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ABSTRACT

In [2], R.Bott and J.Milnor proved that the sphere S^n is a parallelizable manifold if and only if $n = 1,3$ or 7. In [1], J.F.Adams proved above result and proved that S^n is an H-space if and only if $n = 0,1,3$, or 7. Because a Lie group must necessarily be a parallelizable manifold and also an H-space, naturally ones ask that S^n is a Lie group for $n = 0$, 1,3, or 7 ? In this paper we prove that S^7 is not a Lie group, and it is not even a topological group. Therefore, S^n is a Lie group (or a topological group) if and only if $n = 0,1,3$.

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It is known that an n-dimensional Lie group G is a parallelizable manifold (The left invariant vector fields $X_1, ..., X_n$ determined by a base $X_1(e), ..., X_n(e)$ of unit element $e \in G$ are C^∞ basic vector fields of G) and an H-space, where an H-space is a space which admits a continuous product with unit, that is a continuous mapping $\mu : G \times G \to G$ with a point $e \in G$ such that $\mu(x,e) = \mu(e,x) = x$ for any $x \in G$. In [2], R.Bott and J.Milnor proved that the sphere S^n is a parallelizable manifold if and only if $n = 1,3$ or 7. In [1], J.F.Adams proved that S^n is an H-space if and only if $n = 0,1,3$ or 7. A natural question is whether S^n is a Lie group for $n = 0,1,3$, or 7? At first we will prove that S^7 is not a Lie group and even it is not a topological group, then we will prove that S^n is a Lie group (or topological group) if and only if $n = 0,1$, or 3. We give several examples of parallelizable manifolds and H-spaces.

Example 1 It is easy to prove that $S^1 = \{(x_1,x_2) \in R^2 | x_1^2 + x_2^2 = 1\} = \{x_1 + ix_2 \in C = R^2 | x_1^2 + x_2^2 = 1\}$ with a multiplication μ is a Lie group, of course it is a parallelizable manifold and an H-space, where C is the complex field and $\mu : C \times C \to C$, $\mu(x_1 + ix_2,\ y_1 + iy_2) = (x_1 + ix_2)\cdot(y_1+iy_2) = (x_1y_1 - x_2y_2)+i(x_1y_2 + x_2y_1)$. For any $(x_1,x_2) \in S^1$, set vector field $X_{(x_1,x_2)} = (-x_2, x_1)$, then X is a C^∞ basic vector field on S^1, therefore S^1 is a parallelizable manifold.

Example 2 $S^3 = \{(x_1,x_2,x_3,x_4) \in R^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\} = \{x_1 + ix_2 + jx_3 + kx_4 \in Q = R^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, where a quaternion field Q is an algebra of dimension 4 over the field R of real numbers, with a base composed of 4 elements 1, i, j, k whose multiplication table is given by the following formulas :

$1\cdot i = i\cdot 1 = i$, $1\cdot j = j\cdot 1 = j$, $1\cdot k = k\cdot 1 = k$, $i^2 = j^2 = k^2 = -1$, $i\cdot j = -j\cdot i = k$, $i\cdot k = -k\cdot i = -j$, $j\cdot k = -k\cdot j = i$.

Let $\mu : Q\times Q \to Q$, $\mu(x_1 + ix_2 + jx_3 + kx_4, y_1 + iy_2 + jy_3 + ky_4) = (x_1 + ix_2 + jx_3 + kx_4)\cdot(y_1 + iy_2 + jy_3 + ky_4) = (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4) + i(x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3) + j(x_1y_3 + x_3y_1 + x_4y_2 - x_2y_4) + k(x_1y_4 + x_4y_1 + x_2y_3 - x_3y_2)$.

By simple computation, we know that S^3 with multiplication μ is a Lie group (also is a parallelizable manifold and an H-space). For any $(x_1,x_2,x_3,x_4) \in S^3$, set vector fields $X^1_{(x_1,x_2,x_3,x_4)} = (-x_2, x_1, -x_4, x_3)$, $X^2_{(x_1,x_2,x_3,x_4)} = (-x_3, x_4, x_1, -x_2)$, $X^3_{(x_1,x_2,x_3,x_4)} = (-x_4, -x_3, x_2, x_1)$, then $\{X^1, X^2, X^3\}$ is a C^∞ basic vector field on S^3, therefore S^3 is a parallelizable manifold.

Example 3 $S^7 = \{(x_1, ..., x_8) \in R^8 | \sum_{l=1}^{8} x_l^2 = 1 \} = \{x_1 + ix_2 + jx_3 + kx_4 + ex_5 + ie \cdot x_6 + je \cdot x_7 + ke \cdot x_8 \in K = R^8 | \sum_{l=1}^{8} x_l^2 = 1\}$, where the Caley number field $K = \{p + qe = (x_1 + ix_2 + jx_3 + kx_4) + (x_5 + i \cdot x_6 + j \cdot x_7 + k \cdot x_8)e | x_l \in R,\ l = 1, ..., 8\}$ is an algebra of dimension 8 over the field R of real numbers with a base composed of 8 elements 1, i, j, k, e, ie, je, ke, whose multiplication table is given by the following formulas : for any p, $q \in Q$, note $\bar{q} = \overline{x_5 + ix_6 + jx_7 + kx_8} = x_5 - ix_6 - jx_7 - kx_8$, then

$p \cdot q = pq$; $p \cdot e = pe$; $p \cdot (qe) = (qp)e$; $e \cdot q = \bar{q}e$; $e \cdot e = -1$; $e \cdot (qe) = -\bar{q}$; $(pe) \cdot q = (p\bar{q})e$; $(pe) \cdot e = -p$; $(pe) \cdot (qe) = -\bar{q}p$.

Let $\mu : K \times K \to K$, $\mu(p_1 + p_2 e, q_1 + q_2 e) = (p_1 + p_2 e) \cdot (q_1 + q_2 e) = (p_1 q_1 - \bar{q}_2 p_2) + (q_2 p_1 + p_2 \bar{q}_1)e$. From $(i \cdot j) \cdot e = ke \neq -ke = (ji) \cdot e = i \cdot (j \cdot e)$, it follows that the above multiplication of K is not associative. Therefore, S^7 is an H-space, but the multiplication of K does not give the construction of a Lie group.

For any $(x_1, ..., x_8) \in S^7$, set vector fields

$X_x^1 = (-x_2, x_1, -x_4, x_3, -x_6, x_5, x_8, -x_7)$,
$X_x^2 = (-x_3, x_4, x_1, -x_2, -x_7, -x_8, x_5, x_6)$,
$X_x^3 = (-x_4, -x_3, x_2, x_1, -x_8, x_7, -x_6, x_5)$,
$X_x^4 = (-x_5, x_6, x_7, x_8, x_1, -x_2, -x_3, -x_4)$,
$X_x^5 = (-x_6, -x_5, x_8, -x_7, x_2, x_1, x_4, -x_3)$,
$X_x^6 = (-x_7, -x_8, -x_5, x_6, x_3, -x_4, x_1, x_2)$,
$X_x^7 = (-x_8, x_7, -x_6, -x_5, x_4, x_3, -x_2, x_1)$,

then $\{X^1, ..., X^7\}$ is a C^∞ basic vector field on S^7, therefore, S^7 is a parallelizable manifold.

The basic vector fields in above three examples are given respectively by the multiplications of C, Q and K. For example, in K $X_x^1 = ix$, $X_x^2 = jx$, $X_x^3 = kx$, $X_x^4 = e \cdot x$, $X_x^5 = (ie) \cdot x$, $X_x^6 = (je) \cdot x$, $X_x^7 = (ke) \cdot x$.

To prove that S^7 is not a Lie group, we need use the following 8 Lemmas.

Lemma 1 A connected compact Lie group G is a multiplicative product of a center $C(G)$ and a connected simple normal subgroup $G_1, ..., G_s$, i.e. $G = C(G) \cdot G_1 \cdots G_s$, where the subgroup $G^* = G_1 \cdots G_s$ of G is a connected semisimple normal subgroup, every G_i is non-commutative and this decomposition is unique under not counting their order. (See Yan Zhida and Xu Yichao [11] p.110, Theorem 2.2.7.)

Lemma 2 A connected commutative Lie group G is a topological product of a torus and an Euclidean space. That is if G is a complex Lie group, then G and $(C^m/Z^m + iZ^m) \times C^n$ are isomorphic; if G is a real Lie group, then G and $(R^m/Z^m) \times R^n$ are isomorphic. (See Yan Zhida and Xu Yichao [11] p.95, Theorem 1.8.7.)

Lemma 3 Compact simple Lie groups are of the following several types :

$A_l = SU(l+1)$, dim $A_l = l^2 + 2l$, $l \geq 1$;
$B_l = SO(2l+1, R)$, dim $B_l = 2l^2 + l$, $l \geq 2$;
$C_l = Sp(l)$, dim $C_l = 2l^2 + l$, $l \geq 3$;
$D_l = SO(2l, R)$, dim $D_l = 2l^2 - l$, $l \geq 4$

and G_2, F_4, E_6, E_7, E_8, dim $G_2 = 14$, dim $F_4 = 52$, dim $E_6 = 78$, dim $E_7 = 133$, dim $E_8 = 248$. (See Yan Zhida and Xu Yichao [11] pp.158 - 162.)

Lemma 4 Let G be a connected and simple connected Lie group, then any connected normal subgroup of G is closed. (See Yan Zhida and Xu Yichao [11] p.97, Theorem 1.8.8.)

Lemma 5 The factor space G/H of a Lie group G about its closed subgroup H is an analytic manifold, such that the natural projection p : $G \to G/H$ is analytic, and if H is normal, then G/H is a Lie group, which is called a factor group of G about its closed subgroup H. (See Yan Zhida and Xu Yichao [11] p.94, Theorem 1.8.3 and see G.Hochschild [6] p.94, Corollary 2.2.)

Lemma 6 (E.Cartan) Every closed subgroup H of a Lie group G is a Lie subgroup of G and the topology of H must be the induced topology. (See Yan Zhida and Xu Yichao [11] p.40, Theorem 1.4.1 and see G.Hochschild [6] p.92.)

Lemma 7 Let p : $E \to B$ be a fibre mapping, $x_0 \in p^{-1}(b_0) = F$, $b_0 \in B$, then there exists a natural homomorphism of groups $\partial : \pi_{n+1}(B, b_0) \to \pi_n(F, x_0)$, such that the following sequence is exact :

$\to \pi_{n+1}(E, x_0) \to \pi_{n+1}(B, b_0) \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \cdots \to \pi_1(B, b_0) \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to \pi_0(B, b_0)$,

where π_n is a homotopy group, $n = 0, 1, 2, \cdots$. (See R.M.Switzer [10] p.56.)

Lemma 8 Let H be a closed normal subgroup of the Lie group G, then the natural projection p : $G \to G/H$ is a fibre mapping. (See N.Steenrod

[9] pp.28-33.)

We now prove that S^7 is not a Lie group.

Theorem 1 S^7 is not a Lie group.

Proof Suppose that S^7 is a Lie group. Because S^7 is compact and connected, from Lemma 1 we know $S^7 = C(S^7) \cdot G_1 \cdots G_s$, where G_i is connected simple normal subgroup, $G^* = G_1 \cdots G_s$ is connected semisimple normal subgroups. Since S^7 is connected and simple connected, from Lemma 4 it follows that G_i and G^* are both closed subgroups. According to Lemma 6, G_i and G^* are Lie subgroups of G. Moreover, compactness of S^7 implies compactness of closed subgroups G_i and G^*. From Lemma 3 we know that the compact simple Lie group whose dimension is less than 7 must be $SU(2) \cong S^3$ (See C.Chevalley [4] pp.17-18), therefore $G_i \cong S^3$.

Note $S = S^7/G^*$. $p : S^7 \to S$ is a natural projection, then $S = p(S^7) = p(C(S^7))$ is a connected compact commutative Lie group. Hence from Lemma 2 it follows that $S \cong R^m/Z^m = T^m = \underbrace{S^1 \times \cdots \times S^1}_{m}$.

If S^7 is a Lie group, then it must one of the be following cases :

(1) $G^* = \{u\}$ (u is a unit element of G), $S = T^7$, then $S^7 \cong T^7$ and by p.61 in [9] , p52 in [10] we obtain

$0 = \pi_1(S^7) = \pi_1(T^7) = \pi_1(S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1) \cong \pi_1(S^1) \oplus \pi_1(S^1) \oplus \pi_1(S^1) \oplus \pi_1(S^1) \oplus \pi_1(S^1) \oplus \pi_1(S^1) \oplus \pi_1(S^1) = Z \oplus Z \oplus Z \oplus Z \oplus Z \oplus Z \oplus Z$, a contradiction.

(2) $G^* \cong S^3$, $S = T^4$, then by Lemmas 7 and 8, there is an exact sequence :

$$0 = \pi_1(S^7) \to \pi_1(T^4) \to \pi_0(S^3) = 0,$$

which implies $\pi_1(T^4) = 0$, but $\pi_1(T^4) = \pi_1(S^1 \times S^1 \times S^1 \times S^1) = Z \oplus Z \oplus Z \oplus Z \neq 0$, this is a contradiction.

(3) $G^* = S^3 \cdot S^3$. $S = T^l$ $(1 \leq l \leq 3)$. Using the second coordinate we know that the dimension of $S^3 \cdot S^3$ is 6 at least. By Lemmas 7 and 8, there is an exact sequence :

$$0 = \pi_1(S^7) \to \pi_1(T^l) \to \pi_0(S^3 \cdot S^3) = 0$$

which implies $\pi_1(T^l) = 0$, but $\pi_1(T^l) \neq 0$, this is a contradiction.

Remark 1 From J.Milnor [7] p.114 we know that $H^3(G) \neq 0$ if G is a non-abelian compact connected Lie group. (See E.Cartan, "La Topologie des Espaces Représentatives des Groups de Lie," Paris, Hermann, 1936).

Using the above result, ones can prove that S^7 is not a Lie group. In fact, assume that S^7 is a Lie group. If S^7 is a non- abelian compact connected Lie group, then $H^3(S^7) \neq 0$, which contradicts the fact that $H^3(S^7) = 0$ (See R.Bott and Loring W.Tu [3] pp.5 and 36); if S^7 is an abelian compact connected Lie group, it follows from Lemma 2 that $S^7 = T^7$. According to (1) in the proof of Theorem 1 we obtain a contradiction again.

Theorem 2 S^n is a Lie group if and only if $n = 0,1,3$.

Proof ($\Rightarrow$) Obviously, $S^0 = \{u, -u\}$ is considered as a 0-dimensional Lie group. By Examples 1 and 2, S^1 and S^3 are also Lie groups.

($\Leftarrow$) If S^n is a Lie group for $n \geq 1$, then S^n is a parallelizable manifold, from [1] and [2] it follows that $n = 1,3$ or 7. But Theorem 1 tells us that S^7 is not a Lie group, therefore $n = 1,3$.

Theorem 3 S^n is a topological group if and only if $n = 0,1,3$.

Proof ($\Leftarrow$) Since S^n is a Lie group, it is a topological group for $n = 0,1,3$.

($\Rightarrow$) Suppose that the manifold S^n is a topological group for $n \geq 1$. From A.M. Gleason [5], D. Montgomery and L. Zippin [8], we know that S^n is a Lie group. According to Theorem 2, $n = 1,3$.

Remark 2 Because $\pi_1(RP^7) = Z_2 \neq 0$, RP^7 is not a simple connected, where RP^7 is an n-dimensional real projective space. Therefore, we can not use Lemma 4 to prove that RP^7 is not a Lie group.

From R.Bott and Loring W.Tu [3] p.78 it follows that $H^3(RP^7) = 0$. As in Remark 1 we can obtain that RP^7 is not a Lie group. Therefore, it is easy prove that RP^n is a Lie group if and only if $n = 0$, 1,3 and RP^n is a topological group if and only if $n = 0$, 1,3.

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