Salient features of liquid Helium II are being analysed in the light of symmetry breaking mechanism. The excitation spectrum is derived by scaling transformation. A relation between roton excitation and Bogoliubov's quasi particles has been established by giving different structures to the roton operators, similar to quasi boson approximation for phonon-like spectrum. This structure of the roton operators enables us to study the low temperature behaviour of superfluid Helium-4 by introducing the idea of pairing for roton-like excitation for the onset of energy-gap.

Superfluid $^4$He is divided into the states of condensed bosons, oscillating condensed bosons, phonons, axons, rotons and the states of vortices. We consider the transition probabilities between these various states.

Coherent states of various superfluid regions of He II are being obtained and consistently analysed which have deep inter-relationship. As higher excitations have also a group structure and their coherent behaviour has been seen, this formalism seems to be more appropriate to deal with such an interesting problem.
I. Introduction

Liquid helium because of its weak interactive forces and large zero point energy remains a liquid even in the vicinity of absolute zero; gets a title of quantum which has a phase transition at 2.17k into two phases—He-I and He-II, Helium-II behaving as a superfluid with almost zero flow resistance. The transition temperature showing a marked anomaly in specific heat lends it the name \( \lambda \) point. Many investigators visualized the transition phenomenon in different ways on the strength of their experimental and theoretical investigations. While London visualized the two phases as two interpenetrating fluids (superfluid and normal fluid), Landau suggested a theoretical basis giving the phase-I a status of elementary excitations named Phonon \((P = 0)\) and Rotons \((P = P_0)\) in the background of a superfluid (Phase-II), the parameters of the excitation energies of which were deduced from the macroscopic properties like Sp.heat, first and second sound through the liquid. Bogoliubov \( ^3 \) derived the phonon-spectrum by replacing the number operator for particles in zero momentum state by a c-number. Several other modified methods were used by many authors \( ^4-13 \) for computing the spectrum and other associated properties. This aspect is discussed in Section III.

Symmetry, a fundamental attribute of the natural world helps in the study of the particular aspect of a physical system. Symmetry and invariance are synonyms in physics, since the former concept relates more to the intrinsic structure of nature while the latter to the mathematical form of the equations of motion. In many-body theory we find a non-invariance of the entire Hilbert-space of the system as a macroscopic observable which behaves classically and corresponds to a non-invariance of the ground state. This idea first introduced by Heisenberg and latter by Nambu and Jona Lasinio, asserts that the ground or the vacuum state might be less symmetric than the basis dynamical laws i.e the field equation. As a consequence it lead to a remarkable theorem by Goldstone according to which there always is a zero mass particle associated with a broken symmetry of a continuous group. The significance of the relativistic theories though obscured by many a facet is still a matter of rigorous study after the discovery of hadrons containing heavy quarks. A non-relativistic analogue of this theorem implies that if a continuous symmetry of the Hamiltonian is not dealt with by the ground state or the representative ensemble in the case of non-zero temperature, a gapless excitation exists i.e there are states whose excitation energies vanish for momentum.

Superfluid helium does not exhibit all the symmetry properties of its underlying Hamiltonian, however in general it is invariant under a gauge transformation of first kind i.e it is particle number conserving. Evidently superfluid He-4 may be characterised by the lack of gauge invariance or the lack of conservation of particle number. A special version of Goldstone theorem, the Hugenholtz-Pines theorem with interactions of finite range was verified by Lange as also by Katz and Frishman, but the energy spectrum determined \( ^2 \) with self consistent method of first order \( ^2 \) exhibited a gap, contrary to Hugenholtz theorem. It was found however that such a gap was on account of the approximation not taking a satisfactory view of the long range ordering. The gap was however shown to vanish under certain specified conditions. \( ^2 \) This symmetry breaking aspect is discussed in the section-II of the report.

The introduction of a weak external field in the system breaks the phase symmetry of the helium system, and results in the translation of the Boson field operators and creation of coherent superposition of states. Coherent states find an application in the theory of superfluidity because they are well suited to describe a system of interacting particles whose low energy excitations are
bose-modes and have a large occupation number. These coherent states have been seen to behave classically. Thus coherent states play the part of classical fields which describe a set of many bosons as a whole, just as the classical electrodynamic field describes the classical limit of the quantum electrodynamics. In the theory of liquid helium classical description implies superfluidity. Again, the thermodynamical expectation value of the boson field operator is conventionally called as the order parameter $\psi$ for the superfluid helium-4 and the superfluid phase transition is usually associated with the loss of symmetry, which allows the expectation value to be non-vanishing. This order parameter $\psi$ has been conveniently chosen to be the coherent state representation of pure quantum states by many authors. This is discussed in section-V.

In the light of the symmetry breaking the explicit forms of the transition probabilities between various states such as the condensed state, the oscillating condensed boson state, the phonon states and the roton states of superfluid He-4 have been discussed in section-IV. Approximation method making use of the Bogoliubov's canonical transformations has been used for transition between various states. From the forms of transition probabilities, it has been visualized that these states are all the probable ones.

II. SYMMETRY BREAKING AND SUPERFLUIDITY.

Superfluidity is the outcome of the phase transition ($\lambda$-transition) which is characterised by a change of order in the sense of off-diagonal long range order (ODLRO) in the particle density matrix. This $\lambda$-transition being a second order phase transition is associated with breaking of the symmetry of the Lagrangian of system and also with the appearance of the breaking symmetry co-ordinate $\psi$. The broken symmetry co-ordinate is associated with the order parameter of the condensation. As the order parameter is non-vanishing due to the loss of symmetry associated with the superfluid phase transition, so a broken symmetry state can be distinguished by the appearance of a macroscopic order parameter. The macroscopic order parameter obeys the laws of classical mechanics, and determines a continuum disjoint Hilbert subspaces of the full Hilbert space of the system. The breaking of the symmetry establishes multiplicity of the ground states related by the transformations of the broken symmetry group. The multiplicity of the ground states is linked with the degeneracy of vacuum, thus, to the concept of non-invariance of the vacuum and is a common feature of the broken symmetry co-ordinates. The Hamiltonian of superfluid helium is invariant under gauge transformation of first kind or invariant under the phase transformation or is particle number conserving. Thus the specific symmetry which is broken in the system is the gauge invariance or the particle number conservation or the phase invariance. Accordingly the broken symmetry groups under consideration are, the gauge group defined by $u(x) = e^{iN}$ and the translation group defined by $u(x) = e^{iP}$ where $N$ and $P$ are the particle number and momentum respectively. The breaking of these symmetry groups is related by Noether's theorem to non-conservation of a physical quantity acting as the generators of the group (N or P). From the statistical point of view this means that the system is described by a density matrix $\rho$ which although commuting with the Hamiltonian $H$ of the system, does not commute with this group generator $N$ or $P$ the generator however commutes with $H$.

$$[\rho, H] = 0, \quad [\rho, N] \neq 0, \quad [\rho, P] \neq 0$$

These conditions are outcome of the condition that the order parameter is non-vanishing when the symmetry breaks. Thus if the order-parameter is non-vanishing, the Hamiltonian is invariant under the symmetry transformations $N = \int f(x) dx$, but the density matrix is not and we have a broken symmetry.
For the spontaneous symmetry breaking, the non-invariant macroscopic observable can be described by means of the concept of quasi averages introduced by Bogoliubov, who introduced into the treatment an infinitely weak external field which breaks the symmetry of the system. For this purpose a new term $\mathcal{V} n$ is introduced into the Hamiltonian where $\mathcal{V}$ is a small parameter and $n$ is defined by

$$n = -\mathcal{V} \int dx \psi^2(x).$$

The new Hamiltonian $H' = H - \mathcal{V} n$ no longer commutes with the symmetry operation because of the unsymmetrical added term. In this case the average of a dynamical variable equal to zero for $\mathcal{V} = 0$, will be different from zero for $\mathcal{V} \neq 0$ and it can remain different from zero even in the limit $\mathcal{V} \to 0$. The procedure of taking the limit $\mathcal{V} \to 0$ is equivalent to the choice of the ground state or what is the same thing, a concrete value of the macroscopic parameter characterizing the representation. So this state $|\mathcal{V}\rangle$ is defined by

$$\langle\psi| \frac{\partial}{\partial \mathcal{V}} |\mathcal{V}\rangle = \sqrt{\mathcal{N}}.$$

The state $|\mathcal{V}\rangle$ is taken to be the state of lowest energy of $H' = H - \mathcal{V} n$ at finite $\mathcal{V}$ under the constraint $\langle\psi|\mathcal{N}|\psi\rangle = \mathcal{N}$. There is degeneracy of $|\mathcal{V}\rangle$ with all states $|\psi,\mathcal{N}\rangle = \sqrt{\mathcal{N}} |\psi\rangle$ and this degeneracy of the (approximate) ground state is taken synonymously for a broken symmetry.

Recently we discussed the symmetry breaking mechanism with an approach which is akin to that of the relativistic quantum field theory. In this context the main idea is the symmetry exhibited by the basic fields of superfluid He, before the onset of the field interaction is seen to reappear in a different form when the theory is expressed in terms of the physical field operators which describe the system after switching on of the field reactions. As a consequence there exist the generators of the symmetry transformation, which when treated in terms of the physical field operators yield a non-zero value, however when treated in terms of the Fourier components above the transition, they vanish. The Bose-field operator obeys the commutation relations:

$$[\psi(x), \psi^*(y)] = i\delta(x-y)$$

when the interactions are switched on via the interaction Hamiltonian, we have a translation of the operators:

$$\psi(x) \longrightarrow \psi(x) + \alpha, \quad \psi^*(x) \longrightarrow \psi^*(x) + \alpha$$

with the conditions...
where $\xi$ is a spacetime independent c-number because of the translational invariance of the system, $|0>$ is the vacuum or the ground state and the Eq. (2-6) represents the condensation of the single particles in the system. As $\Psi$ and $\Psi'$ are both Bose operators, Eq. (2-5) can be imagined as an outcome of a unitary transformation of the type 

$$\Psi'(x) = \Psi(x) + \xi$$

The theory based on the Eq. (2-4) is invariant under this transformation.

If we restrict $\Omega$ to the local field, then the above transformation represents an infinitesimal unitary transformation in the Hilbert space, hence

$$\delta(\Psi(x)) = \xi$$

and with

$$Q = \int dx \, \Pi(x)$$

we have

$$-i\{\Psi(x), \xi\Omega\} = -i\{\Psi(x), \xi \int \Psi'(y)dy\}$$

$$= -i\int \{\Psi(x), \Pi(y)\}dy$$

$$= \xi$$

i.e. $\delta(\Psi(x)) = \xi = -i\{\Psi(x), \xi\Omega\}$

When we decompose the Bose-field $\Psi(x)$ in terms of its normal modes, to take account of the field reaction we have

$$\Psi(x) = \frac{1}{\sqrt{\omega}} \sum_k \frac{1}{2\sqrt{k}} \left( e^{-ikx} - i |k| t + e^{ikx} i |k| t \right)$$

so that

$$Q = \int dx | \Psi'(x)|^2 = \int \Psi(x) dx dx \Pi(x)$$

$$= \int \Psi(x) dx dx \Pi(x)$$

$$\xi \{ -i |k| t - i |k| t \}$$

$$k \neq 0 \quad 2 \sqrt{k} \quad 2 \sqrt{k} \quad 2 \sqrt{k} \quad 2 \sqrt{k}$$

This is because above is a Hermitian field.

Now as $Q$ is zero

$$\delta(\Psi(x)) = -i\{\Psi(x), \xi\Omega\} = 0$$

From this analysis we conclude here that if the generator $Q$ is valid for the symmetry operation (2-5) then in accordance with Eq. (2-11) it should not vanish. However as in Eq. (2-14), $Q$ vanishes, then it cannot be treated as the generator and hence it is pertinent to explain the contradiction found in Eq. (2-11) and Eq. (2-14). Such a difficulty is seen to exist in the analysis of spontaneous breakdown of symmetry in quantum field theory.

Now to resolve the paradox we have arrived at above, we in the light of scaling of the Fourier Components of the field operator write

$$q(x) = \frac{1}{\sqrt{\omega}} \sum_k q_k e^{ikx}$$

and

$$p(x) = \frac{1}{\sqrt{\omega}} \sum_k p_k e^{ikx}$$

The introduction of the coordinates $q_k$ and its conjugate momentum $p_k$ here is to study the helium system in the classical limit, as the classical description of many boson systems implies superfluidity. The Fourier Components are defined by

$$q(x) = \frac{1}{\sqrt{\omega}} \sum_k q_k e^{ikx}$$

$$p(x) = \frac{1}{\sqrt{\omega}} \sum_k p_k e^{ikx}$$

so that

$$q_k = q_k \quad p_k = p_k$$

and the expansion of the field operators $\Psi$ and $\Pi$ in terms of $p_k$ and $q_k$ representation are:

$$\Psi(x) = \frac{1}{\sqrt{\omega}} \sum_k q_k e^{ikx}$$

and

$$\Pi(x) = \frac{1}{\sqrt{\omega}} \sum_k p_k e^{-ikx}$$

Accordingly we have

$$Q = \int dx | \Psi(x)|^2 = \int dx | \Pi(x)|^2$$

Now if $Q$ vanishes for $k \rightarrow 0$ then $p(t) = 0$

But $p(t) = q(t) k$

so when $p(t) = 0$, $q(t) = constant$

This constancy of $q_k$ is on account of the Bose-Einstein condensation.
In this way the vanishing of \( \Omega \) expresses itself through the vanishing of canonical momentum for the \( k=0 \) mode of the massless Bose-field. This, in other words, means that vanishing of the integral of \( \Pi(x) \) over the whole volume is due to the fact that it picks up only \( k=0 \) Fourier component of the integrand. Thus if \( p=0 \), then \( k=0 \) mode does not represent a true dynamical degree of freedom. Therefore, \( q_{k=0} \), \( p_{k=0} \) represent the spurious coordinates and the commutation relation:

\[
[q_{k=0}, p_{k=0}] = \hbar
\]

is violated. Because of this violation the occupation number

\[
N = \sum_{p=0}^{\infty} a^+_p a_p
\]

is replaced by a c-number in the Bogoliubov's prescription, which neglects the dynamical behaviour of the condensed state. Therefore \( k=0 \) mode does not represent a true dynamical degree of freedom.

Since the massless boson \( p \), (excitation) does not carry energy, the linear homogeneous equation for the stable massless field becomes invariant under the inhomogeneous transformation (2.3). Hence the time independent symmetry generator \( Q \), can mix elements of zero mass spectrum and at the same time induce a constant number by the transformation (2.5). Such a constant represents the Bose-Einstein condensation of the massless bosons, which break the symmetry in superfluid helium system. It is because the condensate involving macroscopic occupation of the single quantum state acts as reservoir of particles and the particles may dissolve in the condensate somewhere in the system and emerge at a remote point leading there by to the non-conservation of particle number. Thus He-II field is quantized in a gauge in which \( q_{k=0} \) mode is cancelled. In this way Eq (2.18) takes the form

\[
\Psi(x) = \frac{1}{\sqrt{2\pi a}} \sum_{k \neq 0} q(t) e^{-ikx}
\]

\[
\Pi(x) = \frac{1}{\sqrt{2\pi a}} \sum_{k \neq 0} p(t) e^{-ikx}
\]

Accordingly the relation (2.4) is modified and with the commutation relation between \( p \)'s and \( q \)'s to

\[
\left[ \Psi(x), \Pi(y) \right]_{k=0} = i \left[ \Psi(x-y) - \frac{1}{2\hbar} \frac{\partial}{\partial x} \right]
\]

This relation follows from the fact that \( f_k \) mode or \( k=0 \) mode has been neglected while \( f_{k \neq 0} \) mode is taken care of by B.E condensation, which is represented mainly by the \( 1/\hbar \) term in this expression. This equation is a modified "delta" function. Thus we find that whenever the components of the field do not represent true dynamical degree of freedom, we get a modified "delta" function and hence the relation:

\[
\left[ q_{k=0}, p_{k=0} \right] = 0
\]

The commutation relation Eq (2.21) depends upon the quantization volume. This volume is to be considered in the infinite limit, as it is the characteristic of the infinitely large systems that they condense into the state of lower symmetry. In this limit \( \Omega \) does not vanish in the Eq (2.21) because we have to ascertain that the integral of this term over the whole space is unity. Accordingly we can say that as \( \Omega \to \infty \), the term \( \Omega \) has an infinitesimal amplitude at every point in space such that its integral over whole of space becomes unity. This in other words means that we have disjoint Hilbert sub-spaces of a full Hilbert space, determined by the volume as a broken symmetry coordinate. The broken symmetry origin of volume lies in the characteristic properties of condensation of liquid helium system which is a second order phase transition. This is also clear from the fact that the number of particles in the condensed phase taken as the broken symmetry coordinate in superfluid helium, which in the infinite volume limit is proportional to the total number of particles in the system, is also proportional to the volume of the system for the reason

\[
\Omega \to \infty, \frac{N}{V} = \text{constant}
\]

The volume as a broken symmetry coordinate is associated with the order parameter of the condensate from the state with full symmetry of the Lagrangian to the state with lesser symmetry. The order parameter describes
the dynamics of the condensate and consequently brings a new degree of freedom. This new degree of freedom has both dynamical and hydrodynamical features. The hydrodynamical aspect is characterized by local equilibrium. The dynamical aspect persists down to absolute zero. This means that the dynamical mode of excitation persists in the collisionless domain. The condensed phase (second fluid) which is related to the microscopic dynamics of the system below the transition exhibits a higher degree of order and develops a condensed phase of long range order at the transition. Hence the infinitesimal amplitude of $\omega^d$ at every point in space such that its integral over the whole quantization volume is unity, depicts the phonon modes, which exhibit the Goldstone bosons and break the symmetry. Accordingly the modified commutation relation Eq(2-21) explains the long-range correlation which carries away the symmetry of the ground state, leaving it frozen in an asymmetric state.

ABELIAN TRANSFORMATION

The boson field operator appears for superfluid $^4\text{He}$ in the form Eq(2-13), when Abelian group symmetries are phenomenologically violated. As $\Psi$ contains both positive and negative frequency parts, the equation for $\Psi(k)$, the Fourier transform of $\Psi(x)$ with momentum vector $'k'$ will be of second order in time derivative as:

$$-\delta^2\mathbf{E}^2 - E_k^2 \Psi(k) = 0 \quad \ldots \ldots 2-23$$

Where $E_k$ denotes the energy of boson ($E_k \to 0$ as $k \to 0$).

So the Lagrangian for the equation is given by

$$L = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2} \frac{\delta^2\mathbf{E}^2}{\delta \dot{k}^2} - E_k^2 \Psi(k)^* \Psi(k) + \frac{1}{2} \mathbf{a}^2 \Psi(k)^* \Psi(k) - E_k^2 \left( \frac{\delta}{\delta \Psi(k)} \right)^2 \right] \ldots \ldots 2-24$$

Where $\mathbf{a}$ is an arbitrary c-number constant. In the p,q representation we have

$$L = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \sum_k \left( \frac{\partial}{\partial \dot{q}_k} - \frac{\partial}{\partial \dot{p}_k} \right)^2 + i \left( \frac{\partial}{\partial q_k} + \frac{\partial}{\partial p_k} \right) \right] - \sum_k E_k^2 \left( \frac{\partial}{\partial \dot{q}_k} - \frac{\partial}{\partial \dot{p}_k} - 1 + 2ip_k \right) + 2a' \sum_k \left( \frac{\partial}{\partial \dot{q}_k} + ip_k \right)^2 \right] \ldots \ldots 2-25$$

Accordingly the Hamiltonian is given by

$$H(\Psi) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ (\Psi(k))^* \Psi(k) - E_k^2 \right] + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\mathbf{a}^2}{2} \Psi(k)^* \Psi(k) \right] \ldots \ldots 2-26$$

Also $H(\Psi) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \Psi(k)^* \Psi(k) + c \cdot \text{number} \right] \ldots \ldots 2-27$

The Canonical transformations which keep the equation invariant are:

$$\Psi(x) \longrightarrow \Psi(x) + c \cdot \text{number}$$

$$\mathbf{\Pi}(x) \longrightarrow \mathbf{\Pi}(x) + c \cdot \text{number}$$

Where $\mathbf{\Pi}(x) = \Psi(x) + a' \ldots \ldots 2-28$

Since $\int \mathbf{\Pi}(x) dx$ commutes with both $\Psi$ and $\Pi$ as $\mathcal{N} \to \mathcal{N}$ and $N$, the particle number is time independent. Therefore $N$ contains only $\mathbf{\Pi}$ linearly.

$$\mathcal{N} = -\chi \int \frac{d^3x}{(2\pi)^3} \left[ \mathbf{\Pi}(x) \right] \ldots \ldots 2-29$$

Where $\chi$ is a c-number constant. This shows firstly that the generators of the transformations Eq(2-28) are linear in the boson operators $\Psi(k)$ and $\Pi(k)$ and secondly that the constant $"a'"$ in Eq(2-28) is not arbitrary but related to the ground state expectation value of the particle density thus $a' = -\mathcal{N}$ so that $\mathbf{\Pi}(x) = \Psi(x) - \frac{\mathcal{N}}{\mathcal{N}} \ldots \ldots 2-30$

Accordingly we see from Eq (2-28) that the canonical conjugate of $'N'$ is $\Psi(x)/\mathcal{N}$. This leads to the transformation of $\Psi(x)$ under the phase transformation:

$$\Psi = e^{i\mathcal{N}} \ldots \ldots 2-31$$

as $\Psi(x) = e^{i\mathcal{N}} \ldots \ldots 2-32$

The phonon field obtained by the transformation Eq(2-31)
exhibits two fluid behaviour. The fields $\mathbf{v}$ and $\mathbf{\Psi}$ describe the depletion of the excitation and $\rho$ denotes the B.E. condensation.

When the system in the ground state is given momentum $p$ without violating translational invariance through the factor $e^{-ip\mathbf{x}}$, then a persistent current of momentum flux $(p, \rho)$ appears in the ground state. This gives rise to the running of condensed phase (boson) current. So, if the superfluid system is contained in a circular cell, the phase factor $\phi(x)$ in helium field starts the running of the current $\rho \mathbf{\Psi}(x)$ in the ground state. If the system is to be rotationally invariant, the phase factor $\lambda$ is represented by:

$$\lambda(x) = \frac{2\pi}{\hbar} \phi$$

Where $\phi$ is a constant denoting vortex strength and $\hbar$ is the angle of rotation. This phase induces a velocity field

$$\mathbf{v} = \frac{1}{m} \nabla \lambda = \frac{1}{m} \frac{\partial \lambda}{\partial x}$$

Where $\mathbf{v}$ is the unit vector tangent to the circulation. According to Onsager $\lambda$ and Feynman $\phi$ is quantized as:

$$\frac{2\pi}{\hbar} n = \phi$$

Where $n$ is an integer.

Thus

$$\mathbf{v} = \frac{2\pi n}{m \hbar} \mathbf{\hat{e}}_\phi$$

From this we find that the circulation of the superfluid velocity is quantized. Since the ground state momentum flux $p$ has the form $\rho \mathbf{\Psi}(x)$ the velocity $\mathbf{v}$ becomes irrotational ie curl $\mathbf{v} = 0$. This is the Landau condition denoting potential flow motion.

Now the basic equations of superfluidity are:

$$\frac{d\lambda}{dt} = \frac{-\hbar}{\nabla N} \mathbf{A}$$

$$\frac{dN}{dt} = \frac{3\hbar}{\nabla \lambda}$$

So from Eq (2-34) we have:

$$\mathbf{F} = m \frac{d\mathbf{v}}{dt} = m \lambda \frac{d\lambda}{dt} \frac{\nabla \lambda}{m} = \nabla \mathbf{A}$$

Where $\mathbf{F}$ is the total force on the particle and $\mathbf{A}$ is the chemical potential. This is the Landau's acceleration equation. From this we infer that superfluid helium cannot sustain a true force in the sense of gradient of therodynamic potential without undergoing acceleration. This means that in the absence of vorticity we must have a flow without frictional damping.

### III. EXCITATION SPECTRUM

Many authors attempted to extend the theory of Bogoliubov and Zubarev by considering the interaction amongst elementary excitations. However, in all these theories, most of which had primarily been developed for absolute zero, the temperature dependence of the excitation spectrum and structure factor of helium-4 was not taken up rigorously, whereas the experiments of Woods, Hallock, Dietrich et al, Achtev and Meyer, Cowley and Woods etc revealed a distinct temperature dependence of phonon spectrum for liquid helium-4. This dependence, although small at very low temperatures became quite significant in the neighbourhood of $\lambda$-point.

Samuloski and Isihara in their microscopic theory for liquid helium-4 for temperature variation of sound velocity and structure factor attempted to justify the Landau phenomenological theory and held the view that the elementary excitations, being due to the collective couplings of particles in the liquid state, should be temperature dependent. Their view was a soft potential for He-He interaction. In view of liquid helium remaining liquid even at absolute zero, a more realistic view was a hard core repulsive interaction. Khanna and Das built a Gaussian equivalent of the Leonard-Jönnies potential having the required fourier transform, the more realistic approach, and with this could derive the energy excitation spectrum of interacting bosons resembling, both qualitatively and quantitatively the experimental excitation spectrum of liquid helium-4. Many other authors have attempted their own approaches from time to time.
An excitation spectrum was recently obtained by us using a scaling transformation and using the Hamiltonian:

\[ \mathcal{H} = -\frac{1}{2m} \int dx \left[ \left( \frac{\partial^2}{\partial x^2} \psi(x) \right)^2 + \left( \frac{\partial^2}{\partial x^2} \psi(x) \right)^2 + V(x) \psi(x) \psi(x) \right] \]

\[ = \frac{1}{2m} \int dx \left[ \left( \frac{\partial}{\partial x} \psi(x) \right)^2 + \left( \frac{\partial}{\partial x} \psi(x) \right)^2 + V(x) \psi(x) \psi(x) \right] \]

\[ + \frac{1}{2m} \int dx \int dy \psi(x) \psi(y) V(x-y) \psi(y) \psi(y) \]

\[ = \mathcal{H} + \mathcal{V} \]

\[ \psi \]

obeys the commutation relation of Eq.(2-4).

Taking:

\[ q_k = \frac{1}{\sqrt{2}} \left[ a_k + a_k^\dagger \right] \]

\[ p_k = \frac{1}{\sqrt{2}} \left[ a_k - a_k^\dagger \right] \]

such that

\[ \psi(x) = \frac{1}{\sqrt{2\pi}} \sum_k a_k \phi_{kx} \]

\[ \psi^\dagger(x) = \frac{1}{\sqrt{2\pi}} \sum_k a_k^\dagger \phi_{kx} \]

In this q, p representation we have the Eq (2-8) as:

\[ \psi(\rightarrow) \rightarrow \psi^+ \rightarrow = \frac{1}{\sqrt{2\pi}} \sum_k \left[ q_k + ip_k + \phi_{kx} \right] \]

\[ \psi^\dagger(\rightarrow) \rightarrow \psi^\dagger \rightarrow = \frac{1}{\sqrt{2\pi}} \sum_k \left[ q_k - ip_k + \phi_{kx} \right] \]

Accordingly the Hamiltonian (3-1) in terms of \( q_k, p_k \) operators up to the constant terms and terms of orders higher than the second term takes the form:

\[ \mathcal{H} = \mathcal{H}_0 - MN \]

\[ = \frac{1}{2} \sum_{k \neq \phi} \left\{ \left( \frac{k^2}{2m} + 2 \phi_k \right) q_k q_k^\dagger + \right. \]

\[ \left. \sum_{k \neq \phi} \left( \frac{k^2}{2m} - \phi_k^2 \right) q_k^\dagger q_k \right\} \]

\[ - \sum_{k \neq \phi} \left( \frac{k^2}{2m} - \phi_k^2 \right) \]

\[ \frac{1}{2} \sum_{k \neq \phi} \left( \frac{k^2}{2m} - \phi_k^2 \right) \]

Where \( N \) is the operator of the number of particles and \( \mu \) is the chemical potential, equal to \( \sqrt{\lambda} \) to the first approximation in this formalism.

Now we make the following scaling change to cast the Hamiltonian suitable for the free field into the form as that of interaction:

\[ Q_k = \frac{1}{\sqrt{2m}} \left( b_k + b_k^+ \right) \]

\[ P_k = \frac{1}{\sqrt{2m}} \left( b_k^+ - b_k \right) \]

where \( \gamma(k) \) is the factor depending upon the nature of momentum dependent interaction.

This scaling transformation is a Bogoliubov transformation of the form

\[ b_k = u_k a_k - u_k^* a_k^\dagger \]

\[ b_k = u_k a_k - u_k^* a_k^\dagger \]

with

\[ u_k = \frac{1}{\sqrt{2}} \left( \gamma(k) + \gamma^{-1}(k) \right) \]

\[ u_k^* = \frac{1}{\sqrt{2}} \left( \gamma(k) - \gamma^{-1}(k) \right) \]

\[ u_k = \frac{1}{\sqrt{2}} \left( \gamma(k) - \gamma^{-1}(k) \right) \]

\[ u_k^* = \frac{1}{\sqrt{2}} \left( \gamma(k) + \gamma^{-1}(k) \right) \]

Provided we associate annihilation and creation operators with \( P_k \) and \( Q_k \) in the form:

\[ Q_k = \frac{1}{\sqrt{2m}} \left( b_k + b_k^+ \right) \]

\[ P_k = \frac{1}{\sqrt{2m}} \left( b_k^+ - b_k \right) \]

So under the scaling transformation Eq(3-6) Hamiltonian can be written as:

\[ \mathcal{H} = \frac{1}{2} \sum_{k \neq \phi} \omega_k (Q_k P_k + P_k Q_k) - \frac{1}{2} \sum_{k \neq \phi} \left\{ \frac{k^2}{2m} - 2 \phi_k^2 \right\} \]

Where \( \omega_k \) the energy of excitation with \( \omega \) is given by:

\[ \omega_k^2 = \frac{k^2}{2m} \left( \frac{k^2}{2m} + 2 \phi_k^2 \right) \]

\[ \omega_k^2 = \frac{k^2}{2m} \left( \frac{k^2}{2m} + 2 \phi_k^2 \right) \]

This is of the form of Bogoliubov spectrum.
\[
q(k) = \frac{1}{2} \left[ Y(k) + \Pi(k) \right]
\]
\[
P(k) = \frac{-1}{2} \left[ Y(k) - \Pi(k) \right]
\]
So that
\[
q_k = \frac{1}{\sqrt{2}} \left[ a_k + a^+_{-k} \right]
\]
and
\[
P_k = \frac{1}{\sqrt{2}} \left[ a_k - a^+_{-k} \right]
\]
and
\[
N = a_k a^+_{k} - a_{-k} a^+_{-k}
\]
We obtain
\[
\Omega_k = \left( \frac{k^2}{2m} \right) + \frac{\hbar V}{2 \sqrt{\gamma(k)}} \gamma(k)^{1/2}
\]
Using the scaling change
\[
D_k = \sqrt{\gamma(k)} q_k
\]
\[
P_k = \frac{1}{\sqrt{\gamma(k)}} p_k
\]
Now in the phonon mediated interaction between \( ^3 \text{He} \), quasiparticles in \( ^3 \text{He} \)-\( ^3 \text{He} \) mixtures a simple parametric form for \( V_k \) is represented by
\[
V_k = -|V_0| \cos \left( \frac{\sqrt{\gamma(k)}}{\gamma(k)} \right)
\]
So assuming that the interaction between phonons in the excited superfluid \( ^4 \text{He} \) to be of the same form as that between \( ^3 \text{He} \) quasiparticles in \( ^3 \text{He} \)-\( ^3 \text{He} \) mixtures so that we can take \( \gamma(k) = \cos(k) \)
Such that \( \gamma(-k) = \cos(-k) = \cos(k) = \gamma(k) \)
\[
V_k = V_{-k}
\]
Accordingly we have
\[
\Omega_k = \left( \frac{k^2}{2m} \right) + \frac{\hbar V}{2 \sqrt{N}} \gamma(k)^{1/2} \cos k
\]
With
\[
n = \frac{N}{\sqrt{\gamma}}
\]
Expansion of \( \cos(k) \) in the powers of \( k \) and taking the \( \sqrt{\gamma} \) root we get
\[
\Omega(k) = ck \left[ 1 - \gamma_{1/2} k^2 + \frac{k^4}{4} \right]
\]
This to the second approximation yields
\[
\Omega(k) = ck \left( 1 - \gamma \frac{k^2}{4} \right)
\]
with \( c = \left( \frac{N \hbar}{m } \right)^{1/2} \) and \( \gamma = 0.5 \pm 0.4992 \times 10^{-2} \) kg M sec
This tallies with that obtained by Landau and Khalatnikov, where \( \gamma = 2.8 \times 10^{-2} \) kg M sec and with that obtained by neutron scattering data, where \( \gamma = 47.2 \times 10^{-2} \) kg M sec.
For small values of momentum, \( \Omega_k \) from Eqs. (3.10) and (3.19) reduced to the form:
\[
\Omega_k = c \left| k \right|
\]
when
\[
c^2 = \left( \frac{\sqrt{\gamma(k)}}{\gamma(k)} \right)
\]
from Eq. (3.10)
It is therefore evident that \( \Omega_k \to 0 \) for \( k \to 0 \) in accordance with the Goldstone theorem. It is also clear that the elementary excitations are the phonons propagated with the speed of sound "c".

Many suggestions have been made to explain the nature of roton minimum. Landau suggested some kind of vortex motion — an idea later taken up by Feynman and Feynman and Cohen, however recent work has shown that the theory does not describe at all well the behavior of roton minimum as a function of pressure. There seems to be no convincing reason to believe that roton motion is necessarily associated with a roton. de-Boer considered rotons as short wave-length longitudinal elastic modes which are completely degenerate, because the wave length is of the order of the magnitude of the average distance between the neighboring atoms. Chester and Miller et al regarded roton as the modified free particle. Onsager called the roton as a ghost of a vanishing vortex ring.

According to Anderson vortex nucleation is the biggest puzzle in the theory of superfluid \( ^4 \text{He} \) and hence the dynamical formulation of the energy gap needed for the roton excitation is still to be understood by a refined microscopic theory. It has been hypothesized that...
the energy gap needed for roton-like excitation, like the quarks in elementary particle theory, can be generated dynamically\(^6^7\) and this energy gap is essential to describe the low-temperature behavior of superfluid \(^4\)He.\(^6^8\) This idea became pertinent due to the fact that there exists a complete analogy between the production of hadrons and creation of excitations in condensed matter physics.\(^6^9\) Encouraged with such a notion, a theory of roton-like excitation in He-II was formulated by exploiting the 3-dimensional rotation group and the introduction of roton operators.\(^3^0\) This way, coupling of roton-like excitation including many states in the roton region of the Landau spectrum of superfluid helium\(^2\) is formulated. This model is justified on two accounts.

1. Liquid helium has a close resemblance to the system of solids.\(^7^0\)
2. Roton is the smallest of the vortices which are produced when He-II moves with a great velocity and the mixture of quantized vortex lines and the superfluid components can mimic the rigid body rotation.\(^7^1\)

On these lines, recently a relation between the roton excitation and Bogoliubov quasiparticles has been established, by giving different structures to the roton operators.\(^3^2\) This formalism is similar to quasiboson or random phase approximation for the phonon-like spectrum. This structure of the roton operators enables to study the low-temperature behavior of superfluid \(^4\)He by the introduction of the idea of pairing for roton-like excitation for the onset of energy gap. This pairing is possible due to a very large density of states of roton-like excitations in the neighborhood of roton minimum of Landau’s model.\(^3^3\) Application of this roton-like excitation model to describe the density dependence and temperature dependence of the energy gap yields the result of the temperature dependence of energy gap more closely in consonance with those obtained experimentally. However, the roton remains a mystery and so does a very large part of the spectrum. The spectrum is very sensitive to pressure changes. Both the velocity of sound which controls the small \(k\)-part of the spectrum and the roton minimum change markedly as the pressure is increased.\(^7^5\) This requires some explanation. Further there seems so far no explanation to what causes the spectrum to end.

A number of authors have in recent past worked with the vorticity aspect of superfluid \(^4\)He. However, it still remains to be understood how actually the vortex lines get into the fluid, since with the absence of viscosity, the circulation is still an invariant of motion. Venin suggested that vortex is a macroscopic excitation which cannot be created at low superfluid velocity and the observed breakdown of pure superflow at very low velocity cannot be well explained by stretching a few lengths of vortex lines that may already be present in the apparently undisturbed helium.

**IV. Transition Probabilities**

Superfluid \(^4\)He may be divided into the states of condensed bosons, oscillating condensed bosons, phonon states, the Maion states, the roton states and the state of vorticities. So there are different states to state transitions in the whole sector of superfluid \(^4\)He. Accordingly, we consider the transition probabilities between various states as is detailed below.

<table>
<thead>
<tr>
<th>Transition from Condensed State to Oscillating Condensed Boson State</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bogoliubov's theory proposed the first ab-initio theory for the excitation spectrum which gave a qualitative explanation of superfluid helium-4. However, the theory failed to give converging results with higher order correction to this spectrum. The reason is that he ignored the quantum fluctuations of the zero-momentum state. Accordingly these fluctuations of the condensed bosons were incorporated into the Hamiltonian to remedy the anomaly.</td>
</tr>
</tbody>
</table>

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\(^2\) Roton is the smallest of the vortices which are produced when He-II moves with a great velocity and the mixture of quantized vortex lines and the superfluid components can mimic the rigid body rotation.

\(^7^0\) Liquid helium has a close resemblance to the system of solids.

\(^7^1\) Roton is the smallest of the vortices which are produced when He-II moves with a great velocity and the mixture of quantized vortex lines and the superfluid components can mimic the rigid body rotation.

\(^3^2\) This formalism is similar to quasiboson or random phase approximation for the phonon-like spectrum.

\(^3^3\) Application of this roton-like excitation model to describe the density dependence and temperature dependence of the energy gap yields the result of the temperature dependence of energy gap more closely in consonance with those obtained experimentally. However, the roton remains a mystery and so does a very large part of the spectrum. The spectrum is very sensitive to pressure changes.
To this end the zero momentum single particle operators $a_0$ and $a_0^{\dagger}$ taken equal to a c-number constant, have been replaced as:

$$a_0 \rightarrow a + c_i$$

Where $c_i$ is a new annihilation operator that describes the small oscillations of the condensed bosons in the ground state. These operators $c_i$ and $c_i^{\dagger}$ obey the usual boson commutation relations:

$$[c_i, c_i^{\dagger}] = 1$$
$$[c_i, c_j] = [c_i, c_j^{\dagger}] = 0$$

However it is approximated that while the shift of the equilibrium point $a_0 \rightarrow a_0^{\dagger} \neq N$ is macroscopic, the matrix elements of $c_i$ remain finite and small even when the system becomes infinitely large. Further the operator $c_i$ may be of quantum origin or it can be due to the interaction of the zero momentum particles with the walls of the container or with $k \neq 0$ particles. The vacuum $|\Omega\rangle$ of $c_i$ and $a_k (k \neq 0)$ is the true ground state and is the eigen vector of $\mathcal{H} = \mathcal{H} - \mu N$, defined by:

$$\mathcal{H} |\Omega\rangle = 0, c_i |\Omega\rangle = 0, a_k |\Omega\rangle = 0 \quad k \neq 0$$

The number of physical particles are to be counted by $N = \sum a_k^\dagger a_k$, the vacuum of which is defined by

$$a_k |0\rangle = 0 \quad (a_k^\dagger |0\rangle) = 1$$

Now the conservation of the particle number implies the invariance of Hamiltonian under the phase Transformation:

$$k \rightarrow e^{-i\beta} \quad \text{and} \quad k \rightarrow e^{i\beta}$$

Accordingly we imagine that we have employed the transformed operators from the beginning and we make the following change of variables:

$$a_k^\dagger = 1/2 \left( a_k^\dagger + a_k \right)$$

Although the replacement Eq (4-1) provides us with a useful approximation scheme, this new form Eq (4-6) will also work well both for free particles as well as for the interacting particles. The apparent forms of $\mathcal{H}'$ before and after the change of variables are different from each other, but their eigen values will appear to coincide. The change of variables can be looked upon as a unitary transformation within the representation space of $a_0$ and $a_0^{\dagger}$. In fact

$$c_i^\dagger a_k = U_\beta a_k^{\dagger} e^{-i\beta}$$

or

$$c_i a_k = U_\beta a_k^{\dagger} + V_\beta e^{i\beta}$$

and

$$c_i^\dagger a_k = U_\beta a_k^{\dagger} + V_\beta e^{i\beta}$$

Where:

$$U_\beta = \exp \left[ \frac{-i\beta}{0} \left( a_0^{\dagger} - a_0 \right) \right]$$
$$V_\beta = \exp \left[ \frac{-i\beta}{0} \left( a_0 + a_0^{\dagger} \right) \right]$$
$$\mathcal{H} = \cosh(\beta) \mathcal{H}_0 + \sinh(\beta) \mathcal{V}$$

and $\{a_0, a_0^{\dagger}\} = \{a_0^{\dagger}, a_0\} = 0$. The true vacuum state is related to the vacuum state of $a_0$ as:

$$|\Omega_\beta\rangle = U_\beta |0\rangle$$

where $\beta$ with the vector $|\Omega_\beta\rangle$ is put to indicate a superposition of the states with different occupation numbers. This relation indicates a transition from the states of the condensed bosons with zero oscillation, to the state of oscillation of the condensed bosons. To compute the transition probability between these states, we first consider the coupling co-efficient between the states of $k$-oscillating condensed bosons and $k^*$-oscillating condensed bosons defined by:

$$B_{k^*, k}(\beta) = \frac{1}{k^*! k!} \int \exp \left[ -2i N \mathcal{H}_0 \right] |0\rangle \langle k^* k|$$

...
Using the Eqs (4-7), (4-8) and (4-11) we get for $k$ and $l$
both non-increasing the following recursion formulae:
\[
-k \cosh(\beta) H_k(\beta) = -\sinh(\beta) H_{k-1}(\beta) + H_{k-2}(\beta),
\]
and
\[
l \cosh(\beta) H_l(\beta) = \sinh(\beta) H_{l-1}(\beta) + H_{l-2}(\beta).
\]
with $H_l(\beta) \neq 0$ for $k$ and $l$ both odd or even,
also $H_{k+1}(\beta) = H_{k}(\beta) = 0$, where $N > 0$ ....4-12

Let the generating function be defined by:
\[
H(a, b, \beta) = \sum_{k,l} H_{k,l}(\beta) a^k b^l
\]
Its differentiation with respect to $a$ and subsequent use of
Eqs. (4-12) gives us:
\[
H(a, b, \beta) = H(\beta) \exp\left[\frac{ab}{\cosh\beta} - \frac{(a - b)^2}{2} \frac{\tanh\beta}{\cosh\beta}\right]
\]
The value of $H(\beta)$ as computed from its
definition, and on its differentiation with respect to
$\beta$ is equal to $e^{-2iN\beta}$ since $H(0) = 1$
So we have
\[
H(a, b, \beta) = e^{-2iN\beta} \exp\left[\frac{ab}{\cosh\beta} - \frac{(a - b)^2}{2} \frac{\tanh\beta}{\cosh\beta}\right]
\]
Expansion of this generating function in powers of "a" and
"b" gives us the transition probability between the vacuum
state of $a_0$ and the true ground state with $k$ oscillating
condensed bosons as:
\[
|G(\beta)|^2 = \left((-1)^{k+1} \frac{\tanh\beta}{\cosh\beta} \sum_{2} \frac{(4N\beta)^2}{(2Z+1)! (1/2k-2-Z)! (Z)!} \right)^2
\]
for even $k$.
\[
\frac{(-1)^{k-1} \frac{\cosh\beta}{\sinh\beta} \sum_{2} \frac{(4N\beta)^2}{(2Z+1)! (1/2k+1-Z)! (Z)!}}{\cosh\beta}
\]
for odd $k$, ....4-15

From this expression we find that the transition
probability is a function of the condensate occupation $N$

\[
\text{TRANSITION FROM OSCILLATING CONDENSED BOSON STATE TO THE PHONON STATES:—}
\]
The states of oscillation of condensed bosons
are harmonic type of vibrations. These independent
oscillations executed by the condensed bosons are with
random amplitude and direction but with the same
frequency, which are denoted by the index $a$ in the operator
$\hat{H}_a$. Such a motion can be imagined as the collective state
that gives rise to a gradient of oscillation. For the states
of oscillation by zero momentum condensed bosons, a state
having such a collective characteristic can be obtained by
choosing the appropriate linear combination of the
degenerate states. As this system of oscillating
condensed bosons has motions of the harmonic oscillator
type, different types of correlations are obtained by
choosing different linear combinations. The lowest state
may have no collective dilational forms. However, the
highest levels in the system of oscillating condensed
bosons with $k=0$, form the excitation spectrum on this
intrinsic fundamental state. This collective behaviour will
finally lead to the formation of the phonon spectrum of the
compressional waves. The highest levels in the system of
oscillating condensed bosons is the result of the
oscillating condensed bosons having acquired different
frequencies. This way the ground state for phonons becomes
asymmetric and changes into a different possible vacuum
state, distinguished from the former by the change of its
phase. This transition from one vacuum state to another is
described by the production of phonons, which break
the symmetry of superfluid helium ground state.

The variance in the frequencies of the oscillating
condensed bosons can be ascribed to the dynamical effect
that takes place between any two particles or subsystems in
the phonon vacuum state, thereby violating the conservation
law of the underlying Hamiltonian, under symmetry operator
$U(\theta) = e^{i\theta c^+}$, i.e., $U(\theta) H U(\theta) \neq H$, ....4-17
where $U(\theta)$ is a unitary operator representing a continuous
symmetry group.

\[
\text{-24-}
\]

\[
\text{-25-}
\]
Accordingly the true vacuum state (state of oscillating condensed bosons) is related to the phonon states as:

$$|\Phi\rangle = U(\theta)|\Omega\rangle$$

where $|\Omega\rangle$ is the vacuum phonon state. The phonon operators $a^+$, which are the transforms of the oscillating condensed boson operators $c$ and $c^+$ are given by:

$$a^+ = Uc + Vc^+ = e^{i\theta}c^+$$

$$a = Uc + Vc = e^{-i\theta}c$$

where

$$S = i\partial c^+c, \quad U = \cosh \theta, \quad V = \sinh \theta$$

$$\{a^+, a\} = \{e^+, e\} = 0$$

$$\{a_i^+, a_j\} = \delta_{ij}$$

Adopting the same procedure as in the previous section, we obtain the transition probability from the zero phonon state to $m$ phonon state as:

$$P_{\text{phonon}} = \left|\frac{G_i(\theta)}{K_{m,0}}\right|^2 \left[\frac{\tanh(\theta)}{1/2} \sum_{Z=1}^{2Z}(1/2^Z(2Z+1)(Z+1/2)(1/2(m-1)-Z))\right]$$

from even $m$, 4-18

$$P_{\text{phonon}} = \sqrt{\frac{(\tanh(\theta))^{m-1}}{\cosh(\theta)}} \sum_{Z=1}^{2Z} \left[\frac{(4/\sinh(\theta))^{2Z}}{Z(2Z+1)(2Z+1/2)(1/2(m-1)-Z)}\right]$$

for odd $m$, 4-19

By the Theorem of compound probability, we have the transition probability of having the oscillating condensed bosons and phonons states simultaneously as:

$$\tau_{\text{phonon}} = \left|\frac{G_i(\theta)}{K_{m,0}}\right|^2 \times \left|\frac{G(\theta)}{m,0}\right|^2$$

where $\left|\frac{G_i(\theta)}{K_{m,0}}\right|^2$ is given by the eq. (3-69)

TRANSITION FROM PHONON STATES TO ROTON STATES:

For low energy phonons, the velocity increases with increasing energy. These phonons with increasing energy, therefore always have velocities greater than the velocities of all phonons of lower energy. Hence the decay process is possible. However the difference in their velocities is very small. For the phonons of higher energy, the velocity starts to decrease during decay process and some of the lower energy phonons have a higher velocity. The decay possibility then becomes restricted and above some critical energy $E$ phonons become totally stable against any decay. In the absence of any decay there is only the interaction of phonons and the interaction of attractive nature can cause the bunching of phonons. This bunching of phonons results in the creation of roton like excitations. Accordingly the critical energy at which the phonons become stable against any decay corresponds to the minimum energy required for the creation of roton like excitations. This is borne out by fact that the value of critical energy at which the phonons become stable against any decay into a large number of phonons, as found by Dyne and Narayanmurti at zero pressure is 9.5 K and as found by Marie is 9.86 K while the minimum energy required for the creation of roton as calculated from the thermodinamical data is given by

$$\frac{\Delta}{k_B} = 9.8 K$$

and this as calculated from Neutron scattering data is

$$\frac{\Delta}{k_B} = 9.9 K$$

To correlate the phonon states with the roton states one can choose the bilinear operators $a^+ a$ due for phonon state and construct out of them the angular momentum operators $L_{ij}$ given by

$$L_{ij} = i (a^+ a - a a^+)$$
The vacuum roton state is considered as a set of eigen states of the ground state with $L = 0$ and the roton state is a well defined angular momentum state formed out of eigen values of the operators of $\hat{L}$ and $\hat{L}_z$, the operator that we use for the transformation of phonon to roton states is

$$L_{ij} = \phi(a_a^+a_a^*)_{ij}, \quad \ldots \quad 4.24$$

where $\phi$ is a phase change as we go from phonon states to roton states. Thus the Bogoliubov transformation is $e$ and transforms $c'$ and $c''$ of the phonon operators $a', a^*$ are:

$$c' = e a e^+ = u a + v a^+ \quad \ldots \quad 4.24$$

$$c'' = e a e^+ = u a + v a^+ \quad \ldots \quad 4.24$$

where

$$(a, a^*) = \delta_{ij}, \quad 1 < j \quad 4.17$$

$$u^* u = \cosh(\phi), \quad \sqrt{v} \sqrt{v} = \sinh(\phi) \quad \ldots \quad 4.25$$

Hence $|\psi\rangle = S(\phi)| 0\rangle \quad \ldots \quad 4.26$

where $|0\rangle$ is the vacuum roton state.

We define the coupling co-efficient between two possible roton states as:

$$B\phi(\phi)_{r,s} = \frac{1}{\cosh(\phi)_{r,s}} \langle 0 | a_a a a^+ a a^+ | 0 \rangle$$

$$= \sqrt{p} \sqrt{q} |r-s| \sinh(\phi)_{r,s} \quad \ldots \quad 4.27$$

Using the equations (4-25) and (4-27), we obtain the following relations with indices all non-vanishing.

$$p \cosh(\phi) H_{p^i q^j r^i s^j} = \sinh(\phi) H_{p^i q^j r^i s^j} + H_{p^i q^j r^i s^j}$$

$$q \cosh(\phi) H_{p^i q^j r^i s^j} = \sinh(\phi) H_{p^i q^j r^i s^j} + H_{p^i q^j r^i s^j}$$

$$r \cosh(\phi) H_{p^i q^j r^i s^j} = \sinh(\phi) H_{p^i q^j r^i s^j} + H_{p^i q^j r^i s^j}$$

As a consequence of momentum conservation and from the definition of $H\phi$ we conclude that

$$H(0)_{p^i q^j r^i s^j} = 0 \quad \text{unless } p' - q' = r' - s' \quad \ldots \quad 4.28$$

Here we take the generating functions as:

$$H(a, b, c, d, \phi) = \frac{1}{\cosh(\phi)} \prod_{p'^i q'^j r'^i s'^j} H_{p'^i q'^j r'^i s'^j} \quad \ldots \quad 4.31$$

Differentiating this with respect to $\phi$ and making the substitution from Eqs. (4-28) and (4-25) we obtain

$$H_{a, b, c, d, \phi} = \frac{1}{\cosh(\phi)} \prod_{p'^i q'^j r'^i s'^j} H_{p'^i q'^j r'^i s'^j} \quad \ldots \quad 4.30$$

The value of $H(0)$ is obtained from its definition and the Eq (4-28). Expansion of the generating function in the powers of $a, b, c$ and $d$ gives the transition probability from phonon to roton states as:

$$\mathcal{K}(\phi) = p(\phi) \cosh(\phi)_{r,s} \quad \ldots \quad 4.32$$

Its dependence on the condensate fraction is obtained by the theorem of compound probability as:

$$\mathcal{K}(0, \phi) = \mathcal{K}_{\phi} \mathcal{K}_{\text{phonon}} \mathcal{K}_{\text{rot}}$$

where $\mathcal{K}_{\phi}$ is given by Eq (4-22).
According to Bogoliubov, the symmetry of a system is broken by the introduction of an infinitely weak external field into the treatment. The interaction term of the total Hamiltonian of the system breaks the phase symmetry of the free Hamiltonian. This condition of broken symmetry leads to the translation of the boson field operators by \( c \)-numbers, Eq (2-5) or equivalently Eq (2-32). Under this situation the Hamiltonian of the system becomes asymmetric so that the conservation law seems to be violated. Now to create a phase variance we have to send the system through some external potential and the magnitude of the phase variance equals the potential times the time it is in the potential. Sending the system through some external potential is to set in the interactions in the form of a dynamical effect taking place between two particles or subsystems. This way we impose some asymmetric condition on the ground state of the system which changes into a different possible ground state. These ground states are distinguished from each other by the change of their phase. Accordingly for each specified phase there exists the corresponding ground state and to each ground state is associated a Fock representation. This transition from one vacuum (ground state) to another vacuum state can be described as the production of quasi-particles (phonons). The term translation or displacement of the field operator was first used by Glauber, because the transformation Eq (2-5) can be induced by a unitary operator:

\[
U(c) = \exp \left[ \frac{i}{\hbar} \int d\mathbf{x} \left( \alpha(x) \Phi^*(x) - \alpha^*(x) \Phi(x) \right) \right]
\]

So that

\[
U(c) \Phi^*(x) U(c) = \Phi^*(x) + \alpha \Phi(x)
\]

The \( c \)-number constant has been generalized to \( \alpha(x) \) when the system is in a container with rigid walls. This dependence of \( \alpha \) on the coordinate space is related to the transformation Eq (2-5) in the sense that the transformation can be viewed as a nonlinear realization of the gauge transformation which can be obtained by imposing the constraint of \( \Phi(x) = f(x) \), viz \( \Phi \Phi^T = 1 \) on the Bose operators \( \Phi \) and \( \Phi^T \). The generalization of \( c \)-number would be the case even for the non-interacting Boson system. However with interaction having the Hamiltonian of the system becomes asymmetric so that its conservation law seems to be violated. Now to create a phase variance we have to send the system through some external potential and the magnitude of the phase variance equals the potential times the time it is in the potential. Sending the system through some external potential is to set in the interactions in the form of a dynamical effect taking place between two particles or subsystems. This way we impose some asymmetric condition on the ground state of the system which changes into a different possible ground state. These ground states are distinguished from each other by the change of their phase. Accordingly for each specified phase there exists the corresponding ground state and to each ground state is associated a Fock representation. This transition from one vacuum (ground state) to another vacuum state can be described as the production of quasi-particles (phonons). The physical significance, as the entire phonon spectrum in superfluid He is to be understood in the limit of \( N \rightarrow \infty \) and if we go out of the volume limit we are going out of the Fock space in which the original boson field operators are defined. Therefore we take recourse to passive transformation, which is a coordinate transformation in the opposite sense and can be performed in any Fock representation in which \( \Phi \) is defined. This ensures the...
rotational invariance by the production of phonon coherent states. Accordingly coherent superposition of states in the phonon region of superfluid helium is obtained by applying a "Block rotation" to an extremal (vacuum) state in a non-compact unitary irreducible representation of the
non-compact group SU(1,1)
\[
|\phi\rangle = R(\theta) |j,0\rangle = e^{i\frac{1}{2}J^z} |j,0\rangle
= \sum (-1)^{j+n/2}(\frac{\theta}{2})^n \left[ \frac{\left(j+n+1\right)!}{\left(j-n+1\right)!} \right]^{1/2} |j+n\rangle
\]
with the ground state defined by
\[
|j,0\rangle = 0
\]
and the generators of the group SU(1,1) obeying the commutation relations:
\[
[J^+, J^-] = i J^z
\]
\[
[J^x, J^y] = i J^z
\]
\[
[J^y, J^x] = i J^z
\]
with
\[
J^z = \frac{1}{2} (j_1j_2 + j_3)
\]
The norm of the state is given by:
\[
\langle \phi | \phi \rangle = \sum (-1)^{n} (\frac{\theta}{2})^{2n} \left[ \frac{\left(j+n+1\right)!}{\left(j-n+1\right)!} \right]^{2} \left( \frac{1}{\left(j+n+1\right)!} \right)
\]
From the finite nature of the norm, we conclude that the coherent states \( |\phi\rangle \) under discussion, for all values of \( n \) span the Hilbert space, as is the case with \( |\Psi\rangle \) which forms the basis of the number operator \( N \) and entails the complete orthonormal set. After the formation of coherent superposition of states the boson field operator is:
\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \left[ \sum L \alpha \langle \alpha | e^{ikx} + \langle \alpha | e^{-ikx} \right]
\]
Accordingly the density matrix becomes
\[
\rho(x,x') = \begin{pmatrix} \Psi^*(x) \Phi(x) \\ \Phi(x) \Psi(x) \end{pmatrix}
= \frac{1}{2\pi} \left[ \sum \langle \alpha | e^{-ikx} \alpha \rangle \exp(-i\alpha x + ikx) \right]
\]
\[
= \frac{1}{2\pi} \left[ \sum \langle \alpha \alpha | \exp(-i\alpha x + ikx) \right]
\]
\[
= \frac{1}{2\pi} \left[ \sum \langle \alpha \alpha | \exp(-i\alpha x + ikx) \right]
\]
This is the form postulated by Penrose and Onsager and discussed by Hohenberg and Martin and Anderson. The first term of this expression describes the normal uncondensed phonons in the system. This term, on account of oscillatory nature of the exponential, vanishes as \( x \to \infty \).

The second term depicts the phenomenon of ODLRO. According to Yang, if the fundamental definition of ODLRO is taken to be the existence of some finite degrees of factorization of an nth order reduced density matrix in the coordinate representation then definitions of ODLRO and coherence become identical. Accordingly the second term in the expression also depicts the coherence phenomenon in phonons in superfluid helium.

Further because of the Eq(5-3) the particle density \( f(x) = (\Psi^+(x), \Phi(x)) \), the Hamiltonian describing the helium system in fock space defined by Eq(3-1) and the associated current density defined by:
\[
J(x) = -\frac{i}{\hbar} \left[ \Psi^+(x) \nabla \Psi - \Psi \nabla \Psi^+ \right]
\]
have their C-number counterparts as:
\[
f_{\phi(x)} = \frac{1}{2\pi} \left[ \Psi^+(x) \nabla \Psi - \Psi \nabla \Psi^+ \right]
\]
\[
f_{\Phi(x)} = \frac{1}{2\pi} \left[ \Psi^+(x) \nabla \Psi - \Psi \nabla \Psi^+ \right]
\]
\[
f_{\Psi(x)} = \frac{1}{2\pi} \left[ \Psi^+(x) \nabla \Psi - \Psi \nabla \Psi^+ \right]
\]
COHERENT STATES IN OSCILLATING CONDENSED BOSONS

Oscillation of the condensed bosons showing high degeneracy is described by the symmetry group SU(n) in N-dimensions. The basic components of quanta emerging from high zero point energy of the symmetry group are
transformed among themselves and accordingly the frequency of the oscillating condensed bosons is assumed to be exact. Thus the ground state of superfluid helium consists of oscillating condensed bosons and is seen to contain SU(2) symmetric structures. If the frequencies of the oscillating condensed bosons are different then the broken symmetry condition is visualized and leads to excited states of phonon type, roton-type and even the higher ones.

Considering the quantum fluctuations in the condensate as a classical source which excite the single mode, the oscillating condensed bosons can be treated as a system of driven oscillators

$$ H = \frac{1}{2} (p + q)^2 - \frac{1}{2} (p - q)^2 - \lambda \tag{5-12} $$

represents the vacuum state $|0\rangle$ of superfluid helium with vanishing mean position & momentum. i.e. $\langle 0| q |0\rangle = 0, \langle 0| p |0\rangle = 0 \tag{5-12a}$

Introduction of c-number terms due to the quantum fluctuations gives the new Hamiltonian that represents the true ground state of superfluid helium. This state is the coherent state of the oscillating condensed bosons with the exact frequencies. This is realized by the operation of the unitary operator

$$ U(p, q) = \exp\left(\frac{i}{\hbar} (p^q - q^p)\right) \tag{5-13} $$

on the ground state. As this operator has the effect of translating the operators $q$ & $p$ by c-numbers, the coherent state obtained by its operation has its coordinates and momentum displaced by $q$ & $p$ from that of the state $|0\rangle$.

$$ \langle p, q | \hat{p} | 0 \rangle = 0, \langle p, q | \hat{q} | 0 \rangle = 0 \tag{5-14} $$

Thus in the state $|p, q\rangle$ the mean position & momentum do not vanish,

$$ \langle p, q | p, q \rangle = \langle 0| q^q + q^q |0\rangle = q, \langle p, q | p, q \rangle = \langle 0| p^p + p^p |0\rangle = p \tag{5-15} $$

The density matrix

$$ f_\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| \langle \psi | e^{i\theta} \right| \langle \psi | \right| d\theta \tag{5-16} $$

describes the average over phases of condensed bosons and provides the particle number:

$$ \langle N \rangle = Tr \left\{ \rho (\hat{c}^\dagger \hat{c}) \right\} = |N|^2 \tag{5-17} $$

and the probability of having "n" oscillating bosons in the state:

$$ P_n = Tr \left\{ \rho |n\rangle \langle n| \right\} = \frac{\langle N \rangle^n}{n!} \tag{5-18} $$

This is poisson's law for the oscillating condensed bosons representing coherent behavior and confining that the state is the coherent state of the oscillating condensed bosons. It is seen that the displaced oscillating condensed boson ground state $|p, q\rangle$ evolves in time from another state of the same nature under the action of displaced Hamiltonian, meaning that only its mean position & momentum changes in accordance with the classical mechanics. Thus the coherent states formed in the oscillating condensed bosons of superfluid He are physically the relevant ground states for the driven type oscillations.

DISCRETE AND CONTINUOUS REPRESENTATION OF COHERENT STATES

Bogoliubov showed that the problem of superfluidity reduces to that of finding the spectrum and wave functions of the Hamiltonian that is quadratic in boson creation and destruction operators. He also indicated how the problem can be solved by diagonalizing the Hamiltonian with the aid of the linear canonical transformations, now called Bogoliubov's canonical transformations. The set of linear canonical transformations for this problem forms a certain group and indeed a direct product of $SU(1,1)$ groups. This group $SU(1,1)$ is isomorphic to $SO(2,1)$ and $SL(2,R)$ groups. The generators of this group as visualized from the Bogoliubov's truncated Hamiltonian and written in terms of creation and annihilation operators obey the commutation relation as indicated in eq.5-6.
The group under consideration has several series of irreducible unitary representations, the discrete, the continuous and the supplementary. It is possible therefore to construct several sets of coherent states associated with the group.

Physically it can be said that the group can describe a large number of states which can be visualized here to be the property of interaction in superfluid helium. This spectrum, generated by non-compact group describes the excited states up to the phonon level. Again as there exists no non-trivial finite-dimensional representation, the excitation can be represented by a unitary representation:

\[ \Pi_k \mathbf{U}(J_k) \text{ where } J_k = [\alpha_k + \frac{1}{2} |\Delta_k|] \]

and for the unitary representation we have

\[ J_k = \frac{1}{2} - \frac{1}{4} |\Delta_k| = -\delta_k \]

In terms of the SU(1,1) generators the reduced Hamiltonian of the Bogoliubov is given by

\[ H = 2N_BV_k (\alpha^\dagger \alpha - J - J_1) \]

considering the Casimir operator:

\[ L = \frac{1}{2} (\Delta^\dagger \Delta - 1) \text{ where } \Delta = (\alpha_k, \alpha^\dagger_k, \alpha_k, \alpha^\dagger_k) \]

From the equations 5.22 and 5.23 it is evident that \( j_3 \) belonging to compact group gives a discrete spectrum which is integer spaced, while the generators \( j_3 \) and \( j_4 \) belonging to the non-compact group give a continuous spectrum.

The Hamiltonian Eq.5.21 which deals with the interaction leading to the excited states, bears a symmetric structure, so the coherent states associated with this group also have a symmetric structure.

Let us now consider the set of coherent states associated with the representations of the discrete series and the continuous series of this group. The basic vectors of the group are the vectors \( |j,\nu\rangle \) forming a complete orthonormal set.

### DISCRETE SERIES REPRESENTATION

The discrete class of unitary representations in Hilbert space with basic vectors \( |j,\nu\rangle \) are given by:

\[ j_3 |j,\nu\rangle = (E + \nu)|j,\nu\rangle \]

\[ j_+ |j,\nu\rangle = \frac{1}{2} \left[(j + E + \nu + 1)(E - j + \nu)|j,\nu\rangle \right] \]

\[ j_- |j,\nu\rangle = \frac{1}{2} \left[(j + E + \nu - 1)(E - j + \nu - 1)|j,\nu\rangle \right] \]

where \( E = \frac{1}{2} - \nu \)

and is related to the universal covering group property of the superfluid system.

There are two discrete series of representations \( U(j) \) and \( U(\delta j) \) for the group under consideration. Although it is enough to consider only \( U(j) \), as all the results obtained with this, transfer to the other, however, we consider them even separately.

As the Casimir operator \( \hat{j} \) is not independent for the discrete series of representation therefore:

a) for the positive discrete representation \( U(j) \) where the spectrum is bounded from below, we have

\[ j + E = 0 \]

b) \( j, \nu = 0 \) for \( j < 0 \)

c) \( j = 0 \) for \( j = 0 \)

Accordingly the coherent states taken as the eigen states of the ladder operator \( \hat{j} \) are given by

\[ |\alpha\rangle = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(\sqrt{m})^m}{m!} \left[\frac{1}{(2\pi)^{\frac{1}{2}}}|j,\nu\rangle \right] \]

where

\[ \alpha = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(\sqrt{m})^m}{m!} \left[\frac{1}{(2\pi)^{\frac{1}{2}}}|j,\nu\rangle \right] \]
so that the norm is given by

$$|\langle \alpha | \rangle|^2 = \langle \alpha | \alpha \rangle = \left| \frac{\sqrt{\pi}}{\Gamma(\nu + 1)} \right|^2 \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + \nu + 1)}$$

$$= \mathcal{F}_1 \left( -\nu, 1; \alpha^2 \right)$$

where \( \mathcal{F}_1 \) is the confluent hypergeometric function. From Eq. 5-28 it is found that the eigenvectors \( |\alpha\rangle \) are complete, but do not form an orthonormal set. The states \( |\alpha\rangle \) and \( |\beta\rangle \) become orthonormal only if the function \( \mathcal{F}_1 \) has a zero at the \( 2 \alpha^2 \) where it vanishes.

Thus it can be said, that the states as formulated above become the Gaussian-wave packets centred around the point \( \alpha^2 \) and zero at the point \( 2 \alpha^2 \) and are very narrow in the classical limit. This result copes with the usual definition of coherence and thus to the phenomenon of superfluidity.

b) For the unitary negative discrete series of representation \( (j) \) bounded from above, we have:

1) \( j - E = 0 \)

2) \( j - E = 0 \) for \( j < (0 - 2j) \)

so that

\( j - E = 0 \), -1, -2, .......

Accordingly the coherent states are given by

$$J \langle \xi | \rangle = \langle \xi | J \rangle$$

where

$$|\xi\rangle = \left[ (\xi^2) \right]^{1/2} \sum_{n=0}^{\infty} \frac{(\sqrt{2} \nu)^{n}}{\sqrt{n! \Gamma(n + \nu + 1)}} |j, n\rangle$$

From this it is clear that coherent behaviour is depicted.

CONTINUOUS SERIES REPRESENTATION:

The continuous bases for \( SO(2,1) \sim SU(1,1) \sim SL(2,R) \) have been studied by many authors. In this representation the coherent states are taken as the eigenstates of

$$L^+ \xi L^- \xi$$

so that

$$|\xi\rangle = \sum_{n=0}^{\infty} \frac{(\xi^2)^n}{\sqrt{n! \Gamma(n + \nu + 1)}} |j, n\rangle$$

and the norm is represented by

$$\langle \alpha | \beta \rangle = \left[ (\alpha^2) \right] \sum_{n=0}^{\infty} \frac{(\alpha^2)^{n}}{\sqrt{n! \Gamma(n + \nu + 1)}}$$

From this finite nature of norm, it can be concluded that the coherent states \( |\xi\rangle \) for all \( \alpha^2 \) do span the Hilbert space, as is the case with \( |\xi\rangle \), \( \langle \xi | \rangle = 0, 1, 2, ... \) forming the basis of the number operator ‘\( \mathcal{N} \)’ and entailing the complete orthonormal set. By the systematic analysis, it has been established that sub-set of all coherent states \( |\xi\rangle \) for which \( |\xi\rangle \) is represented in terms of well defined quantum numbers as in eq.5-32, span the Hilbert space. Experimentally it has been verified that if a subset of coherent states already span a Hilbert space, and forms a complete set, then, the full set of coherent states may become totally complete. This indicates that a certain type of linear dependence should exist among the coherent states of the superfluid helium system. Again from eq. 5-32, it is evident that \( \sum |\xi|^2 \) \( < \infty \). This reveals that the allowed sequences \( \{\xi\} \) themselves form a Hilbert space with an inner product \( \langle \xi | \beta \rangle \). So the norm can be written as:

$$\langle \xi | \beta \rangle = \sum_{\xi \neq \beta} |\xi|^2 \langle \xi | \beta \rangle$$

Evidently the argument of \( \{\xi\} \) of Hilbert space vector \( \langle \xi | \rangle \) is itself a vector in a different but still in infinite dimensional Hilbert space, i.e.,

$$\langle \xi | \rangle \langle \xi | \rangle$$
From this it follows that:
\[ | \langle \eta \rangle \rangle = | \{ \langle \eta \rangle \} \rangle \]
which corresponds to writing the number operators for the superfluid helium-4 by
\[ | n, n, n, \ldots n > \rho n \]
Thus it is clear that such space contains not only the coherent states of the oscillating condensed bosons but also the states \( | \{ \langle \eta \rangle \} \rangle \) defined by:
\[ \langle J \rangle | \{ \langle \eta \rangle \} \rangle = | \langle \eta \rangle | \{ \langle \eta \rangle \} \rangle \]
The above argument leads to
\[ \langle J \rangle | \{ \langle \eta \rangle \} \rangle = \langle \eta \rangle | \{ \langle \eta \rangle \} \rangle = 0 \]
This relation indicates that for higher values of \( n \), i.e., in the higher energy range of photon-like excitations, each coherent state tends to look like an oscillator type ground state with an arbitrary higher value of \( \langle \eta \rangle \). The basic functions in the coherent state continuous representation are contained in eq. S-38 for each \( | \{ \langle \eta \rangle \} \rangle \) since \( | \{ \langle \eta \rangle \} \rangle \) are complete in the analysis of Bogoliubov's quasi-particle spectrum, so too are the coherent states emerging from it. As in the quantum field theory for a single degree of freedom, analyticity argument forms basis to get complete proper subset of coherent states, so does our analysis generalized to several degrees of freedom.

**COHERENT ROTON STATES**

According to Feynman rotons are the smallest vortices. The velocity distribution found by analytical means is similar to that round a vortex ring which may be small, if the curvature of the smallest vortex (roton) is increased beyond a certain fixed radius which is assigned by quantum number \( 2j \), energy is needed. This enables us to assign a rotational level \( j(j + 1) \) for the roton. Thus it can be said that the roton states are merging to form a continuous distribution of states. This is coherent superposition of roton states as intuitively one meaning of the statement, that the states are coherent is that there is, in some sense "one state", which is sum of the states. The phase variance is brought about by the interaction terms of Hamiltonian. The rotation of the system, that produces a phase variance can be regarded as sending the system through some external potential and the change in phase is equal to the potential times, the time, it is in the potential. Sending the system through some external potential can be thought of as any dynamical effect taking place between two particles or sub-systems in the roton ground state. This way the ground state changes into the coherent superposition of roton states. The roton state has been created from the ground state by means of the operator Tₖ
\[ Tₖ | 0 \rangle = | j, m \rangle \]
where \( | j, m \rangle \) is the well defined angular momentum state formed out of the eigen values of the operators \( J^k \) and \( J^l \). The \( (2j+1) \) quantities \( Tₖ \) with the spectrum condition \( -jm < +j \) transforms under the rotation of the frame of reference as components of a tensor with rank \( j \). The coherent superposition of roton states are defined by
\[ Tₖ | \{ \langle \eta \rangle \} \rangle = | \{ \langle \eta \rangle \} \rangle \]
\[ I_{m, m+1} = b^*_{m} j^* - m + m^* (b^*_{m}) \]
\[ I_{m, m-1} = b^*_{m} j^* - m - m^* (b^*_{m}) \]

The coefficients \( a_{j, m} \) in eq. (5-41) can be obtained by the determination of the matrix elements of the tensor operator \( T^{\pm} \), making use of the Wigner - Echartisor operator theorem.

The inner product of the state vectors \( \langle j', m' | j, m \rangle \) is independent of \( m' \). Thus the matrix elements of the tensor operator factorize into two parts. The directional properties are contained in the Clebsch - Gordon coefficients and the dynamics of the system appears only in the scalar matrix element \( \langle j' | T^{\pm} | j \rangle \) called the reduced matrix element given by:

\[ \langle j' | T^{\pm} | j \rangle = (-1)^{j-j'} \sum_{m} \langle j' | m' \rangle \langle j' | m' | j \rangle \]

The time evolution of the coherent roton states is obtained by solving the Schrödinger equation,

\[ \frac{d}{dt} | \psi > = [H + V_J] | \psi > \]

where

\[ H = H^{int} + H^{kin} \]

So the time evolved non-standing states are given by

\[ | \psi(t) > = \sum_{j, m} a_{j, m} \exp(-i E t) | j, m > \]

where \( E \) is the energy of the roton excitation. At time \( t=0 \), the states have the properties characteristic of a coherent rotational state like those of coherent angular momentum states as discussed here:

a) The absolute value of the amplitudes \( a_{j, m} \) are peaked around a mean value of angular momentum \( j \).
b) The energies \( E \) of the states \( | j, m > \) are to a good approximation obey the rule \( j(j+1) \) while in the case of reference \( (\Phi) \) the states \( | j, m > \) are members of an ideal rotational band.
c) The phase \( \Phi_{j, m} \) defined by \( a_{j, m} = a_{j, m} \exp(-i \Phi_{j, m}) \) of the amplitude \( a_{j, m} \) appears roughly equidistant while in the case of coherent rotational state of reference \( (\Phi) \) they are precisely equidistant.

Now if the phases \( \Phi_{j, m} \) are exactly equidistant at \( t=0 \) and energies obey the rule \( j(j+1) \) perfectly, the coherence will disappear after a short time because of time evolution but it will appear again periodically. This is related to the fact that the spread of the coherent states over the spacial angle oscillates in time with the same periodically. However if the roton spectrum \( E \) would follow the rule \( j(j+1) \) only approximately then the coherence will recover quasi-periodically.

VI. CONCLUSION

The preceding section elucidates how the symmetry in liquid \( ^{4}\text{He} \) system is broken. Exploiting the displacement of the boson field operator, we arrived at a paradox. This paradox has been resolved by using the coordinate \( q \) and its conjugate canonical momentum \( p \) representation for the boson field operators. It has been accordingly shown that the \( ^{4}\text{He} \) field is quantized in the gauge in which \( q_0 \) mode is cancelled. This has resulted in a modified commutation relation Eq (2-21). From this relation it is found that whenever the components of the field operator do not represent the true dynamical degree of freedom, we get a modified "delta" function. The commutation relation of boson field operators contains the term \( s_j \), which in the limit of \( s_j \to \infty \) has been found to have an infinitesimal amplitude at every point in space, so that its integral over all space becomes unity, thereby depicting the disjoint Hilbert subspaces of a full Hilbert space.
This relation elegantly explained the long range correlation, which carries away the symmetry of the ground state leaving it frozen in the asymmetric state. Representation theory of groups shows by group decomposition technique that coherent states identified at different regions of excitation spectrum of superfluid $^4\text{He}$ become Gaussian wave packets centered at coordinate "$q$". This identification validates the context of our symmetry breaking mechanism.

Using the displacement of the field operator and a scaling transformation we obtained the Bogoliubov spectrum, as well as, the Landau and Khalatnikov excitation spectrum of superfluid $^4\text{He}$. From the forms of the spectra obtained with our formalism, it is seen that the lowest modes of excitations in the superfluid $^4\text{He}$ are the Phonons and these excitations vanish in the limit $k \rightarrow 0$ in accordance with the Goldstone theorem. From our formalism it is evident that it is not necessary to develop the cumbersome theory of weakly repelling bosons or hard sphere gas methods. This formalism is simpler in the sense that the states of bosons in superfluid helium are treated in such a way that the macroscopic condensation is effected by an invariant inhomogeneous transformation:

$$\Psi(x) \rightarrow e^{ikx} \Psi(x)$$

Treating it as a gauge transformation and analysing it in the light of scaling transformation, we visualized a formalism which gives a general vehicle to the study of special features of superfluid helium. Thus it became evident that such an approach is a more general framework than the collective variable theories for the study of superfluid $^4\text{He}$. Our analysis can lead us to new insights in the superfluid system, establishing a new connection with the infrared divergence problem, quantum electrodynamics and soft pion theorem in high energy physics.

From the study of Abelian transformation, it is seen that in the absence of vorticity we must have a flow without frictional damping in accordance with the Landau's acceleration equation.

In the light of the symmetry breaking, we have found the explicit forms of the transition probabilities between various states, such as the condensed states, the oscillating condensed boson states, the phonon states and the roton states of superfluid $^4\text{He}$. The forms of the transition probabilities reveal that the states of superfluid $^4\text{He}$ considered here, are the probable ones. These transition probabilities have been obtained as the functions of the condensate fraction. Our formalism of the novel extention of the Bogoliubov method will enable us to determine the transition probabilities amongst the other states, like the Vortices, Majoron states, roton pairing states, and Majoron pairing states.

The phenomenological phase variance is the broken symmetry condition for superfluid $^4\text{He}$. As phase variance causes a translation of the bosonfield operator and produces the coherent superposition of the states, the phase symmetry of the free Hamiltonian is broken by the interaction terms of the Hamiltonian. Setting in of the interaction within the system is treated as the dynamical effect taking place between the particles or the subsystems resulting in the creation of phonons and violating thereby the law of conservation of the Hamiltonian thus breaking the symmetry of the ground state.

The phonons produced can decay into a number of low energy phonons or interact among themselves, in which case the phonons are considered stable against any decay and the possibilities of the bunching of phonons leading to the formation of roton-like excitations has been considered. The coherent states obtained in the phonon region bear a deep inter-relationship with those in the other sectors of the liquids-helium system.

A systematic analysis of coherent state phenomenon in superfluid helium-4 has been conducted by exploiting the group theoretic methods. This has been done by laying emphasis on the theory of the group representation obtained from the truncated Hamiltonian of Bogoliubov, incorporating these in the oscillating condensed bosons.
Several coherent states visualized are seen to have deep inter-relationship. As higher excitations have also a group structure and their coherence behavior has been seen, this formalism seems to be more appropriate to deal with such an interesting and challenging problem in the system.

The approximation method using the canonical transformations of Bogoliubov to determine the probability of transition from phonon to roton state yields coherent rotational states identical to the angular momentum states with

i) absolute value of amplitude peaked around a mean value of the angular momentum,

ii) the energy of the states to a good approximation obeying \( j(j+1) \) rule when in the case of coherent rotational states the states are the members of an ideal rotation band,

iii) The phase of the amplitude appearing roughly equidistant while in the case of coherent rotational states it is precisely equidistant.

In the case of phases exactly equidistant at \( t=0 \), and energies obeying the rule \( j(j+1) \) perfectly, the coherence will disappear after a short time because of time evolution, but it will appear again periodically. This is related to the spread of the coherent state over the spacial angle oscillating in time with the same periodicity. However, the roton spectrum following the rule \( j(j+1) \) only approximately implies coherence recovering quasi-periodically.

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