ON THE EFFECT OF COULOMB INTERACTION
ON THE MULTIPHOTON IONIZATION PROBABILITY

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The nonresonant multiphoton ionization problem is considered in the case of one-dimensional Coulomb potential. The continuous spectrum wave function in the presence of electromagnetic field and Coulomb interaction is calculated in the quasiclassical approximation. The Coulomb interaction is taken into account by the use of the perturbation theory in that part of action which arises due to interaction with an electromagnetic field. Criteria of this approximation validity are found and it is shown that such an approach allows the process of nonresonant multiphoton ionization to be described in the field range $\mathcal{E} \ll \mathcal{E}_a \left( \mathcal{E}_a \text{ is the characteristic atomic field} \right)$ for arbitrary values of the adiabaticity parameter $\gamma$. Within the range $\gamma \gg 1$ the Coulomb factor in the ionization probability is independent of the field strength and has to be taken into account. 

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I. INTRODUCTION

A role of the Coulomb interaction in a final state remains not clear in the theory of the nonresonant multiphoton ionization to date. In paper [1,2] the effect of Coulomb interaction was taken into account by the use of the perturbation theory. Within the scope of perturbation theory suggested in [1,2] the effect of Coulomb interaction on the probability of multiphoton ionization manifests itself just as in the static limit $\gamma \ll 1$:

$$\mathcal{W} = \left(\frac{2E_a}{\mathcal{E}}\right)^{2\lambda} \mathcal{W}_{S,-r} \quad , \quad (1)$$

where $\mathcal{W}_{S,-r}$ is the probability of ionization of electron bounded by a short-range potential [3], $\mathcal{E}$ is the amplitude of variable electric field strength, $\mathcal{E}_a = \frac{e^2}{\hbar^2 m I_1 e}$ is the intra-atomic electric field, $\chi = \sqrt{2mI_1} / \hbar$, $\lambda$ is the Coulomb parameter, $\gamma$ is the adiabaticity parameter

$$\lambda = Z \left(\frac{I_H}{I}\right)^{1/2}, \quad \gamma = \frac{\omega \sqrt{2mI}}{e\mathcal{E}} \quad . \quad (2)$$

Here $Z$ is the atomic residual charge ($Z=1$ at a neutral atom ionization), $I$ and $I_H$ are ionization potentials of the considered atom and hydrogen atom ground state, $\omega$ is the variable electric field frequency, $e, m$ are the electron charge and mass, respectively.

According to [1,2] the formula (1) is valid if the criterion $\gamma \ll \gamma_c$ is fulfilled, where $\gamma_c = \frac{2I}{\lambda H \omega} \gg 1$. Within the range of $1 \ll \gamma \ll \gamma_c$ as it follows from the theory in [1,2], the change of the exponent in the dependence on the intensity of the multiphoton ionization probability by the value $\lambda \approx 1$ should take place: $W^{(N)} \sim \gamma^{N-\lambda}$, where $\gamma$ is the electromagnetic wave
Intensity, $N$ is the number of absorbed quanta. In the range of $I \gtrsim I_c$ the proposed perturbation theory is unapplicable and, therefore, nothing can be said about the role of Coulomb interaction in this range.

In the recent years in connection with measurements of electron spectra obtained as a result of the multiphoton atom ionization, a number of new regularities has been revealed in this process (see, for instance, [4,5]). In particular, in the spectrum of above-threshold electron peaks equally spaced by the quantum energy value $\hbar \omega$ were found. At an increase of $I$ up to the values of $\sim 10^{13}-10^{14}$ W/cm$^2$ a disappearance of peaks nearest to the threshold was observed in the electron spectrum. But for every out of peaks observed, corresponding to an absorption of $N$ quanta, the ionization probability obeys the law $W^{(N)} \propto J^N$ [4,5]. The obtained experimental results again have attracted the interest to the multiphoton ionization theory [6,7,8,9,10]. In particular, in [9,10] to consider the one-dimensional hydrogen atom ionization the attempt is made for the calculation of the electron quasiclassical wave function to use the perturbation theory in terms of an electromagnetic field, expanding the wave function phase (classical action) in powers of the field amplitude $\mathcal{E}$. However, on this way to obtain concrete results was not possible. In such a manner, the effect of the Coulomb interaction on this process probability has not yet been clarified in the theory.

The purpose of this paper is the successive accounting of the Coulomb interaction effect upon the process of the nonresonant multiphoton ionization. Here we shall show that within the range of parameters $\hbar \omega \ll I$, $\mathcal{E} \ll \mathcal{E}_0$ and for emitted electron energies $(E - E_{\text{vib}}) \ll I$, where $E_{\text{vib}} = (\mathcal{E})^2/\hbar m \omega = I/2\gamma^2$ is the energy of electron oscillations in the electromagnetic wave field, the Coulomb interaction accounting is possible within the framework of quasiclassical approximation and perturbation theory in terms of the Coulomb interaction at arbitrary values of the adiabaticity parameter $\gamma$. Unlike the approach in [1,2], in the present paper the perturbation theory is applied to that part of action determining the quasi-energy electron's wave function in the final state, which remains after singling out the Coulomb and Volkov actions. By the Coulomb action here is meant the wave function phase arising in the problem of the electron scattering on the Coulomb center without an electromagnetic field.
In order to clarify the questions of principle, related to the accounting of the Coulomb interaction, for simplicity, the problem is considered in the one-dimensional geometry and the field of electromagnetic wave linearly-polarized along the X axis is thought to be uniform.

2. PROBLEM STATEMENT

The electron wave function $\psi(x,t)$ in the strong electromagnetic wave field satisfies the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = (\hat{H}_0 + \hat{H}') \psi,$$  

where

$$\hat{H}_0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

is the unperturbed (atomic) Hamiltonian, the potential of electron interaction with atom $V(x)$ at $|x| \gg x_0$, where $x_0 \sim \epsilon^{-1}$ is the characteristic atomic distance, has the Coulomb asymptote

$$V(x) \approx -\frac{Ze^2}{|x|},$$

and $\hat{H}'(x,t)$ describes the electron interaction with a strong electromagnetic wave. The solution (3) can be written in the form:

$$\psi(x,t) = \psi^{(o)}_{E_0}(x,t) + \sum \lambda_{E E_0}(t) \psi_E(x,t),$$

where $\psi^{(o)}_{E_0}(x,t)$ is the wave function of electron initial bound state, not perturbed by the electromagnetic wave field, which at a distance $|x| \gg x_0$ has the asymptote (for even bound states including the ground one)

$$\psi^{(o)}_{E_0}(x,t) = \mathcal{C} \approx \sqrt{\frac{x}{\hbar}} \exp \left(-\frac{iE_0 t}{\hbar} \right) \left(\frac{\epsilon}{|x|}\right)^2 \exp \left(-\frac{\epsilon |x|}{\hbar} \right),$$

$\mathcal{C}$ is the normalized constant, $E_0 = -1$. The wave function $\psi_E(x,t)$ corresponds to certain values of quasi-energy $E$ \cite{11} and satisfies the equation (3). Coefficients $\lambda_{E E_0}(t)$ are of the form:

$$\lambda_{E E_0}(t) = -\frac{i}{\hbar} \int_0^t \int dx \psi_E^*(x,t) \hat{H}'(x,t) \psi_{E_0}(x,t)$$

$$\int_0^t \int dx \psi_E^*(x,t) \hat{H}'(x,t) \psi_{E_0}(x,t).$$
and for positive quasi-energy values $E = E_p = P^2 / 2m$ represent amplitudes of electron transition from a bound state to a continuous spectrum, from which one can find by the known procedure the ionization probability per unit time.

Such an approach is a natural generalization of the method which was used in the problem on ionization from a short-range potential bound state in the strong electromagnetic wave field by L.V. Keldysh [3].

In this approach the main thing is the determination of electron wave function having a certain quasi-energy value in the continuous spectrum in the electromagnetic wave field with allowance for the Coulomb interaction.

Later on, in describing the electron interaction with a linearly-polarized (in the X-axis direction) monochromatic electromagnetic wave use will be made of an dipole approximation in which an expression for the interaction Hamiltonian is of the form:

$$\hat{H}'(x,t) = -e \mathcal{E} \times \cos \omega t$$

(7)

and we shall assume that $\hbar \omega \ll 1$, $\mathcal{E} \ll \mathcal{E}_0$. Thus, the following equation will be considered

$$i\hbar \frac{\partial \psi_{Ep}}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_{Ep}}{\partial x^2} - \frac{Ze^2}{|x|} \psi_{Ep} - e \mathcal{E} \times \psi_{Ep} \cos \omega t$$

(8)

to determine the electron quasi-energy wave function in a continuous spectrum.

3. DETERMINATION OF QUASI-ENERGY WAVE FUNCTION FOR A CONTINUOUS SPECTRUM.

Let us seek for solving the equation (8) in the form:

$$\psi_{Ep}(x,t) = \exp \left\{ \frac{i}{\hbar} S(x,t) \right\}$$

(9)

and in the future determine the function $S(x,t)$ using the quasi-classical expansion [12] the validity of which for the given problem will be shown below:

$$S(x,t) = S_0(x,t) + \hbar S_1(x,t) + ...$$

(10)
Equations for \( S_0(x,t) \) and \( S_1(x,t) \) are of the known form:

\[
\begin{align*}
- \frac{\partial S_0}{\partial t} &= \frac{1}{2m} \left( \frac{\partial S_0}{\partial x} \right)^2 - eE_x \cos \omega t - \frac{ze^2}{|x|}, \\
- \frac{\partial S_1}{\partial t} &= -i \frac{1}{2m} \frac{\partial^2 S_0}{\partial x^2} + \frac{1}{m} \frac{\partial S_0}{\partial x} \frac{\partial S_1}{\partial x}.
\end{align*}
\] (11) (12)

A solution for the equation (11) we find as follows:

\[
S_0(x,t) = -E_p t + S_c(x) + S'_0(x,t),
\] (13)

where

\[
S_c(x) = \int_0^x \left(2m \tilde{E}_p + \frac{2m ze^2}{|x|} \right)^{1/2} dx,
\] (14)

\[
\tilde{E}_p = \frac{p^2}{2m} - \frac{(E_e)^2}{4m \omega^2} \equiv E_p - \frac{1}{2} \gamma^2.
\] (15)

In the Coulomb action of \( S_0(x) \) the energy of electron oscillations in the electromagnetic wave field is taken into account, so that the value \( \tilde{E}_p \) must be positive. At such a definition even in the non-uniform wave field used to describe real experiments with laser beams localized in a space, the quasi-energy \( E_p \) is constant and corresponds to an electron energy outside the wave field [13]. A part of action of \( S'_0(x,t) \) which vanishes at \( \xi \to 0 \) is a periodic function of time and should not contain addenda resulting in the change in quasi-energy \( E_p \). Besides, without loss of generality let us hold positive the electron momentum \( p \geq 0 \), which was reflected in the selection of sign in (14) (the electron escapes in the positive direction of axis \( X \)).

To determine \( S'_0(x,t) \) let us employ the perturbation theory in terms of the Coulomb interaction at the simultaneous fulfillment of the condition \( \tilde{E}_p \ll 1 \). As a zero-order approximation let us accept the Volkov solution for the action of free electron with energy \( \tilde{E}_p = 0 \) in a variable electric field.

\[
S'_0(x,t) = \frac{ee}{\omega} x \sin \omega t + \frac{(ee)^2 \sin 2\omega t}{8m \omega^3} + \Sigma_2(x) + \Sigma_2(x,t),
\] (16)

Here \( \Sigma_2(x) \) and \( \Sigma_2(x,t) \) are the first- and second-order corrections, respectively, satisfying the equations

\[
- \frac{\partial S_2}{\partial t} = \frac{ee}{mw} \sin \omega t \frac{\partial S_2}{\partial x} + \frac{ee}{mw} \sin \omega t \frac{\partial S_2}{\partial x}.
\] (17)
Let us emphasize once again that the above-formulated perturbation theory differs significantly from the method suggested in [1,2]. This difference is related to the fact that in [1,2] the Coulomb interaction is borne in mind in a complete action in terms of the perturbation theory. In our approach the perturbation theory is used to describe a part of action, which remains after singling out the Coulomb action $S_c(x)$.

The general solution of equation (17) is of the form:

$$\tilde{\sigma}_1(x,t) = -S_c(x) + \Phi_1(y(x,t)),$$

where $\Phi_1(y)$ is an arbitrary function, and

$$y(x,t) = x + \frac{eE}{m \omega^2} \cos \omega t$$

is an integral of motion on a classical free electron trajectory in a variable electric field. Substituting the solution (19) for $\tilde{\sigma}_1(x,t)$ to the equation (18), we shall transform it as follows:

$$-\frac{\partial \sigma_2}{\partial t} = \frac{eE}{m \omega} \sin \omega t \frac{\partial \sigma_1}{\partial x} + \frac{1}{2m} \left( \frac{\partial \Phi_1}{\partial y} \right)^2 - \frac{E_p}{|x|}. \quad (21)$$

The equation (21) has the following general solution

$$\sigma_2(x,t) = \frac{Ze^2}{2eE} \int_1^{\infty} \frac{dy}{[\cos \gamma - \cos \omega t - \frac{m \omega^2 x}{eE}]^2} +$$

$$+ \left[ \frac{E_p}{2} - \frac{1}{2m} \left( \frac{\partial \Phi_1}{\partial y} \right)^2 \right] \sigma_2(y(x,t)),$$

where $\Phi_2(y)$ is also the arbitrary function.

Let us define the function $\Phi_1(y)$ from the condition that in $\sigma_2(x,t)$ summundes were absent which could change the quasi-energy $E_p$. Now the well-known circumstance [3] is used that in the case, when the Coulomb interaction is not taken into account, the main contribution to a transition amplitude (6) in integrating by time is made by the region in the vicinity of saddle points $t_o^{(n)}$ determined (at $E_p = 0$) by ratios

$$\sin \omega t_o^{(n)} = (-1)^n \gamma, \quad \cos \omega t_o^{(n)} = (-1)^n \sqrt{1 + \gamma^2},$$

$$\omega t_o^{(n)} = \pi n + i \sinh^{-1} \gamma,$$  \quad (23)
where \( n = 0, \pm 1, \pm 2, \ldots \). Later on it will be shown that in fulfilling the conditions \( \hbar \omega \ll 1, \varepsilon \ll \varepsilon_{\alpha} \) the displacements of saddle points caused by the accounting of the Coulomb interaction are relatively small as well as small are the displacements of saddle points brought about by the allowance for the kinetic energy at the condition of \( \tilde{E}_{p} \ll 1 \). Therefore, the second-order correction \( \delta_{2}(x,t) \) is sufficient to be known at \( t = t_{0}^{(n)} \). In the saddle points the expression for integral in the first summand of formula (22) can be presented as follows:

\[
\int \frac{dq}{\cos \varphi - \cos wt_{\omega}^{(n)} - \frac{m\omega^{2}x}{\varepsilon \varepsilon'}} = J_{1} + J_{2},
\]

\[
J_{1} = \int \frac{dq}{\cos \varphi - (-1)^{n} \sqrt{1+y^{2}} - \frac{m\omega^{2}x}{\varepsilon \varepsilon'}},
\]

\[
J_{2} = i\int \frac{dq}{\sin^{-1}y^{2} \sqrt{1+y^{2}} - (-1)^{n} \frac{m\omega^{2}x}{\varepsilon \varepsilon'}}.
\]

The integrals (25), (26) are fully defined in ranges of \( \varepsilon \varepsilon' x > 0 \) for even \( n \) and \( \varepsilon \varepsilon' x < 0 \) for odd \( n \), in which, as it will be shown, the main contribution is gained to the transition amplitude (6) while integrated by coordinate. The answer for \( J_{2} \) is of the form:

\[
J_{2} = - \frac{i}{(\sqrt{1+y^{2}} + |m\omega^{2}x|^{2})^{1/2}} \ln \left| \frac{\tanh \left( \frac{\sinh^{-1}y}{2} \right) - A}{\tanh \left( \frac{\sinh^{-1}y}{2} \right) + A} \right|,
\]

where

\[
A = \left\{ \frac{\sqrt{1+y^{2}} - 1 + |m\omega^{2}x|}{\sqrt{1+y^{2}} + 1 + |m\omega^{2}x|} \right\}^{1/2}.
\]

The expression (25) for \( J_{2} \) can be transformed as follows:
The requirement that the second-order contribution $\sigma_2(x,t)$ should not result in the change in quasi-energy $E$, means that in saddle points $t=t_0^{(n)}$ the value $\sigma_2(x,t_0^{(n)})$ does not contain a term depending linearly on the number of saddle points $n$. As a result, taking into account (20), (22), (24), (29) we shall obtain the following expression for $J_4$:

$$J_4 = \frac{|eE|}{m^2} \frac{\pi n}{\left\{ \left[ (x + \frac{eE}{m^2} \cos \omega t_0^{(n)})^2 - (\frac{eE}{m^2})^2 \right]^{1/2} \right\}^{1/2}} \frac{\pi n}{\left\{ \left[ (\frac{m^2}{eE})^2 + \frac{|m^2 x|}{eE} \right]^2 - 1 \right\}^{1/2}} \text{,} \tag{29}$$

The requirement that the second-order contribution $\sigma_2(x,t)$ should not result in the change in quasi-energy $E$, means that in saddle points $t=t_0^{(n)}$ the value $\sigma_2(x,t_0^{(n)})$ does not contain a term depending linearly on the number of saddle points $n$. As a result, taking into account (20), (22), (24), (29) we shall obtain the following expression for $\frac{\partial \Phi_1}{\partial y}$:

$$\frac{\partial \Phi_1}{\partial y} = \left\{ \frac{2mE_p}{y^2 - (eE)^2/m^2} \right\}^{1/2} \left\{ \frac{2mE_p + 2mZ^2}{(y^2 - (eE)^2/m^2)^{1/2}} \right\}^{1/2} \text{.} \tag{30}$$

Let us concentrate upon that for one-dimensional Coulomb interaction ("hydrogen atom") this expression has been defined only at $y^2 > (eE)^2/m^2 \omega^4$, i.e. at same values of $y$ which correspond to free electron trajectories in a variable electric field, not passing through the point $x=0$. Within the range of $y^2 \leq (eE)^2/m^2 \omega^4$ the expression for $\frac{\partial \Phi_1}{\partial y}$ depends on the potential shape at $|x| \leq x_a$, and for all realistic potentials can be determined in the whole region of $y$ variable. Consequently, the function $\Phi_1(y)$ determining the wave function phase is defined not only by the Coulomb potential tail (in contrast to other contributions to an action). Similarly, $\Phi_2(y)$ is found from the condition that quasi-energy changes should not take place when the third-order contribution in terms of the Coulomb interaction $\sigma_3(x,t)$ is taken into account (which affects insignificantly the ionization probability). Similar calculations which are not given here indicate that $\Phi_2(y)=0$. Thus, the expression for the second-order contribution $\sigma_2(x,t)$ in saddle points $t=t_0^{(n)}$ has the form:

$$\sigma_2(x,t_0^{(n)}) = \frac{Z e^2 m \omega}{1eE} \left[ J_2 - \frac{i \sinh^{-1} y}{\left\{ \left[ (\frac{m^2}{eE})^2 + \frac{|m^2 x|}{eE} \right]^2 - 1 \right\}^{1/2}} \right] \text{.} \tag{31}$$

Finally, let us indicate the solution of equation (12) for the function $S_1(x,t)$ defining a pre-exponential factor in the
quasiclassical wave function, if into (12) instead of \( S_0(x,t) \) to substitute an action in the form (13) with \( S_0'(x,t) \), in which only the first-order correction \( \delta_1(x,t) \) is taken into consideration:

\[
S_1'(x,t) = \frac{i}{\gamma} \ln \left[ 1 + \frac{Z e^2}{\bar{E}_p \left( y^2(x,t) - \frac{(eE)^2}{m^2 \omega^2} \right)^{1/2}} \right]. \tag{32}
\]

As a result of combination of formulas (10), (13), (16), (19) we shall obtain the following answer for the function \( S(x,t) \):

\[
S(x,t) = -E_p t + \frac{(eE)^2 \sin \omega t}{8m \omega^3} + \frac{eE}{\omega} x \sin \omega t +
+ \phi_1(x + \frac{eE}{m \omega^2} \cos \omega t) + \frac{i \hbar}{\gamma} \ln \left[ 1 + \frac{Z e^2}{\bar{E}_p \left( x + \frac{eE}{m \omega^2} \cos \omega t \right)^2 - \frac{(eE)^2}{m^2 \omega^2} \right] +
+ \delta_2(x,t). \tag{33}
\]

The region of expression (33) applicability for the action of \( S(x,t) \) from the viewpoint of validity of the quasiclassical expansion (10) is defined by the condition:

\[
\left| \frac{\partial^2 \phi_1}{\partial y^2} \right| \ll \left| \frac{eE}{\omega} \sin \omega t \right| \frac{\partial \phi_1}{\partial y},
\]

which in the proximity to saddle points \( t = t_0^{(n)} \) and under the condition \( E_p \ll I \) is transformed to the form:

\[
\left\{ \left( x + x_0(t_0^{(n)}) \right)^2 - \frac{(eE)^2}{m^2 \omega^2} \right\} \Rightarrow x_2 \left( x + x_0(t_0^{(n)}) \right). \tag{34}
\]

Here the notation is introduced:

\[
x_0(t) = \frac{eE}{m \omega^2} \cos \omega t.
\]

The area of applicability of the perturbation theory in terms of the Coulomb interaction is determined from the condition resulted from the expansion (16):

\[
\left| \frac{\partial \sigma_2}{\partial x} \right| \ll \left| \frac{eE}{\omega} \sin \omega t \right|,
\]

which in saddle points \( t = t_0^{(n)} \) and if \( E_p \ll I \) is equivalent to inequalities
Let us evaluate the value $x_0(t)$ in saddle points $t_0^{(n)}$:

$$|x| \gg x_a,$$

$$\left\{ \left( x + x_o(t_0^{(n)}) \right)^2 - \frac{(eE)^2}{m^2 \omega^2} \right\}^{1/2} \gg x_a. \tag{35}$$

Let us evaluate the value $x_0(t)$ in saddle points $t_0^{(n)}$:

$$|x_o(t_0^{(n)})| = \left| \frac{eE}{m \omega} \sqrt{1 + \gamma^2} \right| \sim \left\{ \begin{array}{l} N_o x_a, \quad \gamma \gg 1 \\ \frac{E}{\hat{E}} x_a, \quad \gamma \ll 1 \end{array} \right. \tag{36}$$

where $N_o = \left[ \frac{1}{\hbar \omega} \right] + 1$ is the minimum number of absorbed quanta in the multiphoton ionization process. From (36) it follows that in the case $\hat{E} \ll E_a, \hbar \omega \ll 1$ interesting for us the condition $|x_o(t_0^{(n)})| \gg x_a$ is always met. Besides, from the following calculation of transition amplitude (6) it will be seen that same characteristic distances which define (6) in the saddle points, have the order of magnitude

$$|x| \sim \left( \frac{\hbar \omega}{m \omega \sqrt{1 + \gamma^2}} \right)^{1/2} \sim \left\{ \begin{array}{l} N_o^{1/2} x_a, \quad \gamma \gg 1 \\ \left( \frac{E}{\hat{E}} \right)^{1/2} x_a, \quad \gamma \ll 1 \end{array} \right. \tag{37}$$

and satisfy the condition

$$|x_o(t_0^{(n)})| \gg |x| \gg x_a. \tag{38}$$

Taking into account (36), (38) it is easy to show that inequalities (34), (35) determining the region of theory applicability are met at any values of the adiabaticity parameter $\gamma$, if only the conditions $\hbar \omega \ll 1, \hat{E} \ll E_a$ have been fulfilled.

The condition (38) makes it possible to simplify the expression (33) for the action $S(x,t)$ near the saddle points $t_0^{(n)}$, in which the function $\Phi_1(x+x_0(t))$ could be expanded in a coordinate up to the first order and in the pre-exponential part determined by the function $S_1(x,t)$ (32) let us assume that $x=0$:

$$S(x,t) = -E_p t + \frac{(eE)^2 \sin \omega t}{8 m \omega^3} + \frac{eE}{\omega} x \sin \omega t + \Phi_1 \left( \frac{eE}{m \omega^2} \cos \omega t \right) +$$

$$+ x \sqrt{2m E_p} (1 + B(t))^{1/2} + \frac{i \hbar}{\gamma} \ln [1 + B(t)] + \sigma_2 (x,t). \tag{39}$$
The expression for the second-order contribution $\sigma_2(x,t)$ in
the saddle points $t = t_0^{(n)}$ in the coordinate region (38) interesting for us can be presented in the form:

$$\sigma_2(x_{\Delta} \ll |x| \ll |x_0(t_0^{(n)})|, t_0^{(n)}) \approx i\hbar \lambda \left[ \ln \left| \frac{eE_x}{\sqrt{g}} \right| + \sinh^{-1} \gamma \right].$$ (41)

Later on, with the aid of the continuous spectrum wave function (9) found in such a way, where $S(x,t)$ is defined from expressions (39)-(41) we shall calculate the multiphoton ionization probability with allowance for the Coulomb interaction.

4. THE MULTIPHOTON IONIZATION PROBABILITY

Making use of the found expression for the electron quasi-energy wave function in a continuous spectrum, let us represent the amplitude of multiphoton transition (6) from the bound state (5) to the continuous spectrum state with an energy $E_p = p^2/2m$ as follows:

$$Z_{EE_0}(t) = \frac{\hbar}{E_p} \frac{eE_x}{\sqrt{2\pi}} \int dx \exp(-i\omega t+i\lambda \ln x) \int_0^\infty d\tau \cos \tau \exp\left[ f(x,\omega)\right],$$ (42)

where $\tau = \omega t$. Taking into account (39) let us write the function $F(x,\tau)$ as a sum of three addenda:

$$F(x,\tau) = F_0(\tau) + F_1(x,\tau) + F_2(x,\tau),$$

where

$$F_0(\tau) = \frac{i}{\hbar \omega} \left[ \tau \frac{\partial}{\partial \omega} + \frac{(\omega E_x)^2}{2m \omega^2} - \frac{(\omega E_x)^2 \sin \frac{\tau}{2}}{8m \omega^2} \right],$$ (43)

$$F_1(x,\tau) = -\frac{i}{\hbar \omega} eE_x \sin \tau - \frac{i}{\hbar} \Phi_2 \left( \frac{eE_x}{m \omega x} \cos \tau \right),$$ (44)

$$F_2(x,\tau) = \frac{i}{\hbar \omega} \tilde{E}_p \tau - \frac{i x}{\hbar} \sqrt{2m \tilde{E}_p} (1 + B(\tau)^{1/2} - \ln \left[ 1 + B(\tau) \right]^{1/2} - \frac{i}{\hbar} \sigma_x(x,\tau).$$ (45)
The integral over time in (42) is calculated by the saddle-point method and saddle points \( \tau_{S}^{(n)} \) can be presented in the form:

\[
\tau_{S}^{(n)} = \tau_{0}^{(n)} + \delta \tau^{(n)},
\]

where \( \tau_{0}^{(n)} \equiv \omega t_{o}^{(n)} \) are defined by the condition \( \dot{P}_{o}(\tau_{0}^{(n)}) = 0 \) coincide with (23) and \( \delta \tau^{(n)} \) are small displacements of saddle points, associated with allowance for the Coulomb interaction, kinetic energy and coordinate dependence of action, which were taken into account in \( P_{1}(x, \tau) \). The equation for \( \delta \tau^{(n)} \) is of the form:

\[
\ddot{P}_{o}(\tau_{0}^{(n)}) \delta \tau^{(n)} + \dot{F}_{1}(\tau_{0}^{(n)}) = 0,
\]

so that

\[
\delta \tau^{(n)} = \frac{m \omega^{2} x}{\kappa \sin \tau_{0}^{(n)}} - \frac{\omega \sqrt{2 m E_{p}}}{\kappa \cos \tau_{0}^{(n)}(1 + B(\tau_{0}^{(n)}))^{1/2}},
\]

the value \( B(\tau_{0}^{(n)}) \) is defined by expression (40). It is easy to show that within the coordinate range (38) under the conditions \( \hbar \omega \ll I, \kappa \ll \kappa_{a} \) the displacements of the saddle points are relatively small:

\[
|\delta(\sin \tau_{S}^{(n)})| \approx |\delta \tau^{(n)} \cos \tau_{0}^{(n)}| \ll |\sin \tau_{0}^{(n)}| = \gamma.
\]

In calculating the function \( P(x, \tau) \) in the vicinity of saddle points \( \tau_{S}^{(n)} \) their displacements should be taken into consideration only in summands \( P_{0}(\tau) \) and \( P_{1}(x, \tau) \). On the whole, for the saddle point with the number \( n \) we have:

\[
F^{(n)}(x, \tau) = \frac{i}{\hbar \omega} \frac{\partial}{\partial \tau} \ln \left( I + E_{p} \right) - \frac{1}{\hbar \omega} \left[ I \left( 1 + \frac{1}{2 \gamma} \right) \sinh^{-1} \gamma - \frac{\hbar \omega \sqrt{1 + \gamma^2}}{2 \gamma} \right] - \lambda \left[ \ln \left( \frac{e E x}{4 I} \right) + \sinh^{-1} \gamma \right] - \frac{i}{\hbar} \frac{\partial}{\partial x} \left( e E \cos \tau_{0}^{(n)} \right) - (-1)^{n} \phi(\kappa \omega) e x|x| - \\
- \frac{m \omega x^2}{2 \hbar \gamma \sqrt{1 + \gamma^2}} - \frac{I \sqrt{1 + \gamma^2}}{\hbar \omega \gamma} (\tau - \tau_{S}^{(n)})^2.
\]
From expressions (42), (48) for the transition amplitude it follows that the main contribution to the integral over coordinate is gained in the regions \( e \in x > 0 \) for even \( n \) and \( e \in x < 0 \) for odd \( n \), where takes place a compensation of damping exponent \( \exp \left(-\frac{2\lambda}{\hbar^2} x^2\right) \) of the bound state. As a result, for characteristic distances being significant in the integral (42) the assessment (37) is valid. Summing-up all the contributions from the saddle points within the time interval \( t \) [14] we shall obtain in the limit \( \omega t \gg 1 \) (but \( \omega t \ll 1 \)) the final expression for the ionization probability per unit time, defined by \( \mathcal{W} = \left| \chi_{E_x} \right|^2 / t \):

\[
\mathcal{W} = \left( \frac{4\pi \hbar^2 \omega}{m \sqrt{1 + \gamma^2}} \right)^{1/2} \exp \left[ -\frac{2}{\hbar^2} \left( \sinh^{-1} \gamma - \frac{\gamma}{\sqrt{1 + \gamma^2}} \right) \right] \sum_{n} \frac{d}{d\chi_{k_{nx}}} \delta \left( I + E_{n} - N \hbar \omega \right).
\]

where \( \chi_{k_{nx}} \) are characteristic only of the one-dimensional geometry, this factor can be substituted by its average value being equal to 2.

5. CONCLUSION

From the expression (49) obtained for the multiphoton ionization probability one can conclude the following:

1. The expression (49) is valid within the entire region of variations of parameter \( \gamma \). At \( \gamma \ll 1 \) \( \mathcal{W} \) coincides with the static
limit (1), and at \( \gamma \gg 1 \) the Coulomb correction is independent of the wave intensity:

\[
W^{(N)} \sim J^N, \quad (\gamma \gg 1). \tag{50}
\]

This conclusion is in agreement with experimental data \([4,5]\).

The dependence (50) of the multiphoton ionization probability on the total number of absorbed photons \( N \) is the consequence of the theory considered. Therefore, deviations from this law can be associated with the non-uniformity of the electromagnetic field and the finite duration of laser pulse.

The expression (49) obtained for the multiphoton ionization probability cannot be presented as a product of ionization probability with absorption of quanta minimum number \( N_0 \) and the probability of successive increase of electron energy in the ion field due to the inverse bremsstrahlung effect. This conclusion is contrary to the assumption contained in \([5]\).

2. The accounting of the Coulomb interaction does not result in the displacement of maxima in the electron spectrum. The position of maxima is defined by the same relation as in \([13]\):

\[
E_p = N \hbar \omega - I.
\]

3. The ionization probability differs from zero only under the condition \( E_p > (Ee^2)/(4n \omega) \). Thus, the energies \( E_p \leq (Ee^2)/(4n \omega) \) are absent in the electron spectrum. This conclusion is consistent with experimental results \([4,5]\) and results of other theoretical papers \([13]\).

4. Within the range of \( \gamma \gg 1 \)

\[
\frac{W^{(N)}}{W_{S-r.}} = \left( \frac{2eI}{\hbar \omega} \right)^{2\lambda} \left[ \frac{E_p - I/2 \gamma \omega}{(E_p - I/2 \gamma \omega) + \lambda \hbar \omega} \right]^{1/2}
\]

(here \( e = 2,71... \)), i.e. the ionization probability in the Coulomb potential with absorption of \( N \) quanta far from the threshold is much greater than the short-range potential ionization probability.

5. The obtained results are applicable for accounting the electrical image forces in the case of nonlinear photoeffect from metals and semiconductors \([16]\). If to assume that the electromagnetic field does not penetrate into a metal, then the multiphoto-
ton photoeffect probability is described mainly by the obtained expression (49). The difference lies in the absence of the last oscillating multiplier, since the integration by coordinate in (6) is limited to the region of half-space beyond the metal, and \( Z = 1/2 \).

A further development of the theory can be associated with a consideration of three-dimensional Coulomb potential, as well as with an allowance for a spatial field non-uniformity and pulse time finiteness. However, one can hope that the Coulomb factor in front of the sum over \( N \) in (49) could be the same for a linearly-polarized light in the three-dimensional geometry too.

REFERENCES
