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АНАЛИТИЧЕСКИЕ МУФАНГ-ПРЕОБРАЗОВАНИЯ

Р е з ю м е

Работа задумана как введение в концепцию аналитического бипредставления аналитической лупы Муфанг.

ANALYTIC MOUFANG-TRANSFORMATIONS

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The present paper is aimed to be an introduction to the concept of an analytic birepresentation of an analytic Moufang loop.

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Introduction. Groups are often said to be an algebraic abstraction of the notion of symmetry. As a slight modification of this, we have introduced in /1/ the notion of the *Moufang-symmetry*. The latter was defined as a hypothetic kind of symmetry associated with Moufang loops. By paraphrasing these words, one can also say that Moufang loops are an algebraic abstraction of the notion of the Moufang-symmetry.

By introducing such notion of symmetry, one finds oneself confronted with the question about its real meaning. If one feels like looking at world affairs from the viewpoint of the Moufang-symmetry, one needs a suitable mathematical machinery for the identification of this symmetry. As in case of groups, one really has to elaborate the representation theory of Moufang loops, and this is the logical way to get an answer to the question.

Guided by this idea, the notion of a *birepresentation of a Moufang* loop have been introduced and considered in /2/. In this paper, we make an attempt to push the problem a bit ahead and, after the presentation (Sec. 1-2) of some elementary notions and facts from the Moufang loops theory, introduce (Sec. 3-4) the notion of an *analytic birepresentation* (S,T) *of an analytic Moufang loop G*. Then, to describe the deviation of (S,T) from associativity, we define (Sec. 5), in full analogy with /1/, the associators of (S,T), and establish (Sec. 6-7) certain constraints for them, called the "minimality" conditions of (S,T).

The first-order minimality conditions of (S,T) read as the differential equations for (S,T), "minimally" generalizing the Lie equations from the Lie transformation groups theory. For the associative (S,T) they directly yield the familiar Lie equations.

The second-order minimality conditions of (S,T) read as a certain open-form set of commutation relations for the generators of (S,T). It turns out (Sec. 8) that these commutation relations can be closed and have the form known from /1/ as well as from the representation theory of alternative algebras /3,4/. As in case of the Lie transformation groups, these commutation relations do not really depend on the particular analytic birepresentation of G. For the associative (S,T) they directly split into two commuting Lie algebras.

<u>1. Moufang loops</u>. A *Moufang loop* /5,6/ is a set G with a binary operation (multiplication) denoted by juxtaposition so that the following axioms are satisfied:

1) in the equation gh = k the knowledge of any two of $g, h, k \in G$ specifies the third uniquely;

2) there is a distinguished element e of G which has the property that $eg=ge=g$ for all $g \in G$;

3) the Moufang identity

$$(gh)(kg) = g((hk)g) \tag{1.1}$$

holds for all $g, h, k \in G$.

A set G with such a binary operation that only the axioms 1) and 2) are satisfied is called a *loop*. An element e is called the *identity* element of the loop. Moufang loops are thus a particular kind of loops: the ones with the Moufang identity (1.1). The most familiar kind of loops are those with the *associative law* $(gh)k = g(hk)$ and they are called *groups*. It is also clear that groups are the associative Moufang loops. A (Moufang) loop is said to be *commutative* if $gh = hg$ for all $g, h \in G$. Only associative commutative (Moufang) loops are said to be *abelian*.

The unique solution of the equation $gx = e$ $(yg = e)$ is called the right (left) *inverse* element of g and denoted as g_R^{-1} (g_L^{-1}). It is a pleasant fact in the Moufang loops theory that the (inverse) elements of every Moufang loop G have the following good properties:

$$g_R^{-1} = g_L^{-1} = g^{-1} , \tag{1.2}$$

$$g^{-1}(gh) = (hg)g^{-1} = h , \tag{1.3}$$

$$(g^{-1})^{-1} = g , \tag{1.4}$$

$$(gh)^{-1} = h^{-1}g^{-1} , \quad g, h \in G, \tag{1.5}$$

Another remarkable property of Moufang loops is their *diassociativity*: the subloop generated by any two elements in a Moufang loop is associative (group). Hence, for any two elements g, h in a Moufang loop G, one has

$$(hg)g = hg^2 , \tag{1.6}$$

$$g(gh) = g^2h , \tag{1.7}$$

$$(gh)g = g(hg) . \tag{1.8}$$

We note also, that thanks to the *elasticity* (1.8), the Moufang identity (1.1) may be written in the symmetric form:

$$(gh)(kg) = g(hk)g .$$

2. Analytic Moufang loops. Let G be an r-dimensional real, analytic manifold which is also a Moufang loop so that both the Moufang loop operation $G \times G \to G$:

$$(g,h) \to gh \,, \qquad g,\, h \in G,$$

and the inversion $G \to G$:

$$g \to g^{-1} \,, \qquad g \in G,$$

are analytic maps between manifolds. Then G is said to be an *analytic Moufang loop*.

Since G is an analytic manifold, one can fix a local coordinate system (U,f) with the coordinate neighbourhood $U \subset G$ of the identity $e \in G$ and with the local coordinate map $f\colon U \to R^r$, which is supposed to be such that $f(e)=0$. Let $g,\, h \in U$ be such that $gh \in U$, and let us denote the coordinates of g, h, gh relative to our chart by g^i, h^i, $(gh)^i$, $i=1,\dots,r$, respectively. Then $(gh)^i$ are the analytic functions of $g^1,\dots,g^r,h^1,\dots,h^r$ and admit the Taylor expansions

$$(gh)^i = h^i + u^i_j(h)g^j + \dots \tag{2.1}$$

$$= g^i + v^i_j(g)h^j + \dots \tag{2.2}$$

$$= g^i + h^i + a^i_{jk}\, g^j\, h^k + \dots\,, \tag{2.3}$$

where the *auxiliary functions* u^i_j, v^i_j *of G* obey the initial conditions

$$u^i_j(e) = v^i_j(e) = \delta^i_j\,. \tag{2.4}$$

One can easily see that $\det(u^i_j) \neq 0$ and $\det(v^i_j) \neq 0$, from which it follows that (u^i_j) and (v^i_j) are invertible matrices.

The anti-symmetric constants

$$c^i_{jk} = a^i_{jk} - a^i_{kj} = -\,c^i_{kj} \tag{2.5}$$

are said to be the *structure constants* of G. By means of these one can, as in case of the Lie groups, introduce the notion of a tangent algebra of an analytic Moufang loop.

Let G be an analytic Moufang loop and let T_e be the tangent space of G at e. For any x, y $\in T_e$, their product $(x,y) \in T_e$ is defined in the component form by

$$(x,y)^i = c^i_{jk}x^j y^k = -(y,x)^i, \quad i = 1,\ldots,r \ . \qquad (2.6)$$

Thus the tangent space T_e of G at e can be considered as an anti-commutative algebra called the *tangent algebra* of G.

The tangent algebra T_e of G need not be a Lie algebra. In other words, there may be a triple x, y, z $\in T_e$, such that the Jacobi identity fails in T_e:

$$(x,(y,z)) + (y,(z,x)) + (z,(x,y)) \neq 0 \ .$$

Instead, one has for any x, y, z $\in T_e$ a more general identity /7/

$$((x,y),(z,x)) + (((x,y),z),x) + (((y,z),x),x) + (((z,x),x),y) = 0 \ , \qquad (2.7)$$

called the *Mal'tsev identity*. The tangent algebra T_e of G is hence said to be the *Mal'tsev algebra*. The Mal'tsev identity reads /8/

$$J(x,y,(x,z)) = (J(x,y,z),x) \ , \qquad (2.8)$$

where the Jacobian $J(x,y,z)$ of x, y, z $\in T_e$ is defined by

$$J(x,y,z) = (x,(y,z)) + (y,(z,x)) + (z,(x,y)).$$

From (2.8) it can be easily seen that every Lie algebra is the Mal'tsev algebra as well.

The reader can find further details on Mal'tsev algebras in /8-10/. Let us note only that the problem of reconstruction of an analytic Moufang loop with the given Mal'tsev algebra has been solved in the series of papers /11-14/, see also /15/.

3. Moufang-transformations. Let $\mathfrak{X}$ be a set and let $\mathfrak{T}(\mathfrak{X})$ be the transformation group of $\mathfrak{X}$, i.e. the group of the bijective maps of $\mathfrak{X}$ onto $\mathfrak{X}$. It is known that the multiplication in $\mathfrak{T}(\mathfrak{X})$ is defined as the composition of transformations and the identity element of $\mathfrak{T}(\mathfrak{X})$ coincides with the identity transformation E of $\mathfrak{X}$.

Let G be a Moufang loop. A pair (S,T) of the maps $g \to S_g$, $g \to T_g$ of G into the group $\mathfrak{T}(\mathfrak{X})$ is said to be an *action* of G on $\mathfrak{X}$ if the following conditions are satisfied:

1) $S_e = T_e = E$; (3.1)

2) $S_g T_g S_h = S_{gh} T_g$, (3.2)

3) $S_g T_g T_h = T_{hg} S_g$ (3.3)

for any g, h $\in$ G. The pair (S,T) of such maps is called also a *birepresentation* of G. The transformations S_g, T_g, g $\in$ G, are called *G-transformations* of $\mathfrak{X}$, or equivalently, its *Moufang-transformations*.

The birepresentation (S,T) of G is said to be *associative* if G-transformations of $\mathfrak{X}$ obey the conditions

$$S_g S_h X = S_{gh} X \, , \qquad (3.4)$$

$$T_g T_h X = T_{hg} X \, , \qquad (3.5)$$

$$S_g T_h X = T_h S_g X \qquad (3.6)$$

for any g, h $\in$ G and X $\in \mathfrak{X}$. By means of (3.2-3), one can easily check that these conditions are equivalent.

The group generated by the set

$$\mathfrak{G}(S,T) = \{S_g, T_g; g \in G\}$$

of all G-transformations of $\mathfrak{X}$ is said to be the *enveloping group* of (S,T) and denoted by $\bar{\mathfrak{G}}(S,T)$. It must be noted that the set $\mathfrak{G}(S,T)$ is not, in general, confined to be a subgroup of $\mathfrak{T}(\mathfrak{X})$, but is only the *generative* subset of $\bar{\mathfrak{G}}(S,T)$. One can also say that the group $\bar{\mathfrak{G}}(S,T)$ is presented by the generative elements S_g, T_g, $g \in G$, and by the relations (3.1-3). That is to say, the action of G on $\mathfrak{X}$ only generates the associative action of $\bar{\mathfrak{G}}(S,T)$ on $\mathfrak{X}$.

Let us check some elementary properties of G-transformations needed in this work. From (3.1-3) one can find that

$$S_g^{-1} = S_{g^{-1}}, \qquad (3.7)$$

$$T_g^{-1} = T_{g^{-1}}, \qquad (3.8)$$

$$S_g T_g = T_g S_g \qquad (3.9)$$

for any $g \in G$. By means of (3.7-9) and (1.5), one can rewrite the defining relations (3.2-3) of (S,T) as follows:

$$S_h\ T_g\ S_g = T_g\ S_{hg}\ , \qquad (3.2')$$

$$T_h\ T_g\ S_g = S_g\ T_{gh}\ , \quad g, h \in G. \qquad (3.3')$$

Combining (3.2'-3') with (3.2-3), one gets the following commutation rule for the Moufang-transformations:

$$S_g S_h\ T_h T_g = T_h T_g\ S_g S_h\ , \quad g, h \in G. \qquad (3.7)$$

4. Analytic G-transformations. Now, let G be an analytic Moufang loop and let $\mathfrak{X}$ be an n-dimensional real, analytic manifold. The action (S,T) of G on $\mathfrak{X}$ is said to be *analytic* if the functions S_gX and T_gX, $g \in G$, $X \in \mathfrak{X}$, are analytic maps of $\mathfrak{X} \times G$ onto $\mathfrak{X}$. In this case, the birepresentation (S,T) of G is also said to be analytic, and the transformations S_g, T_g of $\mathfrak{X}$, $g \in G$, are called its *analytic Moufang-transformations* (G-transformations).

In this paper, we shall consider the analytic G-transformations of $\mathfrak{X}$ only for $g \in G$ sufficiently near to the identity e of G. The action of such $g \in G$ on $\mathfrak{X}$ reads in local coordinates

$$\begin{aligned}(S_gX)^\beta &= S^\beta(X^1,\ldots,X^n;g^1,\ldots,g^r)\\ &= S^\beta(X;g)\ ,\\ (T_gX)^\beta &= T^\beta(X^1,\ldots,X^n;g^1,\ldots,g^r)\\ &= T^\beta(X;g)\ ,\end{aligned}$$

where X^β, $(S_gX)^\beta$, $(T_gX)^\beta$, $\beta=1,\ldots,n$, are the local coordinates of X, S_gX, T_gX, respectively. Due to the analyticity and (3.1), one can expand S_gX and T_gX as follows:

$$(S_gX)^\beta = X^\beta + S^\beta_j(X)g^j + 1/2\ S^\beta_{jk}(X)g^jg^k + \ldots, \tag{4.1a}$$

$$(T_gX)^\beta = X^\beta + T^\beta_j(X)g^j + 1/2\ T^\beta_{jk}(X)g^jg^k + \ldots, \tag{4.1b}$$

$\beta=1,\ldots,n$. The (analytic) functions S^β_j and T^β_j may be called the *auxiliary functions* of (S,T). The further coefficients in the series (4.1a-b) are symmetric with respect to lower indices. For example,

$$S^\beta_{jk} = S^\beta_{kj}\ , \tag{4.2a}$$

$$T^\beta_{jk} = T^\beta_{kj}\ . \tag{4.2b}$$

5. Associators of birepresentation. An action of G on $\mathfrak{X}$ need not be associative even in case G is. The associativity conditions (3.4-6) read in local coordinates

$$(S_g S_h X)^\beta = (S_{gh} X)^\beta , \tag{5.1}$$

$$(T_g T_h X)^\beta = (T_{hg} X)^\beta , \tag{5.2}$$

$$(S_g T_h X)^\beta = (T_h S_g X)^\beta , \quad \beta=1,\ldots,n. \tag{5.3}$$

Thus, to measure the deviation of (S,T) from associativity, we can introduce the formal functions

$$l^\beta(X;g,h) = (S_{gh}X)^\beta - (S_g S_h X)^\beta , \tag{5.4}$$

$$r^\beta(X;g,h) = (T_{gh}X)^\beta - (T_h T_g X)^\beta , \tag{5.5}$$

$$m^\beta(X;g,h) = (T_h S_g X)^\beta - (S_g T_h X)^\beta , \tag{5.6}$$

$\beta = 1,\ldots,n$, called the *associators* of (S,T). One can easily check the initial conditions

$$l^\beta(X;g,e) = r^\beta(X;g,e) = m^\beta(X;g,e) = 0 , \tag{5.7a}$$

$$l^\beta(X;e,g) = r^\beta(X;e,g) = m^\beta(X;e,g) = 0 . \tag{5.7b}$$

We shall treat the associators of (S,T) as the generating functions in the following sense. The *first-order associators* l^β_j, $\overset{*}{l}{}^\beta_j$, r^β_j, $\overset{*}{r}{}^\beta_j$, m^β_j, $\overset{*}{m}{}^\beta_j$ of (S,T) are defined by

$$l^\beta(X;g,h) = l^\beta_j(X;h)g^j + O(g^2) \tag{5.8a}$$

$$= \overset{*}{l}{}^\beta_j(X;g)h^j + O(h^2) , \tag{5.8b}$$

$$r^\beta(X;g,h) = r^\beta_j(X;h)g^j + O(g^2) \tag{5.9a}$$

$$= \overset{*}{r}{}^\beta_j(X;g)h^j + O(h^2) , \tag{5.9b}$$

$$m^{\beta}(X;g,h) = m_j^{\beta}(X;h)g^j + O(g^2) \quad (5.10a)$$

$$= \overset{*}{m}{}_j^{\beta}(X;g)h^j + O(h^2) \quad . \quad (5.10b)$$

As an example, let us find out l_j^{β}. The expansions of $(S_gS_hX)^{\beta}$, $(S_{gh}X)^{\beta}$ in (5.4) with respect to g are given by

$$(S_gS_hX)^{\beta} = S^{\beta}(S_hX;g)$$

$$= (S_hX)^{\beta} + S_j^{\beta}(S_hX)g^j + O(g^2) \; ,$$

$$(S_{gh}X)^{\beta} = S^{\beta}(X;gh)$$

$$= (S_hX)^{\beta} + \frac{\partial(S_hX)^{\beta}}{\partial h^j} u_k^j(h)g^k + O(g^2) \; ,$$

so that

$$l_j^{\beta}(X;h) = u_j^s(h) \frac{\partial(S_hX)^{\beta}}{\partial h^s} - S_j^{\beta}(S_hX) \quad .$$

The remaining first-order associators are figured out similiarly. Let us present the result as follows:

$$l_j^{\beta}(X;g) = u_j^s(g) \frac{\partial(S_gX)^{\beta}}{\partial g^s} - S_j^{\beta}(S_gX) \; , \quad (5.11a)$$

$$\overset{*}{l}{}_j^{\beta}(X;g) = v_j^s(g) \frac{\partial(S_gX)^{\beta}}{\partial g^s} - S_j^{\nu}(X) \frac{\partial(S_gX)^{\beta}}{\partial X^{\nu}} \; , \quad (5.11b)$$

$$r_j^{\beta}(X;g) = u_j^s(g) \frac{\partial(T_gX)^{\beta}}{\partial g^s} - T_j^{\nu}(X) \frac{\partial(T_gX)^{\beta}}{\partial X^{\nu}} \; , \quad (5.12a)$$

$$\overset{*}{r}{}_j^{\beta}(X;g) = v_j^s(g) \frac{\partial(T_gX)^{\beta}}{\partial g^s} - T_j^{\beta}(T_gX) \; , \quad (5.12b)$$

$$m_j^{\beta}(X;g) = - S_j^{\beta}(T_gX) + S_j^{\nu}(X) \frac{\partial(T_gX)^{\beta}}{\partial X^{\nu}} \; , \quad (5.13a)$$

$$\overset{*}{m}{}_j^\beta(X;g) = -T_j^\beta(S_g X) + T_j^\nu(X)\,\frac{\partial(S_g X)^\beta}{\partial X^\nu}\,. \tag{5.13b}$$

Now, one can check the initial conditions

$$l_j^\beta(X;e) = r_j^\beta(X;e) = m_j^\beta(X;e) = 0\;, \tag{5.14a}$$

$$\overset{*}{l}{}_j^\beta(X;e) = \overset{*}{r}{}_j^\beta(X;e) = \overset{*}{m}{}_j^\beta(X;e) = 0\;, \tag{5.14b}$$

and define the *second-order associators* l_{jk}^β, $\overset{*}{l}{}_{jk}^\beta$, m_{jk}^β, $\overset{*}{m}{}_{jk}^\beta$, r_{jk}^β, $\overset{*}{r}{}_{jk}^\beta$ of (S,T) by

$$l_j^\beta(X;g) = l_{jk}^\beta(X)\,g^k + O(g^2)\;, \tag{5.15a}$$

$$\overset{*}{l}{}_j^\beta(X;q) = \overset{*}{l}{}_{jk}^\beta(X)\,g^k + O(g^2)\;, \tag{5.15b}$$

$$r_j^\beta(X;g) = r_{jk}^\beta(X)\,g^k + O(g^2)\;, \tag{5.16a}$$

$$\overset{*}{r}{}_j^\beta(X;g) = \overset{*}{r}{}_{jk}^\beta(X)\,g^k + O(g^2)\;, \tag{5.16b}$$

$$m_j^\beta(X;g) = m_{jk}^\beta(X)\,g^k + O(g^2)\;, \tag{5.17a}$$

$$\overset{*}{m}{}_j^\beta(X;g) = \overset{*}{m}{}_{jk}^\beta(X)\,g^j + O(g^2)\;. \tag{5.17b}$$

After some straightforward effort one gets the following result:

$$l_{jk}^\beta(X) = \overset{*}{l}{}_{kj}^\beta(X) \tag{5.18a}$$

$$= S_{jk}^\beta(X) + a_{jk}^p\,S_p^\beta(X) - S_k^\nu(X)\,\frac{\partial S_j^\beta(X)}{\partial X^\nu}\;, \tag{5.18b}$$

$$r_{jk}^\beta(X) = \overset{*}{r}{}_{kj}^\beta(X) \tag{5.19a}$$

$$= T_{jk}^\beta(X) + a_{jk}^p\,T_p^\beta(X) - T_j^\nu(X)\,\frac{\partial T_k^\beta(X)}{\partial X^\nu}\;, \tag{5.19b}$$

$$m^{\beta}_{jk}(X) = -\overset{*}{m}{}^{\beta}_{kj}(X) \tag{5.20a}$$

$$= S^{\nu}_{j}(X)\,\frac{\partial T^{\beta}_{k}(X)}{\partial X^{\nu}} - T^{\nu}_{k}(X)\,\frac{\partial S^{\beta}_{j}(X)}{\partial X^{\nu}}\,. \tag{5.20b}$$

Let us note that so far we have not exploited the defining relations (3.2-3) of (S,T). We shall do this in the next sextion.

<u>6. First-order minimality conditions.</u> By considering (3.2-3) and (3.2'-3') in local coordinates, one can establish the following constraints for the first-order associators of (S,T):

$$\overset{*}{l}{}^{\beta}_{j}(X;h) \overset{(a)}{=} \overset{*}{m}{}^{\beta}_{j}(X;h) \overset{(b)}{=} -\, l^{\beta}_{j}(X;h)\;, \tag{6.1}$$

$$r^{\beta}_{j}(X;h) \overset{(a)}{=} m^{\beta}_{j}(X;h) \overset{(b)}{=} -\, \overset{*}{r}{}^{\beta}_{j}(X;h)\;. \tag{6.2}$$

As an example, let us check the identity $\overset{*}{m}{}^{\beta}_{j} = -\, l^{\beta}_{j}$. In local coordinates, (3.2) reads

$$(S_g T_g S_h X)^{\beta} = (S_{gh} T_g X)^{\beta}\;,\quad \beta = 1,\ldots,n. \tag{6.3}$$

By expanding the left- and right-hand sides of (6.3) with respect to g, one has

$$\begin{aligned}(S_g T_g S_h X)^{\beta} &= S^{\beta}(T_g S_h X; g)\\ &= (T_g S_h X)^{\beta} + S^{\beta}_{j}(T_g S_h X) g^{j} + O(g^2)\\ &= (S_h X)^{\beta} + T^{\beta}_{j}(S_h X) g^{j} + S^{\beta}_{j}(S_h X) g^{j} + O(g^2)\,,\end{aligned}$$

$$(S_{gh}T_gX)^{\beta} = S^{\beta}(T_gX;gh)$$
$$= (S_hX)^{\beta} + \frac{\partial (S_hX)^{\beta}}{\partial X^{\nu}} T^{\nu}_k(h)g^k +$$
$$+ \frac{\partial (S_hX)^{\beta}}{\partial h^j} u^j_k(h)g^k + O(g^2) \quad ,$$

by means of which one can find desired identity. The remaining identities in (6.1-2) are obtained in the same way, by considering (3.3) and (3.2'-3') in local coordinates.

As it was said above, a birepresentation of G need not be associative even in case G is. However, the deviation of (S,T) from associativity is not at all arbitrary but is highely constrained with (6.1-2), the latter being, in fact, the infinitesimal version of (3.1-3). If (S,T) turns out to be *associative*, one immediately gets the familiar Lie equations for the analytic associative G-transformations:

$$\overset{*}{l}{}^{\beta}_j = \overset{*}{m}{}^{\beta}_j = - l^{\beta}_j = 0 \quad , \tag{6.4a}$$

$$r^{\beta}_j = m^{\beta}_j = - \overset{*}{r}{}^{\beta}_j = 0 \quad . \tag{6.4b}$$

In view of this, it may be said that the birepresentations of G have the attractive property of "minimal" deviation from associativity. Identities (6.1-2) are hence called the *first-order minimality conditions* of (S,T).

Originally, identities (6.1-2) read as the differential equations for the analytic G-transformations, "minimally" generalizing the Lie equations from the Lie transformation groups theory. When defining the functions w^i_j and P^{β}_j by

$$u^i_j(g) + v^i_j(g) + w^i_j(g) = 0 \quad , \tag{6.5a}$$

$$S^{\beta}_j(X) + T^{\beta}_j(X) + P^{\beta}_j(X) = 0 \quad , \tag{6.5b}$$

the differential equations for S_gX read

$$u^s_j(g)\ \frac{\partial (S_gX)^\beta}{\partial g^s} + T^\nu_j(X)\ \frac{\partial (S_gX)^\beta}{\partial X^\nu} + P^\beta_j(S_gX) = 0\ , \tag{6.6a}$$

$$v^s_j(g)\ \frac{\partial (S_gX)^\beta}{\partial g^s} + P^\nu_j(X)\ \frac{\partial (S_gX)^\beta}{\partial X^\nu} + T^\beta_j(S_gX) = 0\ , \tag{6.6b}$$

$$w^s_j(g)\ \frac{\partial (S_gX)^\beta}{\partial g^s} + S^\nu_j(X)\ \frac{\partial (S_gX)^\beta}{\partial X^\nu} + S^\beta_j(S_gX) = 0\ , \tag{6.6c}$$

and the differential equations for T_gX read

$$v^s_j(g)\ \frac{\partial (T_gX)^\beta}{\partial g^s} + S^\nu_j(X)\ \frac{\partial (T_gX)^\beta}{\partial X^\nu} + P^\beta_j(T_gX) = 0\ , \tag{6.7a}$$

$$u^s_j(g)\ \frac{\partial (T_gX)^\beta}{\partial g^s} + P^\nu_j(X)\ \frac{\partial (T_gX)^\beta}{\partial X^\nu} + S^\beta_j(T_gX) = 0\ , \tag{6.7b}$$

$$w^s_j(g)\ \frac{\partial (T_gX)^\beta}{\partial g^s} + T^\nu_j(X)\ \frac{\partial (T_gX)^\beta}{\partial X^\nu} + T^\beta_j(T_gX) = 0\ . \tag{6.7c}$$

These differential equations are not linearly independent since

$$(6.6a) + (6.6b) + (6.6c) = 0\ , \tag{6.8a}$$

$$(6.7a) + (6.7b) + (6.7c) = 0\ , \tag{6.8b}$$

as one can easily see by means of (6.5). Also, it must be said that to consider (6.6-7) as differential equations for S_gX, T_gX one has somehow to specify the auxiliary functions of both (S,T) and G. In /1/, we have done this for the auxiliary functions u^i_j, v^i_j of G, by deriving a certain set of differential equations for these functions. In the following, we shall find the similiar differential equations for the auxiliary functions S^β_j, T^β_j of (S,T).

7. Second-order minimality conditions. By differentiating (6.1-2) with respect to h at h= e, one gets the following constraints for the second-order associators of (S,T):

$$\overset{*}{l}{}^{\beta}_{jk}(X) \overset{(a)}{=} \overset{*}{m}{}^{\beta}_{jk}(X) \overset{(b)}{=} - l^{\beta}_{jk}(X) , \qquad (7.1)$$

$$r^{\beta}_{jk}(X) \overset{(a)}{=} m^{\beta}_{jk}(X) \overset{(b)}{=} - \overset{*}{r}{}^{\beta}_{jk}(X) . \qquad (7.2)$$

Using the relations (5.18a, 19a, 20a), one can write (7.1-2) as

$$\overset{*}{l}{}^{\beta}_{jk}(X) \overset{(a)}{=} m^{\beta}_{jk}(X) \overset{(b)}{=} r^{\beta}_{jk}(X) \overset{(c)}{=} - m^{\beta}_{kj}(X) . \qquad (7.3)$$

For the associative birepresentations of G one has now

$$\overset{*}{l}{}^{\beta}_{jk}(X) = m^{\beta}_{jk}(X) = r^{\beta}_{jk}(X) = - m^{\beta}_{kj}(X) = 0 , \qquad (7.4)$$

and for that reason the birepresentations of G are once more said to have the property of "minimal" deviation from associativity. Constraining identities (7.3) are said to be the *second-order minimality conditions* of (S,T).

The identities $\overset{*}{l}{}^{\beta}_{jk} = - \overset{*}{l}{}^{\beta}_{kj}$, $r^{\beta}_{jk} = - r^{\beta}_{kj}$ read

$$2\, S^{\beta}_{jk} = S^{\nu}_{k} \frac{\partial S^{\beta}_{j}}{\partial X^{\nu}} + S^{\nu}_{j} \frac{\partial S^{\beta}_{k}}{\partial X^{\nu}} - (a^{p}_{jk} + a^{p}_{kj})\, S^{\beta}_{p} , \qquad (7.4a)$$

$$2\, T^{\beta}_{jk} = T^{\nu}_{k} \frac{\partial T^{\beta}_{j}}{\partial X^{\nu}} + T^{\nu}_{j} \frac{\partial T^{\beta}_{k}}{\partial X^{\nu}} - (a^{p}_{jk} + a^{p}_{kj})\, T^{\beta}_{p} . \qquad (7.4b)$$

By substituting (7.4a) and (7.4b) into (5.18b) and (5.19b), respectively, one can easily find

$$S^{\nu}_{k} \frac{\partial S^{\beta}_{j}}{\partial X^{\nu}} - S^{\nu}_{j} \frac{\partial S^{\beta}_{k}}{\partial X^{\nu}} = c^{p}_{jk}\, S^{\beta}_{p} + 2\, \overset{*}{l}{}^{\beta}_{jk} , \qquad (7.5a)$$

$$- T_k^\nu \frac{\partial T_j^\beta}{\partial X^\nu} + T_j^\nu \frac{\partial T_k^\beta}{\partial X^\nu} = C_{jk}^p T_p^\beta - 2 r_{jk}^\beta , \tag{7.5b}$$

from which, in view of (7.3a), (7.3c) and the defining relations (5.20b), one gets the following differential equations for the auxiliary functions of (S,T):

$$S_k^\nu \frac{\partial S_j^\beta}{\partial X^\nu} - S_j^\nu \frac{\partial S_k^\beta}{\partial X^\nu} = C_{jk}^p S_p^\beta + 2 \left(S_j^\nu \frac{\partial T_k^\beta}{\partial X^\nu} - T_k^\nu \frac{\partial S_j^\beta}{\partial X^\nu}\right) , \tag{7.6a}$$

$$- T_k^\nu \frac{\partial T_j^\beta}{\partial X^\nu} + T_j^\nu \frac{\partial T_k^\beta}{\partial X^\nu} = C_{jk}^p T_p^\beta - 2 \left(T_j^\nu \frac{\partial S_k^\beta}{\partial X^\nu} - S_k^\nu \frac{\partial T_j^\beta}{\partial X^\nu}\right) . \tag{7.6b}$$

Now, by defining the *generators* of (S,T) as

$$S_j = S_j^\beta(X) \frac{\partial}{\partial X^\beta} , \tag{7.7a}$$

$$T_j = T_j^\beta(X) \frac{\partial}{\partial X^\beta} , \tag{7.7b}$$

$j = 1,\ldots,r$, one can rewrite (7.6a-b) as commutation relations

$$(S_j,S_k) = - C_{jk}^p S_p - 2 (S_j,T_k) , \tag{7.8a}$$

$$(T_j,T_k) = C_{jk}^p T_p - 2 (T_j,S_k) , \tag{7.8b}$$

where the Lie bracket (A,B) is defined as (A,B) = AB - BA.

The identities $m_{jk}^\beta = - m_{kj}^\beta$ read

$$S_j^\nu \frac{\partial T_k^\beta}{\partial X^\nu} - T_k^\nu \frac{\partial S_j^\beta}{\partial X^\nu} = - S_k^\nu \frac{\partial T_j^\beta}{\partial X^\nu} + T_j^\nu \frac{\partial S_k^\beta}{\partial X^\nu} , \tag{7.9}$$

and can be rewritten as the commutation relations

$$(S_j,T_k) = (T_j,S_k) . \tag{7.10}$$

At this point one can easily note that (7.10) follows also from the identity $(S_j,S_k) = -(S_k,S_j)$, as well as from the identity $(T_j,T_k) = -(T_k,T_j)$.

The above-derived commutation relations for the generators of (S,T) do not really depend on the particular analytic birepresentation of G. It can be easily seen that the mixed-term commutators (S_j,T_k), $j,k = 1,\dots,r$, appear in these commutation relations as a measure of the deviation of (S,T) from associativity. For the associative (S,T), one has the trivial mixed-term commutators and, for that reason, the generators of the associative (S,T) split into two commuting Lie algebras as follows:

$$(S_j,S_k) = -c^p_{jk}\,S_p\ ,$$

$$(T_j,T_k) = c^p_{jk}\,T_p\ ,$$

$$(S_j,T_k) = 0\ ,\quad j,k = 1,\dots,r\ .$$

For the analytic Moufang loop, the commutation relations of form (7.8) have been established in /1/.

8. <u>Closure conditions.</u> At this point one may anxiously ask if it is possible to close (7.8). The answer is affirmative, and here we outline a way of closing these commutation relations into the finite-dimensional Lie algebra.

For the convenience, let us define the *infinitesimal Moufang-transformations* S_x, T_x by

$$S_x = x^p\,S_p\ , \tag{8.1a}$$

$$T_x = x^p\,T_p\ ,\quad x \in T_e. \tag{8.1b}$$

Commutation relations (7.8a,b) and (7.10) can be rewritten as follows:

$$(S_x,S_y) = -S_{(x,y)} - 2\,(S_x,T_y)\ , \tag{8.2a}$$

$$(T_x,T_y) = T_{(x,y)} - 2\ (T_x,S_y)\ , \qquad (8.2b)$$

$$(S_x,T_y) = (T_x,S_y) \qquad (8.2c)$$

for any $x,\ y \in T_e$.

Let us now construct the vector fields $F(x;y)$ as

$$F(x;y) = 1/3\ T_{(x,y)} - 1/3\ S_{(x,y)} - (T_x,S_y)\ , \qquad (8.3)$$

$x,\ y \in T_e$. Commutation relations (8.2a,b) read

$$(S_x,S_y) = 2\ F(x;y) - 1/3\ S_{(x,y)} - 2/3\ T_{(x,y)}\ , \qquad (8.4a)$$

$$(T_x,S_y) = -\ F(x;y) - 1/3\ S_{(x,y)} + 1/3\ T_{(x,y)}\ , \qquad (8.4b)$$

$$(T_x,T_y) = 2\ F(x;y) + 2/3\ S_{(x,y)} + 1/3\ T_{(x,y)}. \qquad (8.4c)$$

Commutation relations (8.2c) read

$$F(x;y) = -\ F(y;x)\ . \qquad (8.5)$$

In addition to (8,5), it can be shown by using (8.4) that

$$F((x,y);z) + F((y,z);x) + F((z,x);y) = 0 \qquad (8.6)$$

for any triple $x,\ y,\ z \in T_e$.

Now, by considering the integrability conditions of (6.6-7), one can establish the commutation relations

$$6\ (F(x;y),S_z) = S_{[x,y,z]}\ , \qquad (8.7a)$$

$$6\ (F(x;y),T_z) = T_{[x,y,z]}\ , \qquad (8.7b)$$

where the triple-product $[x,y,z]$ of $x,\ y,\ z \in T_e$ is defined as in /9,10/:

$$[x,y,z] = (x,(y,z)) - (y,(x,z)) + ((x,y),z)\ . \qquad (8.8)$$

This triple-product obeys the *Sagle-Yamaguti identity* /8,9/ (see also /1/ for another proof of it)

$$[x,y,(z,w)] = ([x,y,z],w) + (z,[x,y,w]) \qquad (8.9)$$

for any x, y, z, w $\in T_e$. Identity (8.9) turns out to be equivalent /8,9/ to (2.7), and can thus be taken, as (2.7), for the defining identity of the Mal'tsev algebra.

By using (8.7) and (8.9), one can establish the following finite-dimensional Lie algebra:

$$6\,(F(x;y),F(z;w)) = F([x,y,z];w) + F(z;[x,y,w]) \qquad (8.10)$$

for any x, y, z, w $\in T_e$. To establish the dimensionality of this Lie algebra one must take into consideration (8.5,6). From (8.5) it follows that the dimensionality of algebra (8.10) does not exceed $r(r-1)/2$.

As a corollary, we can now conclude that the vector fields (8.1), (8.3) close under Lie bracketing. The dimensionality of the Lie algebra (8.4,7,10) does not exceed $2r + r(r-1)/2$. The Lie algebra (8.10) is obviously the subalgebra of (8.4,7,10).

All these commutation relations are valid for every analytic birepresentation of G and are known from /1/ as well as from the representation theory of alternative algebras. What we have achieved in this paper, is that we can now view these commutation relations as the constraining set of differential equations for the auxiliary functions of (S,T). By specifying these functions from the Lie algebra (8.4,7,10), we can return to (6.6-7) for the restoration of the finite G-transformations of $\mathfrak{X}$.

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