab$, then each $\mathcal{H}^*$-class in S is right reversible.

PROOF This is immediate from the fact that each $\mathcal{H}^*$-class of S is a cancellative monoid.

PROPOSITION 4.4 If S is a band of cancellative monoids, then the following statements are equivalent

(1) each $\mathcal{H}^*$-class in S is right reversible.

(2) for any a, b in S, there exist elements x, y in S with $xa = yb$ and $x \mathcal{H}^* y \mathcal{H}^* ab$.

PROOF

(1) $\Rightarrow$ (2) By Lemma (4.1), S is superabundant on which $\mathcal{H}^*$ is a congruence. Let $a \in H^*_e$, $b \in H^*_f$ for some idempotents e, f in S. Then

$$ab \in H^*_{ef}, \quad aba \in H^*_{efe} = H^*_{ef}$$

But H^*_{ef} is right reversible, so there exist u, v in H^*_{ef} such that $uab = vaba$. Notice that $y = ua \in H^*_{efe} = H^*_{ef}$ and $x = vab \in H^*_{efef} = H^*_{ef}$. Therefore

$$xa = yb \quad \text{and} \quad x \mathcal{H}^* y \, H^* ab.$$

(2) $\Rightarrow$ (1) This is Lemma 4.3.

In fact any of the statements of Proposition 4.4 is a consequence of S to have a semigroup Q of left quotients where Q is a left regular band of groups. The following lemma demonstrates this result.

LEMMA 4.5 Let S be superabundant which is a left regular band of right reversible cancellative semigroups. Then for any elements a, b of S, there exist x, y in S with $xa = yb$ and $x \mathcal{H}^* y \mathcal{H}^* ab$.

PROOF Put $S = \bigcup_{\alpha \in Y} S_\alpha$, where Y is a left regular band and S_α is a right reversible, cancellative semigroup for any $\alpha \in Y$. Let $a, b \in S$; $a \in S_\alpha$, $b \in S_\beta$, say. Then $ab \in S_{\alpha\beta}$, $aba \in S_{\alpha\beta\alpha} = S_{\alpha\beta}$ and there exist u, v in $S_{\alpha\beta}$ with $uaba = vab$ where

$$x = uab \in S_{\alpha\beta\alpha\beta} = S_{\alpha\beta} \quad \text{and} \quad y = va \in S_{\alpha\beta\alpha} = S_{\alpha\beta}$$

But every two elements in $S_{\alpha\beta}$ are $\mathbb{R}^*$-related (Lemma 2.4). Then the result follows from the fact that $\mathbb{R}^* = \mathcal{H}^*$ on S.

Now we consider the construction of S_α in S as given in the following proposition.

PROPOSITION [3] 4.6 Let S be superabundant with band of idempotents E and $E = \bigcup_{\alpha \in Y} E_\alpha$ be the maximal semilattice decomposition of E. For each $\alpha \in Y$, define

$$S_\alpha = \{x \in S : x^+, x^* \in E_\alpha\}$$

Then

(1) S_α is the maximal abundant subsemigroup of S which contains E_α as its set of idempotents such that $\mathbb{R}^*(S_\alpha) \subseteq \mathbb{R}^*(S)$ and $\mathcal{L}^*(S_\alpha) \subseteq \mathcal{L}^*(S)$.

(2) $S_\alpha \cap S_\beta = \square$ if $\alpha \neq \beta$

(3) S is a semilattice of S_α; $\alpha \in Y$

(4) $S_\alpha = E_\alpha \times H^*_e$ where H^*_e is the $\mathcal{H}^*$-class in S containing e and $e \in E_\alpha$.

Now let S be superabundant with set of idempotents E. Retain the notations of Proposition 4.6. Assign to each α in Y, a cancellative monoid $M_\alpha = H_e^*$ for some fixed e in E_α. By Lemma 1.4, $M_\alpha \cong H_f^*$ for any $f \in E_\alpha$. By Proposition 4.6, $S_\alpha = E_\alpha \times M_\alpha$. Denote the identity of M_α by e_α and put $M = \bigcup_{\alpha\in Y} M_\alpha$. Define a product $\cdot$ on M by

$$x \cdot y = e_{\alpha\beta}xy \quad \text{for any} \quad x \in M_\alpha, \quad y \in M_\beta$$

It is routine to check that $x \cdot y \in M_{\alpha\beta}$ for any $x \in M_\alpha$, $y \in M_\beta$ and the product is a well-defined associative binary operation makes M a semilattice Y of the cancellative monoids M_α, $\alpha \in Y$. Moreover, we have the the following result

LEMMA 4.7 S is in one-to-one correspondence with $P = \bigcup_{\alpha\in Y} (E_\alpha \times M_\alpha)$.

PROOF Define $\Phi : P \to S$ by $(e, a)\Phi = ea$. It is obvious that Φ is a well-defined map. Let $(e, x) \in E_\alpha \times M_\alpha$, $(f, y) \in E_\beta \times M_\beta$ such that $ex = fy$. It is easy to verify that $e\mathbb{R}^*ex$ and $f\mathbb{R}^*fy$. Therefore $e = f$ and $E_\alpha = E_\beta$, that is, $\alpha = \beta$. Thus

$$ex = fy \Rightarrow e_\alpha ex = e_\alpha fy \Rightarrow e_\alpha x = e_\alpha y \Rightarrow x = Y$$

and Φ is one-to-one. For surjectivity, let $x \in S$ where $x\mathbb{R}^*x^+$; $x^+ \in E_\alpha$, say. Then $(x^+, e_\alpha x) \in E_\alpha \times M_\alpha$ and $(x^+, e_\alpha x)\Phi = x^+e_\alpha x = x^+x = x$.

Hence Φ is a bijection.

Recall that a band E is left normal if $efg = egf$ for any idempotents e, f, g in E. Clearly left normal bands are left regular. To improve the result of Lemma 4.7 we impose the condition of left normality on E.

PROPOSITION 4.8 If E is left normal, then $P = \bigcup_{\alpha\in Y} (E_\alpha \times M_\alpha)$ is isomorphic to S.

PROOF From the proof of Lemma 4.7 we have the bijection $\Phi : P \to S$, defined by $(e,a)\Phi = ea$ for any $(e,a) \in P$. To show that Φ is a homeomorphism, let $(e, x) \in E_\alpha \times M_\alpha$ and $(f, y) \in E_\beta \times M_\beta$. Then

$$((e, x) \cdot (f, y))\Phi = (ef, e_{\alpha\beta}xy)\Phi$$
$$= ef\, e_{\alpha\beta}xy$$
$$= efxy \qquad (ef, e_{\alpha\beta} \in E_{\alpha\beta})$$

and

$$(e,x)\Phi(f,y)\Phi = exfy$$

Notice that $ex\mathbb{R}^*e$ and

$$ex\mathbb{R}^*e \Rightarrow efe\,x\,\mathbb{R}^*efe$$
$$\Rightarrow efx\,\mathbb{R}^*ef$$
$$\Rightarrow efx\,\mathcal{H}^*\,ef \qquad (\mathbb{R}^* = \mathcal{H}^* \text{ on } S)$$
$$\Rightarrow efexef = efx$$
$$\Rightarrow efexfy = efxy$$
$$\Rightarrow efxfy = efxy$$

Now let $i \in E$ such that $xf\,\mathcal{H}^*i$. Then in particular we have $xfi = xf$, that is $xfi = xff$ which implies $i = if$. Therefore

$$efxfy = efixfy$$
$$= eifxfy \qquad (E \text{ is left normal})$$
$$= eixfy \qquad (if = i)$$
$$= exfy$$

and we obtain $exfy = efxy$. Hence Φ is an isomorphism.

As an immediate consequence of Proposition 4.8 we have the following Corollary

COROLLARY 4.9 If E is left normal, then S is a spined product of a left regular band and a semilattice Y of cancellaive monoids M_α; $\alpha\in Y$ where M_α's are $\mathcal{H}^*$-classes of S.

Now directly from Theorem 2.6, Lemma 4.5, Lemma 4.3, Corollary 4.9 and Corollary 3.7 we have

THEOREM 4.10 Let S be superabundant in which the set of idempotents is a left normal band. Then the following statements are equivalent.

(1) S is a left order in a left regular band of groups.

(2) S is a left regular band of right reversible, cancellative semigroups.

(3) for any a, b in S, there exist x, y in S with $xa = yb$ and $x\mathcal{H}^*y\mathcal{H}^*ab$.

(4) each $\mathcal{H}^*$-class in S is right reversible.

(5) S is a spined product of a left regular band and a semilattice of right reversible, cancellative semigroups.

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