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Dynamical Entropy, Quantum K-Systems and Clustering

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Abstract

The two possibilities to define a quantum K-system, either using algebraic relations or using properties of the dynamical entropy, are compared. It is shown that under the additional assumption of strong asymptotic abelianess the algebraic relations imply the properties of the dynamical entropy.

1 Introduction

Ergodic theory is a highly developed field and has brought deep insight into the theory of classical dynamical systems. It is natural to try to translate successful concepts also to the theory of quantum dynamical systems. Also here some promising results have already been obtained. For instance, it was tried to find a quantum analogue to the classical system with the best mixing properties, the K-system [1,2]. The concept is fully based on algebraic relations between the development of subalgebras and though it gives a very transparent structure, some features are extremely sensitive against small perturbations of the dynamics or the observed subsystems. Therefore it seems necessary to find related quantities that serve to characterize a K-system but have better continuity properties. From classical theory we know that a K-system is completely characterized by properties of the dynamical entropy and since an analogue of the classical dynamical entropy can be constructed also for quantum systems [3], we suggest that a K-system in quantum theory should preferably be characterized in terms of its dynamical entropy [4]. The systems of [1,2] satisfy the desired properties under some additional assumption. It is an open problem whether [4] covers a larger range of systems and whether this assumption of asymptotic abelianess is stronger than necessary. In this note we want to concentrate on the continuity problem of the relevant quantities and show how they help to control the properties of the system.

2 K-Systems in the Sense of Emch and Schröder

Following [2] a K-system is defined as follows: Let $(\mathcal{A}, \omega, \alpha)$ be a von Neumann algebra $\mathcal{A}$ with automorphism α and invariant state $\omega \circ \alpha = \omega$. Let $\mathcal{A}_0$ be a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ such that

$$\begin{aligned} \alpha^n \mathcal{A}_0 &\supset \mathcal{A}_0, & n \geq 0, \\ \bigvee_{n=0}^{\infty} \alpha^n \mathcal{A}_0 &= \mathcal{A}, & \mathcal{A} \vee \mathcal{B} \equiv \{\mathcal{A} \cup \mathcal{B}\}'', \\ \bigwedge_{n=0}^{-\infty} \alpha^n \mathcal{A}_0 &= \lambda 1, & \mathcal{A} \wedge \mathcal{B} \equiv \{\mathcal{A}' \cup \mathcal{B}'\}'. \end{aligned}$$

Then $(\mathcal{A}, \omega, \alpha, \mathcal{A}_0)$ is called a W^*-K-system. Evidently, the question arises, given $\mathcal{A}$, ω, α, is it possible to find such an $\mathcal{A}_0$ and how small changes in $\mathcal{A}_0$ spoil the picture. The theory of classical K-automorphisms tells us that we can start with an arbitrary finite-dimensional subalgebra $\mathcal{B}$, construct $\mathcal{B}_0 = \bigvee_{n=-\infty}^{0} \alpha^n \mathcal{B}$. If only $\bigwedge_{n=-\infty}^{0} \bigvee_{k=n}^{-\infty} \alpha^k \mathcal{B} = \lambda 1$, then $\bar{\mathcal{B}} = \bigvee_{n=-\infty}^{+\infty} \alpha^n \mathcal{B}_0$ is a K-system. So it only remains to check whether $\bar{\mathcal{B}} = \mathcal{A}$, i.e. if $\mathcal{B}_0$ resp. $\mathcal{B}$ was chosen sufficiently large.

Let us compare a typical quantum mechanical system: the Fermi algebra $\mathcal{A}$ of creation and annihilation operators $a(f)$, $f \in L^2(\mathbf{R}, dx)$. We consider the continuous automorphism group $\sigma_x a(f(y)) = a(f(x+y))$. Then $\mathcal{A}_0 = \mathcal{A}_{(-\infty,0]}$, built by the creation and annihilation operators $a(f)$ with $f \in L^2(\mathbf{R}_-, dx)$, serves the purpose. We can consider

$\mathcal{A}_0$ to be $\bigcup_{x=0}^{-\infty} \alpha^x \mathcal{A}_{[-1,0]}$. So we can take for $\mathcal{B} = \mathcal{A}_{[-1,0]}$ to construct a K-system. However, if we take $\mathcal{B} = \{a(f), a^\dagger(f)\}''$ with $\tilde{f}(p) \neq 0 \; \forall p$ and $|\tilde{f}(p)| \leq e^{-\lambda|p|}$, so e.g. $\tilde{f}(p) = e^{-p^2}$, then $\mathcal{B}_0 = \bigcup_{x=0}^{-\infty} \alpha^x \mathcal{B} = \mathcal{A}$, and we failed to obtain a suitable $\mathcal{A}_0$. To demonstrate this claim all we have to show is that

$$\bigcup_{y\leq 0} f(x+y) = L^2(\mathbf{R}, dx).$$

Let $g \perp f(x+y) \; \forall y \leq 0$, then

$$\int \bar{\tilde{g}}(p) \tilde{f}(p) e^{ipx} dp = 0 \qquad \forall x \leq 0.$$

Consider

$$F(x+iy) = \int \bar{\tilde{g}}(p) \tilde{f}(p) e^{ipx - py} dp.$$

Due to our assumption on f this function is analytic for $|y| \leq \lambda$. Since it is $= 0$ for $x \leq 0$, $y = 0$, it has to be identical to zero, so also $\bar{\tilde{g}}(p)\tilde{f}(p) \equiv 0$ and this is only possible for $\tilde{g}(p) = 0$.

There is another shortcoming: we want to observe somehow the continuity of the automorphism group, i.e. it should be possible to discretize the steps of the automorphisms and to observe some limiting behaviour if the steps become sufficiently small. If we consider again $\mathcal{B} = \{a(f), a^\dagger(f)\}$, then $\{\bigvee_{k=0}^{n} \alpha^{(1/n)k} \mathcal{B}\}'' \equiv \mathcal{B}_n$ will have dimension 2^n, so we lose completely the fact that $\mathcal{B}_n$ and $\mathcal{B}_{n'}$ should be close to one another if $(n-n')/n \ll 1$.

3 The Dynamical Entropy and K-Systems

In classical theory the K-systems are exactly those for which the dynamical entropy of the automorphism with respect to any non-trivial subalgebra is strictly positive [5]. We repeat the necessary definitions:

Let $\mathcal{B}$ be a finite-dimensional subalgebra, built by the (in $\mathcal{B}$) minimal projections $P_1, \ldots P_n$. Then

$$H_\omega(\mathcal{B}) = -\sum \omega(P_i) \ln \omega(P_i).$$

Further

$$H_\omega(\mathcal{B}, \alpha\mathcal{B}, \ldots, \alpha^k \mathcal{B}) = -\sum_{i_1, \ldots i_k = 1}^{n} \omega(P_{i_1} \alpha P_{i_2} \ldots \alpha^k P_{i_k}) \ln \omega(P_{i_1} \ldots \alpha^k P_{i_k}).$$

Finally $(\omega = \omega \circ \alpha)$,

$$h_\omega(\mathcal{B}, \alpha) = \lim_{k\to\infty} \frac{1}{k} H_\omega(\mathcal{B}, \ldots \alpha^k \mathcal{B}) = \inf_{k\to\infty} \frac{1}{k} H_\omega(\mathcal{B}, \ldots \alpha^k \mathcal{B}).$$

We have to find corresponding definitions in the quantum situation. Now the von Neumann algebra $\mathcal{M}$ as well as the finite-dimensional subalgebra $\mathcal{B}$ are non-abelian. The minimal projections are not uniquely defined. We do not want to repeat the argumentation that led to the definition (see [3], [6]) but simply state ($S(\omega|\nu)$ is the relative entropy of the states ω and ν)

Definition 1:

$$H_{\omega,\mathcal{M}}(\mathcal{B}) = \sup_{\sum \lambda_i \omega_i = \omega} \sum [-\lambda_i \ln \lambda_i + S(\omega|\lambda_i \omega_i)_{\mathcal{B}}]$$

$$H_{\omega,\mathcal{M}}(\mathcal{B}, \alpha\mathcal{B}) = \sup_{\sum \lambda_{ij}\omega_{ij} = \omega, \sum_j \lambda_{ij}\omega_{ij} = \lambda_i^{(1)}\omega_i^{(1)}, \sum_i \lambda_{ij}\omega_{ij} = \lambda_j^{(2)}\omega_j^{(2)}} \sum [-\lambda_{ij} \ln \lambda_{ij} +$$

$$+ S(\omega|\lambda_i^{(1)}\omega_i^{(1)})_{\mathcal{B}} + S(\omega|\lambda_j^{(2)}\omega_j^{(2)})_{\alpha\mathcal{B}}] =$$

$$= \sup \sum [-\lambda_{ij} \ln \lambda_{ij} + \lambda_i^{(1)} \ln \lambda_i^{(1)} + \lambda_j^{(2)} \ln \lambda_j^{(2)}] + \sum \lambda_i^{(1)} S(\omega|\omega_i^{(1)})_{\mathcal{B}} + \sum \lambda_j^{(2)} S(\omega|\omega_j^{(2)})_{\alpha\mathcal{B}}.$$

Generalization for k arguments is obvious. Finally, again

$$h_{\omega,\mathcal{M}}(\mathcal{B}, \alpha) = \lim \frac{1}{k} H_{\omega,\mathcal{M}}(\mathcal{B}, \alpha\mathcal{B}, \ldots, \alpha^k\mathcal{B}).$$

Note that in the abelian case the optimal decomposition is given by

$$\omega(P_{i_1} \alpha P_{i_2} \ldots \alpha^k P_{i_k} \cdot) = \lambda_I \omega_I,$$

so in this case the definitions coincide.

In [4] we used the concept of dynamical entropy to define

Definition 2: A K-system is a quantum dynamical system $(\mathcal{M}, \alpha, \omega)$ for which

$$\inf_{\mathcal{B} \neq c1} m_\omega(\mathcal{B}, \alpha) \equiv \inf_{\mathcal{B} \neq c1, \dim \mathcal{B} < \infty} \lim_{n \to \infty} \frac{h_\omega(\mathcal{B}, \alpha^n)}{H_\omega(\mathcal{B})} = 1.$$

The advantage of the definition is given by the following fact:

Theorem 1: Let $\mathcal{M}$ be a hyperfinite von Neumann algebra and $\mathcal{B}_k$ a sequence of finite-dimensional subalgebras, such that $\mathcal{M} = \overline{\lim} \mathcal{B}_k$ in the weak sense. Then

$$1 = \inf_{\mathcal{B} \neq c1} \lim_{n \to \infty} \frac{h_\omega(\mathcal{B}, \alpha^n)}{H_\omega(\mathcal{B})}$$

iff

$$1 = \lim_{n \to \infty} \frac{h_\omega(\mathcal{A}_{d,k}, \alpha^n)}{h_\omega(\mathcal{A}_{d,k})} \qquad \forall \mathcal{A}_{d,k} \subset \mathcal{B}_k, \dim \mathcal{A}_{d,k} = d \geq 2.$$

This means it suffices to check $(\cdot)$ for a countable set of subalgebras.

The theorem is a consequence of the following

Lemma 1: Monotonicity Let $\mathcal{B} \subset \mathcal{C}$. Then

$$H_\omega(\mathcal{B}, \alpha\mathcal{B}, \ldots, \alpha^n\mathcal{B}) \leq H_\omega(\mathcal{C}, \ldots, \alpha^n\mathcal{C}).$$

Proof: We take the optimal decomposition for $\alpha^k\mathcal{B}$ and use it for $\alpha^k\mathcal{C}$. The relative entropy $S(\omega|\varphi)_{\mathcal{B}} \leq S(\omega|\varphi)_{\mathcal{C}}$ is monotonic in $\mathcal{B}$ (see [7,8]).

Lemma 2: Continuity ([9]) in one subalgebra Let B be an algebra of dimension $d > 8$. Then

$$|H_\omega(B) - H_{\bar\omega}(B)| \le 12\varepsilon^{1/3} \ln \frac{2d}{\varepsilon} \qquad \text{with } \varepsilon = \|\omega - \bar\omega\|.$$

Remark: We do not claim that 12 is optimal. For taking into account smaller d, the number would change. But the essential interest lies in the way, how the continuity depends on the dimension for large algebras. The ε-dependence might also not be optimal and it is a challenge to improve it.

The estimates are based on the following facts:

$$\lambda S_\omega + (1-\lambda) S_{\bar\omega} \le S_{\lambda\omega + (1-\lambda)\bar\omega} \le \lambda S_\omega + (1-\lambda) S_{\bar\omega} - \lambda \ln \lambda - (1-\lambda)\ln(1-\lambda)$$

together with the scaling law

$$S_{\lambda\omega} = \lambda S_\omega - \omega(1)\lambda \ln \lambda$$

and

$$S_\omega(B) \le \ln d$$

leads to

$$|S_\omega - S_{\bar\omega}| \le \varepsilon \ln \frac{2d}{\varepsilon}$$

using order relations between ω, $\bar\omega$ and $\nu = \dfrac{\omega + \bar\omega}{2} + \dfrac{|\omega - \bar\omega|}{2}$. To use this estimate for $H_\omega(B)$ we first observe that due to the continuity of S we may assume that we are arbitrarily close to the optimal decomposition with a finite decomposition $\omega = \sum \omega_i$, where $\omega_i(A) = \langle \Omega | x_i^* A x_i | \Omega \rangle$ with $x_i \in \mathcal{M}'$ and $\sum x_i^* x_i = 1$.

We use the same decomposition of unity to construct $\bar\omega_i = \langle \bar\Omega | x_i^* A x_i | \bar\Omega \rangle$. Since by assumption $\sum \|x_i|\Omega - \bar\Omega\rangle\|^2 = \|\Omega - \bar\Omega\|^2 < \varepsilon$, we split the x_i into those for which the normalized $\hat\omega_i$ and $\hat{\bar\omega}_i$ are close to one another (up to $\varepsilon^{2/3}$) and show that the rest only gives a contribution of $O(\varepsilon^{1/3})$. This gives the estimate of Lemma 2.

For $H_\omega(B)$ estimates using the norm difference of the state and the difference of two d-dimensional subalgebras (to every $b_i \in B_i \, \exists \, b_j \in B_j$ such that $\|(b_i - b_j)\Omega\| < \varepsilon$, $\|b_i^* - b_j^*)\Omega\| < \varepsilon$) are rather related. For an arbitrary number of subalgebras this does not hold anymore and in this case only a less explicit continuity property is available.

Lemma 3 [3]: Let B_n be a sequence of finite-dimensional subalgebras such that to every $b_{n_0} \in B_{n_0}$ exists a sequence $b_n \in B_n$, $b \in B$, with st $\lim_{n\to\infty} b_n = b$. Then

$$\lim_{n\to\infty} \frac{1}{k} H_\omega(B_n, \alpha B_n, \ldots, \alpha^k B_n) = \frac{1}{k} H_\omega(B, \ldots \alpha^k B).$$

Sketch of the proof: Due to the continuity of the entropy $S_\omega(B_n)$ it suffices to consider only decompositions $\omega_{i_1,\dots,i_k}(\cdot) = \omega(x_{i_1,\dots,i_k}\cdot)$ with $i_k \le r(d,\varepsilon)$ to come within ε to the optimal result. By the same argument it suffices to take into account only those contributions where $\omega(x_{i_\ell}) > \varepsilon$ with $x_{i_\ell} = \sum_{I_k, i_\ell\, fixed} x_{I_k}$ with $I_k = \{i_1,\dots,i_k\}$. Now the convergence of the B_n gives

$$\lim_n \sup_{x,\omega(x)>\varepsilon} \left| \frac{\langle\Omega|xb_n|\Omega\rangle}{\langle\Omega|x|\Omega\rangle} - \frac{\langle\Omega|xb|\Omega\rangle}{\langle\Omega|x|\Omega\rangle} \right| = 0.$$

Lemma 4: Uppersemicontinuity Assume $B_n \to B$ in the above sense. Then

$$\lim_{n\to\infty}\lim_{k\to\infty}\frac{1}{k}H_\omega(B_n,\alpha B_n,\dots,\alpha^k B_n) \le \lim_{k\to\infty}\frac{1}{k}H_\omega(B,\dots,\alpha^k B).$$

Proof:

$$H_\omega(B_1,\dots,B_\ell,B_{\ell+1},\dots,B_n) \le H_\omega(B_1,\dots,B_\ell) + H_\omega(B_{\ell+1},\dots,B_n),$$

which can be seen in the second form of Definition 1, if we notice that $\sum -\lambda_{ij}\ln\lambda_{ij} + \sum \lambda_i \ln\lambda_i + \lambda_j \ln\lambda_j \le 0$ due to the subadditivity of the entropy. Standard arguments tell us that $\lim_{k\to\infty}$ is in fact an infimum. Thus the continuity in Lemma 3 implies semicontinuity in Lemma 4.

Lemma 5: Let B_k be a sequence of finite-dimensional subalgebras with $\mathcal{M} = \overline{\lim B_k}$. Let B be a finite-dimensional subalgebra. Then there exists for every $\varepsilon > 0$ some k such that

$$H_\omega(B) \le H_\omega(B_k) + \varepsilon,$$

$$\frac{1}{n}H_\omega(B,\dots\alpha^n B) \le \frac{1}{n}H_\omega(B_k,\dots\alpha^n B_k) + \varepsilon.$$

Proof: Let E_k be the conditional expectation $\mathcal{M} \to B_k$, τ_k the imbedding of $B_k \to \mathcal{M}$ and γ the imbedding of B in $\mathcal{M}$. Then st lim $\tau_k E_k \gamma = \gamma$ and the continuity gives the desired result.

We combine the results to prove Theorem 1.

We may assume that $\inf \lim \dfrac{h_\omega(B,\alpha^n)}{H_\omega(B)}$ is attained within ε by some finite-dimensional B. This B is the strong limit of $\mathcal{A}_{d,k}$. Therefore

$$1 = \lim_{k\to\infty}\lim_{n\to\infty}\frac{h_\omega(\mathcal{A}_{d,k},\alpha^n)}{H_\omega(\mathcal{A}_{d,k})} \le \lim \frac{h_\omega(B,\alpha^n)}{H_\omega(B)} \le 1$$

due to Lemmas 2 and 4.

It should be noted that it is not sufficient to calculate $m_\omega(\alpha, B_n)$ only for an increasing set of subalgebras. Here we have the counterexample of the crossed product of an abelian von Neumann algebra with a Bernoulli shift [10,11] where it can be shown that

$m_\omega(\alpha, \mathcal{B}_k) = 1$ for an appropriately chosen sequence of $\mathcal{B}_k$ (see [11]) but nevertheless there exist invariant elements. This is easily seen since the Bernoulli shift is an inner automorphism for the crossed product. Therefore there exist invariant subalgebras. For them $h_\omega(\alpha, \mathcal{A}) = 0$ and also $m_\omega(\alpha, \mathcal{A}) = 0$.

4 The Shift on the Quantum Lattice System

We consider a quantum lattice system $\mathcal{M} = \bigotimes_{x=-\infty}^{+\infty} \mathcal{A}_x$, where $\mathcal{A}_x$ are $d \times d$ matrix algebras. The K-automorphism maps $\alpha^n \mathcal{A}_x = \mathcal{A}_{x+n}$. It is obvious that it leads to a K-system in the sense of [1,2]. We consider some extremal invariant state ω (local correlations are permitted) and take

$$\mathcal{B}_0 = \bigotimes_{x=-\infty}^{0} \mathcal{A}_x.$$

Since ω is extremal invariant the state is clustering. Therefore w lim $\alpha^n b = \omega(b)$ for all $b \in \mathcal{M}$ and $\alpha^n b\Omega = b\Omega$ only holds for $b = \lambda 1$. Let P_n be the projection on $\alpha^{-n}\mathcal{B}_0|\Omega)$. Then st $\lim_{n\to\infty} P_n = |\Omega)(\Omega|$ and

$$\bigwedge_{n=0}^{-\infty} \alpha^n \mathcal{B}_0 = \lambda 1.$$

It was less trivial to show that $(\mathcal{M}, \omega, \alpha)$ is also a K-system in the sense of [4]. The argument runs as follows:

We know that for any finite-dimensional local algebra $\mathcal{B}$ a finite decomposition is sufficient. Without losing more than ε we can assume that this finite decomposition is given by $1 = \sum x_i$, $i = 1, \ldots, r$, where $x_i \in \tilde{\mathcal{B}}$ is again a strictly local algebra. So with $\eta(\lambda) = -\lambda \ln \lambda$

$$H_\omega(\mathcal{B}) = \sum \eta(\omega(x_i)) + \sum S(\omega|\omega(x_i \cdot))_{|\mathcal{B}}.$$

If $\mathcal{B} \neq c1$, there exists some x with $\omega(xb) \neq \omega(x)\omega(b)$ for $b \in \mathcal{B}$, so $H(\mathcal{B}) > 0$. For $\frac{1}{n} H_\omega(\mathcal{B}, \alpha^\ell \mathcal{B}, \ldots \alpha^{\ell n}\mathcal{B})$ we consider

$$\geq \sum \frac{1}{n}\eta(\omega(x^\ell_{I_n}) - \sum \eta(\omega(x_i)) + \sum \eta(\omega(x_i)) + \sum S(\omega|\omega(x_i \cdot))_{\mathcal{B}},$$

where $x^\ell_{I_n} = \prod_{k=0}^n \alpha^{\ell k} x_{i_k}$ is a positive operator if only ℓ is large enough so that $[x_{i_1}, \alpha^\ell x_{i_2}] = 0$. We can consider the index set I_n as an r-dimensional Bernoulli shift, $i \in \{1, \ldots, r\}$ and probabilities $\omega(x_i)$. So $\lim_{n\to\infty} \frac{1}{n} \sum \eta(\omega(x^\ell_I)) = h_\omega(\alpha^\ell, \{1, \ldots, r\})$ and $\sum \eta(\omega(x_i)) = H_\omega(\{1, \ldots, r\})$. In the classical case we have strong subadditivity which gives

$$\lim_{n\to\infty} \frac{1}{n} H(\mathcal{B}, \ldots, \alpha^{n\ell}\mathcal{B}) = \lim_{n\to\infty} (H(\mathcal{B}, \ldots, \alpha^{n\ell}) - H(\alpha^\ell \mathcal{B}, \ldots, \alpha^{n\ell}\mathcal{B}) \equiv \lim_{n\to\infty} H(\mathcal{B} \setminus \bigvee_{k=1}^{n} \alpha^{k\ell}\mathcal{B}).$$

Now K-mixing implies the triviality of the tail [5]:

$$\lim_{\ell\to\infty} \lim_{n\to\infty} \bigvee_{k=\ell}^{n} \alpha^{k\ell}\mathcal{B} = \lambda 1.$$

Therefore we continue

$$\lim_{n\to\infty}\frac{1}{n}H(B,\ldots,\alpha^{n\ell}B)=H(B).$$

In our case

$$\lim_{\ell\to\infty}\lim_{n\to\infty}\sum_{I_n}\frac{1}{n}\eta(\omega(x^{\ell}_{I_n})-\sum_{i}\eta(\omega(x_i))=0$$

or

$$\lim h_\omega(\alpha^\ell,B)=H_\omega(B).$$

5 Clustering and K-Systems

In [2] it was shown that a K-system in the sense of [2] satisfies the following clustering property: Let x be given. Then to every $\epsilon>0\ \exists\ n_0$ such that for all $n\geq n_0$ and all $y\in\mathcal{A}_0$

$$|\omega(x\alpha^{-n}y)-\omega(x)\omega(y)|\leq\epsilon\|y\|.$$

It should be noted that this is the well-known clustering with respect to the shift given in [12]. We expect that this clustering is the tool to find a connection between K-systems in the sense of [2] and [4]. At least we have the following result:

Theorem: Let $(\mathcal{M},\alpha,\mathcal{A}_0)$ be a K-system in the sense of [2]. Assume further that α is strongly asymptotically abelian. Let ω be an α invariant state which is KMS with respect to an automorphism τ. Then it is also a K-system in the sense of [4].

Remark: The clustering together with the KMS property already implies weak asymptotic abelianess, i.e. [13]

$$\lim\omega(a[b,\alpha^n c]d)=0\qquad\forall\, a,b,c,d.$$

We assume therefore in addition that

$$\lim_{n\to\infty}\|[b,\alpha^n c]d\Omega\|=0.$$

Proof: More or less we have only to repeat the arguments of the preceding paragraph. We only have to be careful in the construction of x_I because we do not have strict abelianess. Therefore we choose

$$x^\ell_{I_n}=\sqrt{x_{i_0}}\alpha^\ell\sqrt{x_{i_1}}\ldots\alpha^{\ell n}x_{i_n}\ldots\sqrt{x_{i_0}}\in\mathcal{M}$$

which is obviously positive.

Take y_s a testelement of $\alpha^{\ell s}\mathcal{A}$. Then

$$\lim_{\ell\to\infty}\sum_{I_n,\hat{i}_s}\omega(\sqrt{x_{i_0}}\ldots\alpha^{\ell s}\sqrt{x_{i_s}}\alpha^{\ell(s+1)}\sqrt{x_{i_{s+1}}}\ldots\alpha^{\ell s}\sqrt{x_{i_s}}\ldots\sqrt{x_{i_0}}\tau^{i/2}y_s)=$$

$$= \lim_{\ell\to\infty} \sum \omega(\alpha^{\ell(s+1)}\sqrt{x_{i_{s+1}}}\ldots\alpha^{\ell n}x_{i_n}\ldots\alpha^{\ell s}\sqrt{x_{i_s}}\ldots\tau^{i/2}y_s\tau^i(\sqrt{x_{i_0}}\ldots\alpha^{\ell s}\sqrt{x_{i_s}}))$$

due to the τ-KMS property. The operator $\alpha^{\ell(s+1)}\sqrt{x_{i_{s+1}}}\ldots\alpha^{\ell n}x_{i_n}\ldots\alpha^{\ell(s+1)}\sqrt{x_{i_{s+1}}}$ is separated from the rest by α^ℓ. So we use K clustering to obtain, again with the KMS condition,

$$= \lim_{\ell\to\infty} \sum \omega(\alpha^{\ell(s+1)}\sqrt{x_{i_{s+1}}}\ldots\alpha^{\ell(s+1)}\sqrt{x_{i_{s+1}}})\omega(\sqrt{x_{i_0}}\ldots\alpha^{\ell s}x_{i_s}\ldots\sqrt{x_{i_0}}\tau_{i/2}y_s) =$$

$$= \lim_{\ell\to\infty} \sum \omega(\cdot)\{\omega(\alpha^{\ell s}x_{i_s}\sqrt{x_{i_0}}\alpha^\ell\sqrt{x_{i_1}}\ldots\sqrt{x_{i_0}}\tau_{i/2}y_s)+\omega([\sqrt{x_{i_0}}\ldots,\alpha^{\ell s}x_{i_s}]\alpha^{\ell(s-1)}\sqrt{x_{i_{s-1}}}\ldots\tau_{i/2}y_s)\}.$$

The last term goes to zero, since we assume strong asymptotic abelianess. In the first term we can perform the summation and arrive at

$$= \omega(x_{i_s}\tau_{i/2}y_s)$$

which gives the optimal decomposition for $\alpha^{\ell s}\mathcal{A}$. In calculating $h_\omega(\alpha^\ell, \mathcal{A})$ we have therefore to show that

$$\lim_{\ell\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{I_n}\eta(\omega(x^\ell_{I_n})) - \sum_i \eta(\omega(x_i)) = 0.$$

We can use again K clustering for

$$\lim_{\ell\to\infty}\omega(\sqrt{x_{i_0}}\alpha^\ell\sqrt{x_{i_1}}\ldots\alpha^\ell\sqrt{x_{i_1}}\sqrt{x_{i_0}}) = \lim_{\ell\to\infty}\omega(x^\ell_{I_n}) =$$

$$= \omega(x_{i_0})\omega(\sqrt{x_{i_1}}\ldots\sqrt{x_{i_1}}) \equiv \omega(x_{i_0})\omega(x^\ell_{I_{n-1}}).$$

Notice that the limit is attained uniformly in n. Therefore

$$\lim_{\ell\to\infty}\lim_{n\to\infty}\frac{1}{n}[\sum\eta(\omega(x^\ell_{I_n}) - n\sum\eta(\omega(x_{i_0}))] =$$

$$= \lim_{\ell\to\infty}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}[\sum(\omega(x^\ell_{I_k}) - \eta(\omega(x_{i_0})) - \eta(\omega(x^\ell_{I_{k-1}})].$$

We consider $\omega(x^\ell_{I_k})$ to be a state ω^k_ℓ on the abelian algebra $\mathcal{B} = \mathcal{B}_0 \otimes \mathcal{B}^{k-1}$, $\mathcal{B}_0$ finite-dimensional, and K clustering tells us that $\lim_{\ell\to\infty}\|\omega^k_\ell - \omega_0\otimes\omega^{k-1}_\ell\| = 0$ uniformly in k. It remains to show that this suffices to guarantee that

$$\lim_{\ell\to\infty}[S(\omega^k_\ell) - S(\omega_0) - S(\omega^{k-1}_\ell)] = 0$$

uniformly in k.

ω^k_ℓ corresponds to a density matrix

$$\sum_{a,j}(\lambda_a\mu^\ell_j + \varepsilon^\ell_{aj})P_aQ_j = \sum\alpha^\ell_{aj}P_aQ_j, \qquad P_a\in\mathcal{B}_0, \qquad Q_j\in\mathcal{B}^{k-1},$$

with $a \in \{1,\ldots,n\}$ and $j \in \{1,\ldots\}$. Further $\sum_j\alpha^\ell_{aj} = \lambda_a$, $\sum_a\alpha^\ell_{aj} = \mu^\ell_j$, $\alpha^\ell_{aj} \geq 0$. Therefore

$$|\varepsilon^\ell_{aj}| \leq \max\{\lambda_a\mu^\ell_j, \alpha^\ell_{aj}\} \leq \max\{\lambda_a\mu^\ell_j, \mu^\ell_j\} \leq \mu^\ell_j$$

and also

$$|\varepsilon^{\ell}_{aj}| \leq \lambda_a.$$

Therefore we can write $\varepsilon^{\ell}_{aj} = \gamma^{\ell}_{aj}\mu^{\ell}_j$ with $|\gamma^{\ell}_{aj}| \leq 1$. We split the index set j into $I_a = \{j, |\gamma^{\ell}_{aj}| < \sqrt{\varepsilon}\}$ and $I^c_a = \{j, |\gamma^{\ell}_{ij}| \geq \sqrt{\varepsilon}\}$. $\|\omega^k_\ell - \omega_0 \otimes \omega^{k-1}_\ell\| < \varepsilon$ implies that $\sum |\varepsilon^{\ell}_{aj}| < \varepsilon$. It follows that

$$\sum_{j \in I^c_a} \mu^{\ell}_j < \sqrt{\varepsilon}.$$

Now we use

$$\begin{aligned} 0 &\leq S(\omega_0) + S(\omega^{k-1}_\ell) - S(\omega^k_\ell) = S(\omega_0 \otimes \omega^{k-1}_\ell | \omega^k_\ell) = \\ &= \sum_{a,j} \alpha^{\ell}_{aj} \ln \frac{\alpha^{\ell}_{aj}}{\lambda_a \mu^{\ell}_j} \leq \sum_{\gamma^{\ell}_{aj} \geq 0} (\lambda_a + \gamma^{\ell}_{aj})\mu^{\ell}_j \ln \frac{(\lambda_a + \gamma^{\ell}_{aj})}{\lambda_a} = \\ &= \sum_{a,\gamma^{\ell}_{aj} \geq 0} \sum_{j, j \in I^c_a} + \sum_{a,\gamma^{\ell}_{aj} \geq 0} \sum_{j, j \in I_a} . \end{aligned}$$

λ_a runs through a finite set and $\neq 0$. Therefore

$$\ln(1 + \frac{\gamma^{\ell}_{aj}}{\lambda_a}) \leq \ln(1 + \frac{1}{\lambda_a})$$

is bounded and

$$\sum_{j \in I^c_a} \cdots \leq c_1 \sqrt{\varepsilon}.$$

For $j \in I_a$,

$$\ln(1 + \frac{\gamma^{\ell}_{aj}}{\lambda_a}) \leq \frac{\sqrt{\varepsilon}}{\lambda_a},$$

therefore

$$\sum_{j \in I_a} \cdots \leq c_2 \sqrt{\varepsilon}.$$

All these estimates are independent of k, which proves the theorem.

Remark: It should be noted that a K-system in the sense of [2] need not be strongly asymptotically abelian. We have the example of the Fermi lattice system and the shift automorphism that is only weakly asymptotically abelian. On the other hand, it was only necessary that sufficiently many operators – those that are needed to construct optimal decompositions – asymptotically commute, and in fact this was necessary, otherwise we could not construct the positive x_I. So one can see that the C^*-algebra of fermions on a lattice is a K-system in the sense of [2], but only its even part a K-system in the sense of [4]. The next task that is under investigation [14] is to classify this class of operators that is needed for good decompositions for arbitrary decompositions for arbitrary subalgebras and so to see what kind of clustering is the consequence for a K-system in the sense of [4].

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