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ABSTRACT

Li and Yorke originally introduced the notion of chaos for continuous self-map of the interval $I = [0,1]$. In the present paper we show that an interval self-map with positive topological entropy has a chaoticity more complicated than the chaoticity in the sense of Li and Yorke. The main result is that if $f : I \to I$ is continuous and has a periodic point with odd period > 1 then there exists a closed subset K of I invariant with respect to f such that the periodic points are dense in K, the periods of periodic points in K form an infinite set and $f|K$ is topologically mixing.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Li and Yorke [4] originally introduced the notion of chaos for continuous self-maps of the interval $I = [0,1]$, and show that if a continuous map $f : I \to I$ has a periodic point with period 3 then the following condition (*) is satisfied.

(*) There exists an uncountable subset C of I such that for any two different points y_1 and y_2 of C

$$\liminf_{i\to\infty} |f^i(y_1) - f^i(y_2)| = 0 \text{ and } \limsup_{i\to\infty} |f^i(y_1) - f^i(y_2)| > 0$$

i.e., there exist two increasing sequences $\{m_i\}$ and $\{k_i\}$ of positive integers such that

$$\lim_{i\to\infty} f^{m_i}(y_1) = \lim_{i\to\infty} f^{m_i}(y_2) \text{ and } \lim_{i\to\infty} f^{k_i}(y_i) \neq \lim_{i\to\infty} f^{k_i}(y_2)$$

Definition 1 A continuous map $f : I \to I$ is said to be chaotic in the sense of Li and Yorke if the above condition (*) is satisfied.

A theorem of Sarkovski [7] (see also [1]) guarantees that f^m has a periodic point with period 3 for some $m > 0$ if the continuous map $f : I \to I$ has positive topological entropy (equivalently, if f has a periodic point with period which is not a power of 2 (see [6])). Therefore, the following theorem A works.

Theorem A [4] Every continuous self-map of the interval I with positive topological entropy is chaotic in the sense of Li and Yorke.

Some other conditions characterizing chaos of interval self-maps are given in [2] and [8].

The main aim of the present paper is to show that a continuous self-map of the interval I with positive entropy has a chaoticity more complicated than the chaoticity in the sense of Li and Yorke.

Definition 2 Suppose $f : X \to X$ is continuous, where X is a topological space f is said to be topologically mixing if for any two non-empty open sets U and V of X there exists $N > 0$ such that $f^{-n}(U) \cap V \neq \emptyset$ for every $n \geq N$.

Definition 3 A continuous map $f : I \to I$ is said to be strongly chaotic if there exists a closed subset K of I invariant with respect to f such that

(1) the set of periodic points is dense in K.

(2) the periods of periodic points in K form an infinite set, and

(3) $f|K$ is topologically mixing.

To explain the highly complicated chaoticity of a topologically mixing self–map we quote a theorem (see Theorem B below) from [9].

Definition 4 A subset Y of a topological space X is said to be everywhere uncountable if for every non–empty open set U of X we have $U \cap Y$ is uncountable.

Theorem B [9] Suppose $f : X \to X$ is continuous, where X is a compact metric space having infinitely many points. Then f is topologically mixing if and only if for any increasing sequence $\{q_i\}$ of positive integers and any countable dense subset S of X there exists everywhere an uncountable subset C of X satisfying the following conditions.

(1) For any $s \in S$ there exists a subsequence $\{m_i\}$ of the sequence $\{q_i\}$ such that $\lim_{i\to\infty} f^{m_i}(y) = s$ for every $y \in C$.

(2) For any $n > 0$, any n distinct points $y_1, y_2, \dots y_n$ of C and any n points $x_1, x_2, \dots x_n$ of X there exists a subsequence $\{k_i\}$ of the sequence $\{q_i\}$ such that $\lim_{i\to\infty} f^{k_i}(y_j) = x_j$ for every $j = 1, 2, \dots, n$.

In this paper we show the following.

Theorem Suppose $f : I \to I$ is continuous. Then f has a periodic point with odd period > 1 if and only if f is strongly chaotic.

Therefore, f has a periodic point with period $2^n \cdot d$, where $d > 1$ is odd, if f^{2^n} is strongly chaotic.

Corollary 1 Suppose $f : I \to I$ is continuous. Then the following conditions are equivalent.

(1) f has positive topological entropy.

(2) f^{2^n} is strongly chaotic for some $n > 0$.

(3) There is an uncountable subset C of I densen in itself such that for every $y \in C$ the set of limit points of the sequence $\{f^{2^i}(y)\}$ is exactly the closure $\bar{C}$ of C.

(4) There is a point $x \in I$ such that the set of limit points of the sequence $\{f^{2^i}(x)\}$ contains at least 3 distinct points.

Corollary 2 Suppose $f : I \to I$ is a continuous map with positive topological entropy. Then for any increasing sequence $\{q_i\}$ of positive integers there exists an uncountable subset C of I such that for any two different points y_1 and y_2 of C and any periodic point p of f

$$\lim_{i\to\infty} \inf |f^{q_i}(y_1) - f^{q_i}(y_2)| = 0, \ \liminf_{i\to\infty} |f^{q_i}(y_1) - f^{q_i}(y_2)| > \delta,$$

and

$$\limsup_{i\to\infty} |f^{q_i}(y_1) - f^{q_i}(p)| > \delta$$

where $\delta > 0$ is a constant.

Especially, there exists an uncountable set C of I such that for any two different points y_1 and y_2 of C and any periodic point p of f,

$$\liminf_{i\to\infty} |f^{2^i}(y_1) - f^{2^i}(y_2)| = 0, \ \limsup_{i\to\infty} |f^{2^i}(y_1) - f^{2^i}(y_2)| > \delta$$

and

$$\limsup_{i\to\infty} |f^{2^i}(y_1) - f^{2^i}(p)| > \delta$$

where $\delta > 0$ is a constant.

We put the proofs of the results above in Sec. 3.

2. PRELIMINARIES

Let $Y = \{1, 2, \dots, n\}$ with the discrete topology and let $\sum_n = \Pi_{i=1}^{\infty} Y_i$ with the product topology, where $Y_i = Y$ for every $i > 0$. The topological space $\sum_n$ is compact and metrizable and a metric d on $\sum_n$ is given by

$$d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|/2^i$$

where $x = x_1 x_2 \dots$ and $y = y_1 y_2 \dots$ are in $\sum_n$. The shift $\sigma : \sum_n \to \sum_n$ is defined by $\sigma(x_1 x_2 \dots) = x_2 x_3 \dots$, where $x_1 x_2 \dots \in \sum_n$.

For a given $n \times n$ matrix $A = (a_{ij})$ consisting of 0's and 1's let $\sum_n(A)$ be the subsets of all $x_1 x_2 \dots \in \sum_n$ such that $a_{x_i x_{i+1}} = 1$ for every $i > 0$. It is clear that $\sum_n(A)$ is a closed subset of $\sum_n$ invariant with respect to σ. The map $\sigma_A = \sigma|\sigma_n(A) : \sum_n(A) \to \sum_n(A)$ is called a subshift of finite type determined by the matrix A.

Lemma 1 Suppose A is an $n \times n$ matrix consisting of 0's and 1's. Then the subshift σ_A of finite type determined by the matrix A is topologically mixing if and only if there exists $N > 0$ such that each coefficient of the matrix $A^m = \underbrace{A \times A \times \dots \times A}_{m \ times}$ is positive for every $m \geq N$.

For proof see [5, pp. 71-72].

Suppose $f : I \to I$ is continuous. Let $J_1, J_2, \dots J_n$ be n non–trivial closed subintervals of I whose interiors are disjoint. We will call an $n \times n$ matrix $A = (a_{ij})$ consisting of 0's and 1's a covering matrix with respect to the intervals $J_1, J_2, \dots J_n$ if $a_{ij} = 1$ implies $f(J_i) \supset J_j$. A matrix A is said to be a covering matrix of f if it is a covering matrix with respect to some subintervals of I.

Lemma 2 If a continuous map $f: I \to I$ has a periodic point with odd period > 1 then there exists an even number $n > 0$ and a covering $n \times n$ matrix A such that

(1) the set of periodic points of σ_A is dense in $\sum_n(A)$,

(2) the periods of periodic points of σ_A form an infinite set, and

(3) σ_A is topologically mixing.

Proof From Lemma 2.1 in [1, pp. 22–24] it follows that for some even number $n > 0$ there exists n non-trivial subintervals $I_1, I_2, \ldots, I_n$ of I whose interiors are disjoint such that $f(I_1) \supset I_1 \cup I_2$, $f(I_i) \supset I_{i+1}$ for $i = 2, 3, \ldots, n-1$, and $f(I_n) \supset I_1$. Therefore, f has a covering $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if and only if $(i,j) = (1,1)$, $(i,j) = (i, i+1)$ for $i = 2, 3, \ldots, n-1$, or $(i,j) = (n, 1)$. We now prove that the matrix A is required.

Suppose $x_1 x_2 \ldots x_m$ is a finite sequence, where $x_i \in \{1, 2, \ldots, n\}$ such that $a_{x_i x_{i+1}} = 1$ for $i = 1, 2, \ldots, m-1$ and $a_{x_m x_1} = 1$. We denote $(x_1 x_2 \ldots x_m)^\infty$ the sequence constructed by repeating infinitely the sequence $x_1 x_2 \ldots x_m$, i.e.,

$$(x_1 x_2 \ldots x_m)^\infty = \underbrace{x_1 x_2 \ldots x_m}\ \underbrace{x_1 x_2 \ldots x_m}\ \underbrace{x_1 x_2 \ldots x_m} \cdots$$

It is easy to see that $(x_1 x_2 \ldots x_m)^\infty$ is a periodic point of σ_A. We now verify the condition (1)–(3) of this lemma.

(1) Suppose $x = x_1 x_2 \ldots \in \sum_n(A)$. For any $N > 0$

$$x' = (x_1 x_2 \ldots x_N (x_N + 1)(x_N + 2) \ldots n12 \ldots (x_1 - 1))^\infty$$

is a periodic point of σ_A such that $d(x, x') < \frac{1}{2^N}$. Hence, the set of periodic point of σ_A is dense.

(2) For each $m \geq n$ the periodic point $(234 \ldots n \underbrace{11 \ldots 1}_{(m-n+1)times})^\infty$ of σ_A has period m. Hence the set of periods of periodic points is infinite.

(3) Let $A^m = (a_{ij}^{(m)})$. Then for $m \geq 2n$

$$\begin{aligned} a_{ij}^{(m)} &= \sum_{k_1, k_2, \ldots, k_{m-1}} a_{ik_1} \cdot a_{k_1 k_2} \cdot \ldots \cdot a_{k_{m-1} j} \\ &\geq a_{i(i+1)} \cdot a_{(i+1)(i+2)} \cdot \ldots \cdot a_{(n-1)n} \cdot a_{n1} \cdot a_{11} \cdot \ldots \cdot a_{11} \cdot a_{12} \cdot \ldots \cdot a_{(j-1)j} > 0 \end{aligned}$$

By Lemma 1, σ_A is topologically mixing.

Lemma 3 Suppose $f: I \to I$ is continuous. If $f([a,b]) \supset [c,d]$ for two non-trivial subintervals $[a,b]$ and $[c,d]$ of I, then there exists a non-trivial closed interval $[a', b'] \subset [a,b]$ such that $f([a',b']) = [c,d]$ and $f((a',b')) = (c,d)$. Consequently, $f(\{a',b'\}) = \{c,d\}$.

Proof Since $f([a,b]) \supset [c,d]$ there are two points $x, y \in [a,b]$ such that $f(x) = a$ and $f(y) = b$. Without loss of generalities suppose $x < y$. Let $b' = \min\{y' \in [x,y] | f(y') = d\}$ and $a' = \max\{x' \in [x, b'] | f(x') = a\}$. Then, $[a', b']$ is required.

Lemma 4 Suppose $f: I \to I$ is continuous and $A = (a_{ij})$ is a covering $n \times n$ matrix with respect to n non-trivial subintervals $J_1, J_2, \ldots, J_n$ of I whose interiors are disjoint. Then there exists a correspondence D which, for each point x of $\sum_n(A)$, determines a closed subinterval $D(x)$ of I satisfying the following conditions

(1) $f(D(x)) = D(\sigma_A(x))$ for every $x \in \sum_n(A)$

(2) D is at most two-to-one, and if $x, x' \in \sum_n(A)$ with $x \neq x'$ then the interiors of $D(x)$ and $D(x')$ are disjoint,

(3) the set $\{x \in \sum_n(A) | \|D(x)\| > 0\}$ is countable, where $\|D(x)\|$ denotes the length of the interval $D(x)$.

(4) If $\{x^i\}$ is a sequence of points of $\Sigma_n(A)$ such that $\lim_{i \to \infty} x^i = x$, then $\lim_{x \to \infty} p(D(x^i), D(x)) = 0$, where $p(D(x^i), D(x))$ denotes the distance between two intervals $D(x^i)$ and $D(x)$

(5) $f(\partial(D(x)) = \partial(f(D(x))$ for every $x \in \sum_n(A)$, where $\partial(D(x))$ and $\partial(f(D(x))$ denotes the sets of end points of the intervals $D(x)$ and $f(D(x))$ respectively.

(6) if $x \in \sum_n(A)$ is a periodic point of σ_A with period m then each point of $\partial(D(x))$ is a periodic point of f with period $\geq m/2$.

Proof For the matrix A, a finite sequence $x_1 x_2 \ldots x_m$, where $x_i \in \{1, 2, \ldots n\}$, is called A-sequence with length m if $m = 1$ or if $m > 1$ and $a_{x_i x_{i+1}} = 1$ for every $i = 1, 2, \ldots, m-1$.

For each A-sequence x_1 with length 1, let $J(x_1) = J_{x_1}$. For $m > 1$ if $x_1 x_2 \ldots x_m$ is an A-sequence by Lemma 3 we choose, inductively, a non-trivial closed subinterval $J(x_1 x_2 \ldots x_m)$ of I such that $J(x_1 x_2 \ldots x_m) \subset J(x_1 x_2 \ldots x_{m-1})$, $f(J(x_1 x_2 \ldots x_m)) = J(x_2 x_3 \ldots x_m)$ and $f(\mathring{J}(x_1 x_2 \ldots x_m)) = \mathring{J}(x_2 x_3 \ldots x_m)$ where $\mathring{J}$ denotes the interior of an interval J. Consequently, we have $f(\partial(J(x_1, x_2 \ldots x_m))) = \partial(J(x_2 x_3 \ldots x_m))$.

We show the following claim first.

Claim If $x_1 x_2 \ldots x_m$ and $x'_1 x'_2 \ldots x'_m$ are two different A-sequences then the interiors of $J(x_1 x_2 \ldots x_m)$ and $J(x'_1 x'_2 \ldots x'_m)$ are disjoint.

We prove this claim by induction. For $m = 1$ the claim comes from the assumption on the intervals $J_1, J_2, \ldots J_n$. Suppose for some $m > 0$ the claim is true. If for two different A-sequences $x_1 x_2 \ldots x_{m+1}$ and $x'_1 x'_2 \ldots x'_{m+1}$ the intervals $J(x_1 x_2 \ldots x_{m+1})$ and $J(x'_1 x'_2 \ldots x'_{m+1})$ have a common interior point, then $J(x_1 x_2 \ldots x_m)$ and $J(x'_1 x'_2 \ldots x'_m)$

would have a common interior point, so that $x_j = x'_j$ for every $j = 1, 2, \dots m$ by inductive assumption. On the other hand, $\mathring{J}(x_2x_3\dots x_{m+1}) = f(\mathring{J}(x_1x_2\dots x_{m+1})$ and $\mathring{J}(x'_2x'_3\dots x'_{m+1}) = f(\mathring{J}(x'_1x'_2\dots x'_{m+1})$ would also have a common point, so that $x_{m+1} = x'_{m+1}$. Hence $x_1x_2\dots x_{m+1} = x'_1x'_2\dots x'_{m+1}$, a contradiction. Therefore, the claim works for $m+1$. By induction the claim is proved.

We now define the correspondence D as follows. For each $x = x_1x_2\dots \in \sum_n(A)$, $x_1x_2\dots x_i$ is an A-sequence for every $i > 0$ and we have $J(x_1) \supset J(x_1x_2) \supset J(x_1x_2x_3) \supset \dots$ Let $D(x) = \cap_{i=1}^{\infty} J(x_1x_2\dots x_i)$. $D(x)$ is a closed subinterval of I. It is clear that $\lim_{i\to\infty} \|J(x_1x_2\dots x_i)\| = \|D(x)\|$.

We now show that the correspondence D satisfies the conditions (1)–(6) of this lemma.

(1) Suppose $x = x_1x_2\dots$ is a point of $\sum_n(A)$. Then

$$\begin{aligned} f(D(x)) &= f(\cap_{i=1}^{\infty} J(x_1x_2\dots x_i)) \\ &= f(\cap_{i=2}^{\infty} J(x_1x_2\dots x_i)) \\ &\subset \cap_{i=2}^{\infty} f(J(x_1x_2\dots x_i)) \\ &= \cap_{i=2}^{\infty} J(x_2x_3\dots x_i) = D(\sigma_A(x)) \end{aligned}$$

On the other hand if $y \in D(\sigma_A(x))$ then for each $i > 1$ there exists $y_i \in J(x_1x_2\dots x_i)$ such that $f(y_i) = y$. Let y' be a limit point of the sequence $\{y_i\}$. It is easy to see that $y' \in D(x)$ and $f(y') = y$. Hence $f(D(x)) \supset D(\sigma_A(x))$.

(2) If $x = x_1x_2\dots, x' = x'_1x'_2\dots$ and $x'' = x''_1x''_2\dots$ are three distinct points of $\Sigma_n(A)$, then $x_1x_2\dots x_m, x'_1x'_2\dots x'_m$ and $x''_1x_2\dots x''_m$ are distinct for some m large enough, so that $J(x_1, x_2\dots x_m) \cap J(x'_1x'_2\dots x'_m) \cap J(x''_1x''_2\dots x''_m)$ is empty by the claim above. Hence, $D(x) = D(x') = D(x'')$ is not true. This proves the first statement of condition (2).

If $x = x_1x_2\dots$ and $x' = x'_1x'_2\dots$ are two points of $\sum_n(A)$ such that $D(x)$ and $D(x')$ have a common interior point then $J(x_1x_2\dots x_i)$ and $J(x'_1, x'_2\dots x'_i)$ have a common interior point for every $i > 0$, so that $x_1x_2\dots x_i = x'_1x'_2\dots x'_i$. Hence, we have $x = x'$. This proves the second statement of the condition (2).

(3) Since every family of disjoint open subintervals of I is countable, the family $\{D(x) | \|D(x)\| > 0,\ x \in \sum_n(A)\}$ and the set $\{x \in \sum_n(A) | \|D(x)\| > 0\}$ are also countable by (2).

(4) Suppose $\{x^i\}$ is a sequence of points of $\sum_n(A)$ such that $\lim_{i\to\infty} x^i = x$, where $x^i = x^i_1x^i_2\dots$ and $x = x_1x_2\dots$. Recall that $\lim_{i\to\infty} \|J(x_1x_2\dots x_i)\| \Rightarrow \|D(x)\|$. Given $\varepsilon > 0$, choose $N > 0$ such that $\|J(x_1x_2\dots x_N)\| - \|D(x)\| < \varepsilon$. Then, choose $M > 0$ such that for every $i > M$ we have $d(x^i, x) < \frac{1}{2^N}$, so $x^i_1x^i_2\dots x^i_N = x_1x_2\dots x_N$. Hence, $D(x^i)$ and $D(x)$ are contained in $J(x_1x_2\dots x_N)$ for every $i > N$. Then we have $\rho(D(x^i), D(x)) \leq \|J(x_1x_2\dots x_N)\| - \|D(x)\| < \varepsilon$ for every $i > N$ and $\lim_{i\to\infty} \rho(D(x^i), D(x)) = 0$.

(5) Suppose $x = x_1x_2\dots$ is a point of $\sum_n(A)$. If the f-image of an end point of $D(x)$ is an interior point of $f(D(x))) = D(\sigma_A(x)))$, then for i large enough there is an end point of $J(x_1x_2\dots x_i)$ whose f-image is an interior point of $J(x_2x_3\dots x_i)$. This is a contradiction, so $f(\partial(D(x)) \subset \partial(f(D(x))$. On the other hand, an end point of $D(\partial_A(x))$ is a limit point of a sequence of end points of $J(x_2x_3\dots x_i)$, so that it is an f-image of a limit point, which is an end point of $D(x)$, of a sequence of end points of $J(x_1x_2\dots x_i)$. Hence $\partial(D(\partial_A(x))) = \partial(f(D(x))) \subset f(\partial D(x)))$.

(6) Suppose $x \in \sum_n(A)$ is a periodic point of σ_A with period m. By (1) we have $f^m(D(x)) = D(\sigma_A^m(x)) = D(x)$. Hence, if $D(x)$ is a singleton, then it is a periodic point with period, say d. If $d < m/2$, then $D(x) = f^d(D(x)) = f^{2d}(D(x))$, so that $D(x) = D(\sigma_A^d(x)) = D(\sigma_A^{2d}(x))$, where $x, \sigma_A^d(x)$ and $\sigma_A^{2d}(x)$ are distinct, a contradiction with (2). Hence, $d \geq m/2$. Suppose $D(x) = [y, y']$, where $y < y'$. It follows from (5) and $f^m(D(x)) = D(x)$ that either $f^m(y) = y$ and $f^m(y') = y'$ or $f^m = y'$ and $f^m(y') = y$, so that in both cases we have $f^{2m}(y) = y$ and $f^{2m}(y') = y'$. Hence y and y' are periodic points of f with periods, say d and d' respectively. We now prove $d, d' \geq m/2$. If not, suppose without loss of generality that $d \leq m/2$. In this case, the intervals $D(x), f^d(D(x)) = D(\sigma_A^d(x))$ and $f^{2d}(D(x)) = D(\sigma_A^{2d}(x))$, which are non-trivial by (5), have a common end point y. Hence, there are two among the three $D(x), D(\sigma_A^{2d}(x))$ and $D(\sigma_A^{2d}(x))$ which have a common interior point, a contradiction with (2).

Lemma 4 is proved.

3. PROOF OF RESULTS

We need one more Lemma. (For proof see [3])

Lemma 5 Suppose $f : I \to I$ is continuous. Then f has a periodic point with odd period > 1 if and only if there is a point $x \in I$ and an odd number $n > 1$ such that either $f^n(x) \leq x < f(x)$ or $f(x) < x \leq f^n(x)$.

Proof of Theorem

Necessity If f has a periodic point with odd period > 1, then by Lemma 2 for some $n > 0$ f has a covering $n \times n$ matrix $A = (a_{ij})$ such that the subshift $\sigma_A : \sum_n(A) \to \sum_n(A)$ of finite type satisfying the conditions (1), (2) and (3) in Lemma 2, and by Lemma 4 there exists a correspondence D which, for each x of $\sum_n(A)$, determines a closed subinterval $D(x)$ of I, satisfying the conditions (1)–(6) in Lemma 4.

Let $P = \{x \in \sum_n(A) | \|D(x)\| > 0\}$ and $Q = \sum_n(A) - \cup_{i=-\infty}^{+\infty} \sigma_A^i(P)$. By Lemma

4 (3) the set P is countable, so that the set $U_{i=-\infty}^{+\infty}\sigma_A^i(P)$ is also countable, because σ_A is at most an n–to–one map. Hence, Q is the complement of a countable subset of $\sum_n(A)$. It is easy to see that Q is invariant with respect to σ_A. We have

Claim 1 $\sigma_A|Q$ is topologically mixing.

To show this claim let U and V be two non–empty sets open in Q, and let $\tilde{U}$ and $\tilde{V}$ be two sets open in $\sum_n(A)$ such that $\tilde{U} \cap Q = U$ and $\tilde{V} \cap Q = V$. Since σ_A is topologically mixing by Lemma 2 (3), there is $N > 0$ such that $f^i(\tilde{U}) \cap \tilde{V} \neq \phi$ for every $i > N$. It follows from Theorem B that every non–empty open set of $\sigma_n(A)$ is uncountable, so is the non–empty open set $f^{-i}(\tilde{U}) \cap \tilde{V}$. Hence, $f^{-i}(U) \cap V = f^{-i}(\tilde{U}) \cap \tilde{V} \cap Q \neq \emptyset$, because Q is the complement of a countable subset.

Let $L = U_{x \in Q} D(x)$. Recall that for each $x \in Q$ the interval $D(x)$ is a singleton, so that D can be regarded as a map from Q to L. Then, it follows from Lemma 4 the following

Claim 2 (1) the map $D : Q \to L$ is continuous and at most two–to–one.

(2) The subset L of I is invariant with respect to f, and

(3) $D \circ (\sigma_A|Q) = (f|L) \circ D$.

If U and V are two non–empty sets open in L, then by Claim 2(3) for each $i > 0$ we have

$$(\sigma_A|Q)^{-i}(D^{-1}(U)) \cap D^{-1}(V) = D^{-1}((f|L)^{-i}(U) \cap V)$$

Hence $(\sigma_A|Q)^{-i}(D^{-1}(U)) \cap D^{-1}(V) \neq \phi$ implies $(f|L)^{-i}(U) \cap V \neq \phi$. $\sigma_A|Q$ is topological mixing, so is $f|L$.

To complete the proof of necessity part it is sufficient to verify that the closed subset $K = \bar{L}$ of I satisfies the conditions (1), (2) and (3) of Definition 3.

Obviously, K is invariant with respect to f. Since every non–empty set open in K contains a non–empty set open in L and $f|L$ is topologically mixing it follows that $f|K$ is topologically mixing.

We now claim that $\partial D(x) \cap K \neq \emptyset$ for each $x \in \sum_n(A)$. The reason is that if $x \in Q$ then $D(x) \in L \subset K$ and if $x \bar{\in} Q$, then by Lemma 4(4) for a sequence $\{x^i\}$ of points of Q such that $\lim_{i \to \infty} x^i = x$ the sequence $\{D(x)\}$ has at least one limit point in $\partial(D(x)) \cap K$.

By Lemma 4 (6) if x is a periodic point of σ_A with period m then there exists a periodic point of f in $\partial(D(x)) \cap K$ with period $\geq \frac{m}{2}$. Consequently, by Lemma 2 (2) the periods of periodic points in K form an infinite set.

Suppose y is a point of K. Given $\varepsilon > 0$, there is $y' \in L$ such that $|y - y'| < \varepsilon/2$. Let $x' \in Q$ such that $D(x') = y'$. By Lemma 2(1) and Lemma 4 (1 and 3) we can choose a periodic point p of σ_A such that $|t - y| < \varepsilon/2$ for each $t \in D(p)$. There is a periodic point, say q, of f in $D(p) \cap K$. Then, we have $|q - y| < \varepsilon$. This shows that the set of periodic points of $f|K$ is dense in K.

Sufficiency Suppose $f : I \to I$ is strongly chaotic and $K \subset I$ is a closed subset of I, invariant with respect to f, satisfying the conditions (1), (2) and (3) in Definition 3. By Definition 3 (2) K has infinitely many points, and by Definition 3(3) $f|K$ is topologically mixing. Let S be a countable subset dense in K containing the maximum b and the minimum b' of K and let $\{q_i\}$ be the sequence such that $q_i = 2i - 1$ for every $i > 0$. By Theorem B there is a point y of K with $b' < y < b$ such that $\lim_{i \to \infty} f^{k_i}(y) = b$ and $\lim_{i \to \infty} f^{k'_i}(y) = b'$ for some two subsequences $\{k_i\}$ and $\{k'_i\}$ of the sequence $\{q_i\}$. Since k_i and k'_i are odd numbers it follows from Lemma 5 that f has a periodic point with odd period > 1 in any case either $y < f(y)$ or $f(y) < y$.

Proof of Corollary 1 (1) $\Rightarrow$ (2). By the above theorem and a theorem of Misiurewicz mentioned in Sec. 1 (see [6])

(2) $\Rightarrow$ (3) By Theorem B

(3) $\Rightarrow$ (4) Obviously

(4) $\Rightarrow$ (1) Suppose $y_1 < y_2 < y_3$ are three limit points of the sequence $\{f^{2^i}(x)\}$ for some $x \in I$. Then there are $i_0 < i_1 < i_2$ such that $z_2 < z_0 < z_1$, where $z_j = f^{2^{i_j}}(x)$, $j = 0, 1, 2$. Let $g = f^{2^{i_0}}$ and $u = z_0$. Then, we have $z_2 = g^{\ell_2}(z_0) < z_0 < z_1 = z^{\ell_1}(z_0)$, where $\ell_1 = 2^{i_1 - i_0} - 1$ and $\ell_2 = 2^{i_2 - i_0} - 1$ are odd. By Lemma 5 g has a periodic point with odd period > 1, so that f has a periodic point with period which is not a power of 2. Hence the topological entropy of f is positive (see [6]).

Proof of Corollary 2 Suppose f has a periodic point with period $2^n \cdot d$, where d is odd > 1. Suppose $\{q_i\}$ is an increasing sequence of positive integers. For each i let $q_i = p_i 2^n + r_i$, where $0 \leq r_i < 2^n$. Since r_i's take finitely many values, the sequence $\{q_i\}$ has a subsequence for which each element takes a constant $r_i = r$. Hence, without loss of generality, we suppose that $q_i = p_i 2^n + r$ for each $i > 0$. (If not, substitute a suitable subsequence for the original sequence.)

In the case $r = 0$ this Corollary is an immediate consequence of the Theorem. Hence for the sequence $\{(p_i + 1)2^n\}$ there is an uncountable subset C satisfying this Corollary. If $r \neq 0$, then it is easy to see that the subset $f^{2^n - r}(C)$ is required.

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