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STUDYING SHOCKS IN MODEL ASTROPHYSICAL FLOWS

Sandip K. Chakrabarti

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Sandip K. Chakrabarti **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We briefly discuss some properties of the shocks in the existing models for quasi two-dimensional astrophysical flows. All of these models which allow the study of shock analytically have some unphysical characteristics due to inherent assumptions made. We propose a *hybrid* model for a thin flow which has fewer unpleasant features and is suitable for the study of shocks.

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** Permanent address: Theoretical Astrophysics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba 400 005, India.

Consider a one dimensional, transonic, adiabatic flow with a fixed boundary, say conical, undergoing a shock transition. The dimensionless specific energy of the flow can be written as,

$$\mathcal{E} = \frac{M^2a^2}{2} + na^2 + G(x) \tag{1}$$

where, a is the non dimensional sound speed, M is the Mach number of the flow defined by $M = \vartheta/a$, ϑ being the non-dimensional radial velocity, and n is the polytropic index of the flow for the equation of state $p = K\rho^{1+1/n}$, p and ρ being the pressure and the density of the fluid respectively. Here K is the constant of proportionality (related to the entropy of the flow) which is assumed to be separately constant on both sides of the shock, but is allowed to vary at the shock due to the generation of entropy. The first term is the kinetic energy of the radial motion and the second term is the specific enthalpy. The final term $G(x)$ is a generalized potential in which the flow moves, x being the dimensionless radial coordinate, measured in units of typical length scale of the system. Since the flow is one dimensional, all the physical quantities such as ϑ, a .. etc. are constant in the transverse direction of the flow.

Apart from a geometric constant, the conserved accretion rate $\dot{\mu}$ is given by,

$$\dot{\mu} = Ma\rho x^2 \tag{2}$$

Along with the conservation of the specific energy $\mathcal{E}$ and the baryon number flux $\dot{\mu}$, the pressure balance equation must be satisfied at the stationary shock, i.e.,

$$P_+ + \rho_+ M_+^2 a_+^2 = P_- + \rho_- M_-^2 a_-^2 \tag{3}$$

where, by − and + signs we refer quantities before and after the shock respectively.

From eqns. (1), (2) and (3) one derives a Mach number relation which is invariant across the shock,

$$C = \frac{(\gamma M_- + \frac{1}{M_-})^2}{2 + (\gamma - 1)M_-^2} = \frac{(\gamma M_+ + \frac{1}{M_+})^2}{2 + (\gamma - 1)M_+^2} \tag{4}$$

where C is a constant, and $\gamma = 1 + 1/n$ is the adiabatic index. One obtains a number of important properties from this invariant function. In particular, the product of the Mach numbers at the shock is,

$$M_+ M_- = \frac{1}{[\gamma^2 - (\gamma - 1)C]^{1/2}} \tag{5}$$

Here C is minimum for $M_- = 1 = M_+$ and as $M_- \rightarrow 1_+$, $C \rightarrow \gamma + 1$, so that $M_+ \rightarrow 1_-$ for *all* adiabatic index γ. This implies that the minimum shock strength $S = M_-/M_+$ is unity. The standard relation for the post-shock Mach number given by Landau and Lifshitz (1955)

$$M_+^2 = \frac{2 + (\gamma - 1)M_-^2}{2\gamma M_-^2 - (\gamma - 1)} \tag{6}$$

is easily derived from eqn. (4). It is important to note that both the eqns. (4) and (6) are valid for strictly one dimensional flow (more correctly, when the shock is planer) although in the literature unwarrented use of eqn. (6) is made even when the flow is divergent.

A real astrophysical flow is, unfortunately, three dimensional. Even when the assumptions of the axisymmetry, small transverse height and the independency of the radial velocity with height are adopted, there is no model in the literature which can deal with shocks fully self-consistently. The reason is that at the shock the flow changes its thickness

(unless very efficient radiative losses are present), and satisfying the shock conditions is no longer an easy task. We show this below with the existing models.

A relatively useful stationary model of the rotating accretion flow near a compact object, in which the quantities are vertically averaged assuming the flow in vertical equilibrium, consists of the following set of equations (see, e.g., Matsumoto *et al.* 1984):

$$\dot{\mu} = x\Sigma\vartheta, \tag{7a}$$

and

$$\vartheta\frac{d\vartheta}{dx} + \frac{1}{\Sigma}\frac{dW}{dx} = \frac{\lambda^2 - \lambda_K^2}{x^3} - \frac{W}{\Sigma}\frac{dln\Omega_K}{dx}, \tag{7b}$$

where, λ and λ_K are the conserved angular momentum and the angular momentum of the Keplerian orbit at x respectively. Σ and W are the vertically averaged density and pressure, i.e.,

$$\Sigma = \int_{-h}^{h} \rho dz = 2\rho_e I_n h \tag{8a}$$

$$W = \int_{-h}^{h} p dz = 2p_e I_{n+1} h. \tag{8b}$$

The equations are written neglecting viscosity ($\alpha = 0$). Here h is the transverse thickness of the flow calculated using the pseudo-Newtonian potential $\psi = -1/2(x-1)$ [$\lambda_K^2 = x^3/2(x-1)^2$ in this model] of the compact object (Paczynski and Wiita, 1980) and is given by,

$$h = \sqrt{(2n)}a_e x^{1/2}(x - 1) \tag{9}$$

The subscript e is used for quantities in the equatorial plane and $I_n = (2^n n!)^2/(2n+1)!$. The equation of state along the vertical direction is assumed to be polytropic.

It is easily observed that this model can consistently deal with the shocks only if the entire flow is isothermal, i.e. $W/\Sigma = a_0^2$, where a_0 is the constant sound speed. The exact momentum balance condition at the shock is obtained as,

$$W + \Sigma v^2 = const. \tag{10}$$

and using this one obtains the Mach number relation at the shock as,

$$M_- + \frac{1}{M_-} = M_+ + \frac{1}{M_+}, \tag{11a}$$

the non trivial solution of which is,

$$M_- M_+ = 1. \tag{11b}$$

This can also be verified by putting $\gamma = 1$ in eqn. (4). A second analytically tractable model could be constructed if one adopts a rather unrealistic equation of state $W = K\Sigma^\gamma$, (where constancy of K *does not* imply isentropicity). One again obtains the same momentum balance condition as above (eqn. 10) and the energy and accretion rate equations similar to eqns. (1) and (2) *provided* the final term of the radial momentum equation (7b) is ignored (which is a good assumption for very thin flows) and $A = \sqrt{\gamma W/\Sigma}$ is used as the sound speed. This *squashed* disk model is more like one dimensional, and the Mach number relation at the shock is the same as eqn. (4) presented above.

A relatively "cleaner" model is recently proposed by Abramowicz and Chakrabarti (1988). In this so called 1.5 dimensional model radial equation is treated exactly *only* in the equatorial plane. No vertical motion is assumed (as in the other models) and the

thickness of the flow is determined from the vertical equilibrium condition (eqn. 9). The flow cross-section is therefore proportional to the adiabatic sound speed and the accretion rate $\dot{\mu}$ is given by,

$$\dot{\mu} = M a_e^2 \rho_e f(x) \tag{12}$$

where, $f(x) = x^{3/2}(x-1)$ in the pseudo-Newtonian description of the compact object. The equation for the specific energy and the momentum balance condition at the shock are kept the same as eqn. (1) and eqn. (3) respectively, provided only the *equatorial* quantities are used. This model is oversimplified but one can study the properties of the shocks completely (Chakrabarti, 1988a). However, shocks in this model can not be arbitrarily weak. To see this, we derive the Mach number relation by conserving the accretion rate (12) and balancing pressure at the shock (eqn. 3) as,

$$2C = \gamma M_+ + \frac{1}{M_+} = \gamma M_- + \frac{1}{M_-}, \tag{13}$$

where, C is a constant. From the above equation, the product of the Mach numbers at the shock is,

$$M_+ M_- = \frac{1}{\gamma}. \tag{14}$$

The shock strength is $S = M_-/M_+ = \gamma M_-^2$. If the minimum M_- allowed for a supersonic to subsonic transition is unity, the minimum shock strength is γ. However, the acoustic perturbation propagates in the flow with speed $a_e \nu \pm \vartheta$ (where, $\nu = \sqrt{2/(\gamma+1)}$) and the minimum pre-shock Mach number for a transonic flow is $M_- = \nu$ and the minimum shock strength is,

$$S_{min} = \frac{2\gamma}{\gamma + 1}. \tag{15}$$

Thus, S_{min} goes to unity only if the flow is isothermal, $\gamma = 1$. It is to be emphasized that above relation is *independent* of the third shock condition which we must employ in order to solve the problem completely. What the limit S_{min} implies so far is that the enhancement of the cross-section of the post-shock flow reduces the Mach number (and increases the temperature and the entropy) and the pressure balance (using only the equatorial quantity) is achieved only if this reduction is sufficient. It is easily shown that in addition when one imposes the condition of constancy of the energy flux across the shock, the lower limit of shock strength is pushed even higher (Chakrabarti, 1988).

Because of these *unplesant* features in both the quasi two dimensional flow models described above, we propose here a new *hybrid* model in studying shocks. Here, we keep the energy equation and the accretion rate equation as given by the 1.5 dimensional model (eqns. 1 and 12) employing equatorial quantities and keep the momentum balance condition (10) employing the averaged quantities. The later choice treats the change of thickness at the shock quite accurately. Substituting relations (8a) and (8b) in eqn. (10) one easily obtains the momentum balance condition in terms of the equatorial quantities. In this model one derives the Mach number relation as,

$$C = \frac{[M_-(3\gamma-1)+\frac{2}{M_-}]^2}{2+(\gamma-1)M_-^2} = \frac{[M_+(3\gamma-1)+\frac{2}{M_+}]^2}{2+(\gamma-1)M_+^2}, \tag{16}$$

where, C is a constant. The product of the Mach numbers at the shock is,

$$M_+M_- = \frac{2}{[(3\gamma-1)^2-(\gamma-1)C]^{1/2}} \tag{17}$$

The weakest shock which may form in a transonic flow with vertical equlibrium has the pre-shock Mach number $M_- = \sqrt{\frac{2}{\gamma+1}}$ (see discussions following eqn. 14). One easily checks that the post-shock Mach number is also the same. Thus, the minimum shock strength is again unity in this model. It is also clear that for isothermal flow one recovers eqn. (11b) from above.

In this *letter* we have shown that a *hybrid* model which partly uses the equations on the equatorial plane and partly uses averaged quantities gives more realistic results in the study of the shocks in quasi two dimensional adiabatic flows. The errors involved in the energy and the accretion rate equations are corrected exactly by the momentum balance equation which is assumed in an ad-hoc way. The detail properties of the shocks along the line of what is done for the 1.5 dimensional flow (Chakrabarti, 1988) will be discussed elsewhere.

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