NONLINEAR QUANTUM EFFECTS IN THE MASER MODEL: A LARGE N SCALING BEHAVIOUR

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ABSTRACT

The time evolution of the mean photon number is numerically studied for different initial conditions. We find an approximate scaling of the curves for the various values of \( \gamma \) when they are plotted against \( r = t/N^\alpha \) and the coefficient \( \alpha \), empirically determined, \( \alpha \) found to be directly related to the spectral distribution of the initial condition over the exact states. Its value can be obtained from the \( N \)-dependence of the relevant eigenvalues. The approximate periodicity of the curves can also be understood in terms of the spectral distribution of the initial condition.
I. INTRODUCTION

In the study of quantum systems involving many degrees of freedom we are usually interested in the description (or measurement) of a few of them only. This is the main motivation for setting up formalisms to describe the effective dynamics of quantum subsystems. It is well known that such descriptions lead to nonlinear, nonmarkovian equations for the subsystems, which are very difficult to solve. However one can find systems whose dynamics are described by very simple, nontrivial and exactly soluble hamiltonian's. Examples can be found in the context of nuclear physics (the Lipkin model) and also in the context of quantum optics where Dicke's Maser Model has been widely studied. Their physical relevance and simplicity turn these models into very useful tools to study the exact dynamic of quantum subsystems and their nonlinear time evolution.

In this paper we report a numerical study of the effective dynamics of the radiation field in Dicke's Maser Model. We show that the time evolution of observables scale to a good approximation as behaviour is a function of $t/N^\alpha$ where $\alpha$ is empirically determined. We also investigate the dependence of this systematics as a function of initial conditions. A similar study has been performed by M.C. Cambiaggio, G.G. Dussel and M. Sarraceno in the context of the Lipkin model with similar conclusions. We are aware of the fact that both models belong to a wide class of models, the Curie—Weiss systems for which the mean field approximation becomes exact in the limit of large number of particles. It would be an interesting and important topic of research to investigate whether the empirically observed scaling constitutes a general property of the above quoted class of models.

We also find that the key ingredient to understand the dynamics is the exact spectrum and the spectral decomposition of the initial conditions. In the case of the Maser Model the spectrum has an asymptotic limit as a function of the number of particles and
this limit is reached very quickly for large enough values of the coupling constant. The empirically observed value for $\alpha$ can be obtained from the $N$-dependence of the relevant eigenvalues.

In section II we present some general considerations on the dynamics of the model and study its static properties. In section III the results are presented and analyzed on the light of the spectral decomposition of given initial conditions. In this way a partial understanding of the systematics obtained is achieved. Conclusions are given in section IV.

II. GENERAL CONSIDERATIONS ON THE STATIC PROPERTIES AND DYNAMICS OF THE MODEL

II.A. STATIC PROPERTIES

We consider a system of $N$ two level atoms which interact with one mode of the radiation field (single mode laser). The Hamiltonian describing this system was first studied by Dicke$^{(4)}$ and various quantum optical phenomena were found to be connected to it since then$^{(6,7)}$. It is given by$^{(4)}$

$$H = \epsilon J_z + \alpha \hat{a}^* \hat{a} + \frac{\epsilon^2}{N} (\hat{a}^* J_y + \hat{a} J_x)$$  \hspace{1cm} (1)

where

$$J_z = J_x = i J_y$$  \hspace{1cm} (2)

and the operators $J_j$ obey the usual commutation relations for angular momentum. They represent the $N$ level atoms in a cavity. The radiation field is represented by the operators $\hat{a}$ and $\hat{a}^*$ which obey the commutation relation

$$[\hat{a}, \hat{a}^*] = \mathbb{1}$$
and $G$ is a coupling constant. It is convenient to write the Hamiltonian (1) in the form

$$H = H_o + H_{int}$$

(3)

where

$$H_o = \epsilon J_z + \epsilon a^+ a$$

(4a)

and

$$H_{int} = \frac{G}{\sqrt{N}}(J_+ a + J_- a^+)$$

(4b)

and easy to check that

$$[H, H_o] = 0$$

(5a)

and

$$[H, J^2] = 0$$

(5b)

If we denote by $p$ the eigenvalues of $H_o$ and by $J(J+1)$ the eigenvalues of $J^2$ where $p = 0, 1, 2, \ldots$ and $J = 0, 1, \ldots, \frac{N}{2}$ for even $N$ and $J = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N}{2}$ for odd $N$ then the full Hamiltonian can be diagonalized in the subspace corresponding to a fixed value of $J$ with the condition that $p$ remains constant. A standard basis in this subspace can be defined by the kets $|J, M, n\rangle$ where

$$J_z |J, M, n\rangle = M |J, M, n\rangle$$

(6a)
Throughout this paper we shall work in the multiplet where \( p = n + M = N \) and \( J = N/2 \). In such a multiplet the state corresponding to \( | \frac{N}{2}, - \frac{N}{2}, 0 \rangle \), i.e., all atoms in their ground states and no photons remains uncoupled after diagonalization of \( H \) in the basis we defined above. The other states, exact eigenstates of \( H \) can be easily obtained by numerically diagonalizing the Hamiltonian (1),

\[
\sum_{M=1}^{N} \left[ \langle J', M, m, \frac{N}{2}, 0 | H | J, M, n \rangle - \delta_{m, M, J} \lambda_{\frac{N}{2}} \right] \langle J', M, m, \frac{N}{2}, 0 | | J', M, m, \frac{N}{2}, 0 \rangle = 0
\]  

The properties of the eigenvalues \( \lambda_{\frac{N}{2}} \) have been studied in ref. (10); they are not equally spaced but vary slowly with \( N \), in such a way that they can be considered as locally equidistant. In fig. 1 we show moreover that there is an asymptotic limit for the spectrum which is very rapidly reached for values of the coupling constant which are of the order of one or larger. The spectrum tends to a universal curve (independent of \( N \)) which should correspond to the values of the energy in the classical limit. It would be interesting if corrections of order \( 1/N \) could also be calculated. This has been done by G. Scharf\(^{(9)}\) for a limited region of the spectrum. A comparison with his calculation is given in table I.
We notice that although the asymptotic limit of the curve (Fig. 1) seems to have been reached already for \( N = 20 \) the eigenvalues seem to show large discrepancies, even for the cases \( N = 40 \) and \( 100 \). This might be due to the rather small values of \( N \).

II.B. TIME EVOLUTION

We now proceed to derive the equations we use in order to investigate the kinetic behaviour of the quantum subsystem corresponding to the radiation field. Let us take as an initial condition one of the eigenstates of \( H_0 \)

\[
|\psi(t=0)\rangle = |J, M, n\rangle
\]

At time \( t \) this state will be

\[
|\psi(t)\rangle = \sum_{\tilde{n} = 0}^{N} e^{-i\lambda_{\tilde{n}} t} |\tilde{n} \times n; J, M, n_0\rangle.
\]

The coefficients \( <\tilde{n}|JM_0n_0> \) determine the spectral distribution of the initial state, and can provide for a (partial) explanation of the time evolution in the following sense: If the initial condition is such that its spectral distribution remains confined to a region of the spectrum in which the eigenvalues are equidistant to a very good approximation then the time evolution of observables will exhibit a periodic behaviour for times which are not too long. This is precisely the case for the initial condition

\[
|\frac{N}{2}, \frac{N}{2}, 0\rangle
\]

as can be seen in fig. 2. We can predict in this case an approximate periodic behaviour of
observables with period given by (see table 1)

\[ T = \frac{2\pi}{\Delta \lambda} \]  

(11)

This is in fact confirmed in what follows. However for the initial condition

\[ | \frac{N}{2}, 2, \frac{N}{2} - 2 \rangle \]

(12)

the spectral distribution will be spread over various states and this spreading increases as a function of \( N \) (see fig. 3). Therefore we might expect a much more complex time evolution for this state.

It is now easy to construct the reduced density matrix corresponding to the radiation field. We define

\[ \rho(t) \equiv \mathcal{T}_{\gamma^0} \left( \psi(t) \right) \]

(13)

Using eq. (10) we get

\[ \rho(t) = \sum_{n=0}^{\infty} |n\rangle C_n(t) \langle n| \]

(14)

where the kets \( |n\rangle \) correspond to eigenstates of the free radiation field and \( C_n(t) \) is given by

\[ C_n(t) = \sum_{j, \sigma} \epsilon^{-i(\lambda e - \lambda \sigma) t} \langle j, p - n, n | J, M_0, n_0 \rangle \]

\[ \times \langle J, J, \sigma | \hat{\mathcal{W}} | \tilde{M}_0, n_0 \rangle \]

(15)
and the occupation probability of the n-photons state is

\[ p_n(t) = \left| c_n(t) \right|^2 \]  \hspace{1cm} (16)

III. RESULTS

The mean photon number eq. (15) for the initial condition (10) is shown in fig. 4. We found out numerically that the curves for the various values of \( N \) seem to present a scaling behaviour if the mean number of photons is plotted against \( t/N^\alpha \). \( \alpha = 0.217 \). In fact all observables should present the same scaling, since the relevant occupation probabilities as a function of \( t/N^\alpha \) exhibit a similar behaviour (see fig. 5). This scaling can be shown to be directly related to the \( N \)-dependence of the distance between relevant eigenvalues (according to the spectral distribution of the initial condition as we saw in the preceding section). Let us assume that the adjacent eigenvalues depend on \( N \) in the following way

\[ \Delta \lambda = A N^\alpha \]  \hspace{1cm} (17)

and determine the values of \( \alpha \) and \( A \) from the exact values in table 1. We get

\[ A = 0.9 \text{ and } \alpha = -0.2 \]

According to this and eq. (11) we achieve an understanding of the observed scaling property as a direct consequence of the \( N \)-dependence of eigenvalues. Interestingly enough the same value for \( \alpha \) can be obtained from the theoretical values in table 1, in
spite of the fact that the eigenvalues themselves are not as well predicted. This systematic behaviour is of course strongly dependent on initial conditions, although averaged quantities like the mean photon number seem to be less sensitive to this. In fig. 6 we show the mean photon number as a function of $t/N^\alpha$ for the initial condition (12). The occupation probabilities are shown in fig. 7 and are strongly influenced by the spectral decomposition discussed in the last section. Recurrences for non-equidistant frequencies show up very clearly. Still, the periodicity strictly connected to the $N$-dependence of the most relevant eigenvalues is approximately maintained.

IV. CONCLUSIONS

The nonlinear quantum evolution of the mean number of photons was studied for various initial conditions. An interesting approximate scaling with the variable $r = t/N^\alpha$ is found empirically and explained in terms of the $N$-dependence of the distance between neighbouring relevant eigenvalues. G. Scharr(3) studied extensively the properties of the spectrum of this model and obtained analytical expressions in the asymptotic limit (large values of $N$). However the $N$-dependence of neighbouring eigenvalues has not been studied in detail. As shown in this paper this might have interesting dynamical consequences and therefore such a theoretical study remains still as a fascinating open question.

ACKNOWLEDGMENTS

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REFERENCES

1. In this figure we plot the ratio \( \frac{\lambda_n - \lambda_0}{\lambda_{\text{max}} - \lambda_0} \) where \( \lambda_n \) is the eigenvalue, \( \lambda_{\text{max}} \) the maximum eigenvalue and \( \lambda_0 \) the eigenvalue around which the spectrum is symmetric, as a function of \( n/n_{\text{max}} \) for various values of \( N \).

2. This figure shows the same as fig. 1 plus the spectral distribution of the indicated initial condition.

3. Same as above.

4. Here we show the time evolution of the average number of photons as a function of \( t = t/N^2 \) for the initial condition \( |N, N, 0> \).

5. a) Here we show the time evolution of the occupation probability for the initial state \( |N, N, 0> \).
   
b) Same for a neighbour state.

6. Same as fig. 4 for the initial condition \( |N, 2, N/2 - 2> \).

7. a) b) Same as fig. 5a) for the initial state \( |N, 2, N/2 - 2> \).
### Table 1 - Eigenvalue Differences

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\[
\frac{\lambda n - \lambda_0}{\lambda_{\text{max}} - \lambda_0}
\]
\[
\frac{\lambda n - \lambda_0}{\lambda_{\text{max}} - \lambda_0}
\]

Fig. 2
\[
\frac{\lambda_n - \lambda_0}{\lambda_{\text{max}} - \lambda_0}
\]

\hspace{1cm} n/n_{\text{max}}

\hspace{1cm} \triangle N = 20

\hspace{1cm} \bullet N = 40

\hspace{1cm} \circ N = 100

\hspace{1cm} \text{fig. 3}
\[ \langle n \rangle / N \]

- \( N = 8 \)
- \( N = 20, 40 \)
- \( N = 100 \)

Fig 4
\[ P(\tau) \]

- $N = 8$
- $N = 20$
- $N = 40$

Fig. 5a
Fig. 5b
Fig. 7a