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VECTOR FIELDS AND GRAVITY ON THE LATTICE

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Vector Fields and Gravity on the Lattice

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ABSTRACT

The problem of discretization of vector field on Regge lattice is considered. Our approach is based on geometrical interpretation of the vector field as the field of infinitesimal coordinate transformation. A discrete version of the vector field action is obtained as a particular case of the continuum action, and it is shown to have the true continuum limit.

I. INTRODUCTION

A discrete version of general relativity (GR) which deals with the piecewise-flat manifolds composed of flat simplices was suggested by Regge [1]. Mathematical basis for this construction and it's connection with the continuum theory were clarified in Ref. [2—4], but long before that practical applications of Regge calculus to approximating classical problems of GR had been started [5—7]. Also discretization of other fields on piecewise-flat manifolds and related problems were studied [8—12]. The quantum applications of Regge calculus were considered mainly in the framework of functional integral approach [13—19]. The Hamiltonian formalism [20] provides a complementary approach. It's Regge discretization yields the most direct way to fix the functional integral measure and is interesting also from another points of view [21—23].

In practical applications of Regge calculus one often encounters the problem of discretization of vector (and, in general, tensor) fields in gravity background. If the latter is taken in the form of Regge manifold the difficulties arise connected with ambiguity of tangent space at singular points [24] where the curvature residues. These are the points of $(n-2)$-dimensional subsimplices (the bones) of Regge lattice. Formally, this singularity displays as a divergence of the action. Namely, suppose the action is quadratic in the covariant derivatives and depends nontrivially on the Christoffel connection involved in these derivatives, $\nabla_\mu \eta_\nu = \partial_\mu \eta_\nu - \Gamma^\lambda_{\mu\nu}\eta_\lambda$. In the piecewise-affine coordinate system $\Gamma^\lambda_{\mu\nu}$ takes the form of δ-functional distribution (with support on $(n-1)$-dimensional faces) so the action includes the singularities of the form of δ-function squared under the integral sign. In the particular case of electromagnetic field Christoffel connection cancels in the expression for the action. Therefore this and similar systems admit an elegant formulation on Regge lattice [8—10] (in terms of some integral variables).

Now consider vector field η^μ with the action of the form

$$S=4\int M^{\mu\nu\lambda\rho}(\nabla_\mu\eta_\nu)(\nabla_\lambda\eta_\rho)\sqrt{g}\,d^n x\,,$$
$$M^{\mu\nu\lambda\rho}=M^{\nu\mu\lambda\rho}=M^{\mu\nu\rho\lambda} \qquad (1)$$

where $M^{\mu\nu\lambda\rho}$ depends on metric and, perhaps, on another matter fields. We consider $M^{\mu\nu\lambda\rho}$ be piecewise-constant on Regge lattice. List some examples of (1). First, (1) is the general form of the Faddeev—Popov vector ghost field action in gravity if the gauge is chosen in the form of algebraic relations between the components of metric tensor. In this case $\eta^\mu(x)$ is the field of infinitesimal coordinate transformations $x'^\mu=x^\mu-\eta^\mu(x)$, $\nabla_{(\mu}\eta_{\nu)}=\nabla_\mu\eta_\nu+\nabla_\nu\eta_\mu$ is the corresponding variation of metric, $\delta_\eta g_{\mu\nu}$, (1) is some (degenerate) infinitesimal norm of the type of that of the deWitt one [25] on the space of metrics. In particular, for the conformal gauge in the two-dimensional gravity relevant to the Polyakov string quantization [26] $M^{\mu\nu\lambda\rho}=g^{\mu\lambda}g^{\nu\rho}+g^{\mu\rho}g^{\nu\lambda}-g^{\mu\nu}g^{\lambda\rho}$. Second, the so-called shift vector N_i enters bilinearly (in the form of symmetrized derivative $\nabla_{(i}N_{k)}$) the 3+1 action arising when constructing Hamiltonian formalism in GR [20]. Finally, if the symmetry conditions (1) were not imposed on $M^{\mu\nu\lambda\rho}$ one would decompose the derivative

$$\nabla_\mu\eta_\nu=\frac{1}{2}(\partial_\mu\eta_\nu-\partial_\nu\eta_\mu)+\frac{1}{2}\nabla_{(\mu}\eta_{\nu)}$$

so we see that the main difficulty is connected with it's symmetric part containing Christoffel connection.

If this note we obtain a discrete action for vector field on Regge lattice as a particular case of the continuum one. Then we show that this action is a formal finite-difference approximation to the continuum action in the leading order in a lattice spacing. In this aspect there is an analogy with the derivation of Regge action as a particular case of the Einshtein one [2] and subsequent proving that the latter is the continuum limit of the former [3]. To resolve the singularity at the bones we use a geometrical interpretation of the vector field as that of infinitesimal coordinate transformation and appropriate choice of field variables at the singular points. The paper is organized as follows. In the next section the procedure of discretization is described. This discretization has a freedom connected with the choice of an anzats for vector field being parametrized by a discrete set of variables. In sect. 3 a particular choice of this anzats is described. Using this anzats it is proven in sect. 4 that the resulting action is, first, quadratic in the discrete variables and, second, it has the true continuum limit. Then we conclude. The following notations are used throughout the paper. K is the (n-dimensional) Regge lattice (simplicial complex; piecewise-flat manifold). σ^k, s^k are the k-dimensional simplices (for $k=n-2$ and $k=n-1$ these will be called the bones and the faces, respectively). $\Gamma^j(\sigma^k)$ ($\Gamma^j(K)$) is the j-dimensional skeleton of σ^k (of K), i. e. the union of subsimplices $\sigma^j \subset \sigma^k$ ($\sigma^j \subset K$), in particular, $\Gamma^{k-1}(\sigma^k)$ is $\partial\sigma^k$, the boundary of σ^k. $St^j(\sigma^k)$ is the j-dimensional star of σ^k, i. e. the union of simplices $\sigma^j \ni \sigma^k$. In particular, we denote $St^n(\sigma^{n-1}) \equiv St(\sigma^{n-1})$. $\sigma(\vec{l}_1, \ldots, \vec{l}_k)$ is the simplex σ^k spanned by the vectors $\vec{l}_1, \ldots, \vec{l}_k$. $\varphi=\varphi(\sigma^{n-2})$ is the deficite angle which characterizes the curvature at the bone σ^{n-2}. $|\sigma^k|$ is the (k-dimensional) volume of σ^k. As a rule we use a piecewise-affine coordinate system x^μ in K so that $g_{\mu\nu}=\text{const}$ in each σ^n.

2. THE EFFECTIVE ACTION FOR THE VECTOR FIELD ON REGGE LATTICE

According to the previous discussion (1) can be rewritten as

$$S=\int M^{\mu\nu\lambda\rho}\delta_\eta g_{\mu\nu}\,\delta_\eta g_{\lambda\rho}\sqrt{g}\,d^n x' \tag{2}$$

where

$$\delta_\eta g_{\mu\nu}(x')=g_{\lambda\rho}(x(x'))\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}-g_{\mu\nu}(x'). \tag{3}$$

Here $x(x')$ is the inversed to $x'^\mu=x^\mu-\eta^\mu(x)$ function. If one were able to expand (3) in the Taylor series over η^μ he would reproduce (1). Now on Regge lattice such an expansion is impossible because of the discontinuity of Regge metric*).

*) Using infinitesimal form of the norm of $\delta_\eta g_{\mu\nu}$ in our representation of S (1) implies the smallness of changes of metric $g_{\mu\nu}$ from simplex to simplex (a more detailed analysis below shows that it is sufficient to keep deficite angles small). Otherwise one should proceed from an expression for the finite distance between the two metrics [25] (when $\delta_\eta g_{\mu\nu}$ is not infinitesimal).

Our task is to reduce (2) to some field theory action with the appropriately chosen field variables. Besides the choice of the variables we also perform partial functional integration and insert an anzats for vector field being parametrized by a discrete set of variables. It is convenient to divide this procedure into the following three points.

a). Changing the field variables. This includes the following two steps. First, to each point with *old* coordinates y we assign the variable $\tilde{\eta}(y)=x(y)-y$ instead of the previous one $\eta(y)=y-x'(y)$. Here $x(x'(y))\equiv y$. At a nonsingular point infinitesimal $\tilde{\eta}$ and η coincide. It is seen that $\eta(y)$ is the difference of the coordinates taken in the different coordinate systems. At a singular point $y\in\sigma^{n-2}$ one cannot give the sence of a vector to $\eta(y)$ because of the absence of a locally affine system at this point. Unlikely $\tilde{\eta}(y)$ is the difference of some coordinates both taken in the *same* (namely, *old*) coordinate system. Therefore to $\tilde{\eta}(y)$ the sence of a vector can be given: it is the vector which connects the given singular point with *old* coordinates y to the point y_η whose *new* coordinates (x') are y. (Here y_η is a particular case of more general notation used in the following: by A_η we denote the image of a set of points A at the mapping $y\to x(y)$.)

Second, there should be a correspondence with the continuum case. In the continuum limit we should have a vector field on a smooth Riemann manifold. However, no object of this kind arises when tending lattice spacing to zero. Therefore we introduce an auxiliary smooth Riemann manifold M and parametrize $\tilde{\eta}$ by a vector field ζ on it. The manifold M can be chosen so that K be the piecewise-flat approximation to it in the sence of Ref. [4]. Namely, K is mapped onto M and the length of an edge connecting the two vertices in K is precisely the geodesic distance between the images of these points in M. It is convenient to introduce a piecewise-affine coordinate system in K and a curvilinear system in M such that the coordinates of the vertices be fixed points of this mapping. To put correspondence between ζ and $\tilde{\eta}$ consider a vertex O in K and a simplex $\sigma(\vec{l}_1, ..., \vec{l}_n)$ to which O_η belongs. Here each $\vec{l}_a$ connects O to some another vertex O_a. Let $\mathcal{O}$ and $\mathcal{O}_a$ be the images of O and O_a in M. There is the natural basis at $\mathcal{O}$ consisting of unit tangent vectors u_a^α to geodesics $\mathcal{O}\mathcal{O}_a$. If l_a^α is the difference between the coordinates of $\mathcal{O}_a$ and $\mathcal{O}$ in M then

$$u_a^\alpha = l_a^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma} l_a^\beta l_a^\gamma + O(l^3)\,. \tag{4}$$

For $\tilde{\eta}^\alpha = \eta^a l_a^\alpha$ we put

$$\zeta^\alpha = \eta^a u_a^\alpha\,. \tag{5}$$

Then

$$\tilde{\eta}^\alpha = \zeta^\alpha - \sum_a \frac{1}{2}\Gamma^\alpha_{\beta\gamma} l_a^\beta l_a^\gamma \frac{[\vec{l}_1, \ldots, \vec{l}_{a-1}\vec{\zeta}\vec{l}_{a+1}, \ldots, \vec{l}_n]}{[\vec{l}_1, \ldots, \vec{l}_n]} + O(l^2) = \zeta^\alpha + O(l) \tag{6}$$

where $[\vec{l}_1, \ldots, \vec{l}_n]$ is the determinant built of the components of $\vec{l}_a$.

b). Integrating out free fields. The distinctive advantage of Regge-discretized gravity is it's being a particular case of continuum gravity and, further, the flatness of the geometry in the interior of simplices. Therefore the dynamics of fields in the simplices considerably simplifies and it is possible to perform functional integration over the values of the fields in the interior of simplices σ^n. So we get an effective action being a functional of the boundary values of fields (on the faces). Thereby we pass from R^n field theory to $Z\times R^{n-1}$ one with Z being the set of faces $\sigma^{n-1}\subset K$. There is a local UV divergent term of the form $\Lambda^n\int f(g_{\mu\nu})\sqrt{g}\,d^nx$ in this action where Λ is UV cut-off, $f(g_{\mu\nu})$ — a local function. Omitting this term we get the effective action less singular than the original one (it leads to the divergences in loops not exceeding $O(\Lambda^{n-1})$).

Gaussian integration effectively reduces to minimization of the action. Therefore for the considered vector field of infinitesimal coordinate transformations the action reduces to the sum of the squared distances between the orbits of geometries in σ^n and in σ^n_η, $\sum_\sigma \mathrm{dist}^2(\mathrm{orb}\, g'_{\mu\nu}(\sigma^n_\eta),\ \mathrm{orb}\, g_{\mu\nu}(\sigma^n))$ (in the deWitt terminology [25]). These distances depend only on the geometries in σ^n (flat) and in σ^n_η and on the mappings between the boundaries $\partial\sigma^n\to\partial\sigma^n_\eta$. Thus we get an effective action as a functional of these mappings.

c). To discretize the action we «freeze» some degrees of freedom of the field parametrizing it by a discrete set of variables. Now it is the mappings $\sigma^{n-1}\to\sigma^{n-1}_\eta$ which should be parametrized. If it were the case of scalar field $\Phi(x)$ there would be a particularly simple anzats at ones disposal with $\Phi(x)$ chosen as a linear function on the faces. This function is completely determined by its values at the vertices and would lead to the discretization of Ref. [11]. Analogous anzats for the problem at hand would be the linear mappings between the *hyperplanes* $\sigma^{n-1} \to \sigma_{\eta}^{n-1}$. But generally speaking σ_{η}^{k} cannot be a hyperplane at $k > 1$ because it intersects with the bones carrying the curvature (in case of two-dimensional gravity it is sufficient to consider one-dimensional simplices σ_{η}^{1} which all can be chosen to be straight lines; at $n > 2$ complications arise). However, we can choose the mappings $\sigma^{n-1} \to \sigma_{\eta}^{n-1}$ such that at zero deficits they reduce to the linear mappings between the hyperplanes. Below explicit construction of such the anzants for $n = 3$ is given.

3. ANZATS FOR DISCRETIZATION OF THE VECTOR FIELD

Here we construct the faces σ_{η}^{n-1} and the mappings $\sigma^{n-1} \to \sigma_{\eta}^{n-1}$ for the case $n = 3$. Generalization to arbitrary n offers no difficulties except for the notational ones

Given the vectors $\tilde{\eta}$ at the vertices we construct the mappings $\sigma^2 \to \sigma_{\eta}^2$ in the three-dimensional case which can be called «almost linear» mappings of «almost flat faces». First, we construct the face σ_{η}^2 itself. For that we draw the edges σ_{η}^1 as geodesics (straight lines) connecting the vertices σ_{η}^0. Then we draw a surface $\tilde{\sigma}_{\eta}^2$ which is a plane in some flat submanifold of K and passes through σ_{η}^0's. Deforming $\tilde{\sigma}_{\eta}^2$ near it's boundary we get a surface σ_{η}^2 passing also through the edges σ_{η}^1.

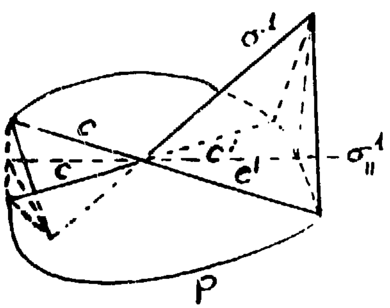

Fig. 1. The two alternative cuts c and c' relevant to the two different maximal continuous plane-like extensions of the surface P around the curvature support σ^1 (in the three dimensional space); $\sigma_{\parallel}^1$ is the projection of σ^1 onto P.

The surface $\tilde{\sigma}_{\eta}^2$ can be chosen to be a plane in a flat submanifold of K obtained by cutting K through the bones σ^1 i. e. by removing the points of halfplanes passing through the bones σ^1. In order that after glueing together the edges of a cut the continuous surface arise from this plane the cut should be symmetric with respect to the plane. In the degenerate case of σ^1 orthogonal to the plane any cut is suitable but generally there are the two possible such cuts running along the projection of σ^1 onto the plane, see Fig. 1. In the assumption of convexity of $St(\sigma^2)$ the cuts can be chosen not to intersect the interior of $St(\sigma^2)$.

The surface $\tilde{\sigma}^2_\eta$ can be drawn to pass through the vertices σ^0_η. Then it can be bounded by geodesics *in it*, $\tilde{\sigma}^1_\eta$, connecting the vertices σ^0_η. Generally $\tilde{\sigma}^1_\eta$ does not coincide with σ^1_η. By construction $\tilde{\sigma}^2_\eta$ possesses the following properties. 1). It passes through the given three vertices σ^0_η. 2). It is piecewise-flat so that the dihedral angles between the different flat pieces can differ from π by $O(\varphi)$ in the exterior of $St(\sigma^2)$, i. e. in the narrow strip at the distances $O(\eta)$ from the boundary $\partial\tilde{\sigma}^2_\eta$. Here η and φ are typical scales of the vector field variables $\vec{\eta}(y)$ and of the deficite angles at the bones, respectively. 3). It is a plane in $St(\sigma^2)$ (the plane DGEHF in Figs 2,*a*,*b*).

It follows from the second property that $\tilde{\sigma}^2_\eta$ can be approximated by the straight lines up to deformations $O(\eta)O(\varphi)$. Therefore we can deform $\tilde{\sigma}^2_\eta$ in the neighbourhood of it's boundary to get another surface σ^2_η which also can be approximated by the straight lines up to deformations $O(\eta)O(\varphi)$. It possesses the following properties. 1). It passes through the given vertices σ^0_η and the edges σ^1_η connecting them. 2). It is piecewise-flat, it can be composed of flat triangles s^2 whose vertices are either σ^0_η or intersections of σ^2_η with the bones. 3). It is a plane in $St(\sigma^2)$ (the plane DGEHF in Fig. 2,*c*) and one of the triangles $s^2_{(0)}$ (DEF in Fig. 2,*c*) has the edge lengths up to $O(\eta)$ the same as those of σ^2_η.

Next we construct the mappings $\sigma^2 \to \sigma^2_\eta$ as a boundary values of the mappings $\sigma^3 \to \sigma^3_\eta$ which we shall describe as follows. Let us triangulate each σ^3_η by a set of flat simplices s^3 with the following properties. 1). The vertices of s^3 which belong the face σ^2_η are the vertices of the triangles s^2 which form this face. 2). Another vertices of s^3 are the vertices σ^0 of original Regge lattice K. 3). One of the tetrahedra $s^3_{(0)}$ has the edge length up to $O(\eta)$ the same as those of σ^3_η. It follows from this that there are also the simplices $s^3_{(m)}$, $m = 1, 2, 3$ in which m dimensions are of the order of η and 3-m ones are $O(1)$. Let us embed thus triangulated complex into an Euclidean space of sufficiently large dimension E^N. In E^N we can assign to each simplex σ^k_η the flat one σ^k_g spanned by $\Gamma^1(\sigma^k_\eta)$. Consider the following operation of projecting the points of σ^k_η orthogonally onto σ^k_g. First, we project the vertices $s^0 \in \sigma^3_\eta$ onto σ^3_g, $s^0 \to s^{0\prime}$. Second, if s^0 belongs the boundary, $s^0 \in \sigma^2_\eta \subset \partial\sigma^3_\eta$, it is additionally projected onto σ^2_g, $s^{0\prime} \to s^0_\parallel$. Despite of the fact that boundary vertex belongs the two simplices σ^3_η, the whole operation is correctly defined because the result of the second projecting is independent of the simplex σ^3_g on which the first projecting is made. The resulting projecting $s^0 \to s^0_\parallel$ can be linearly extended to each s^3 to give some

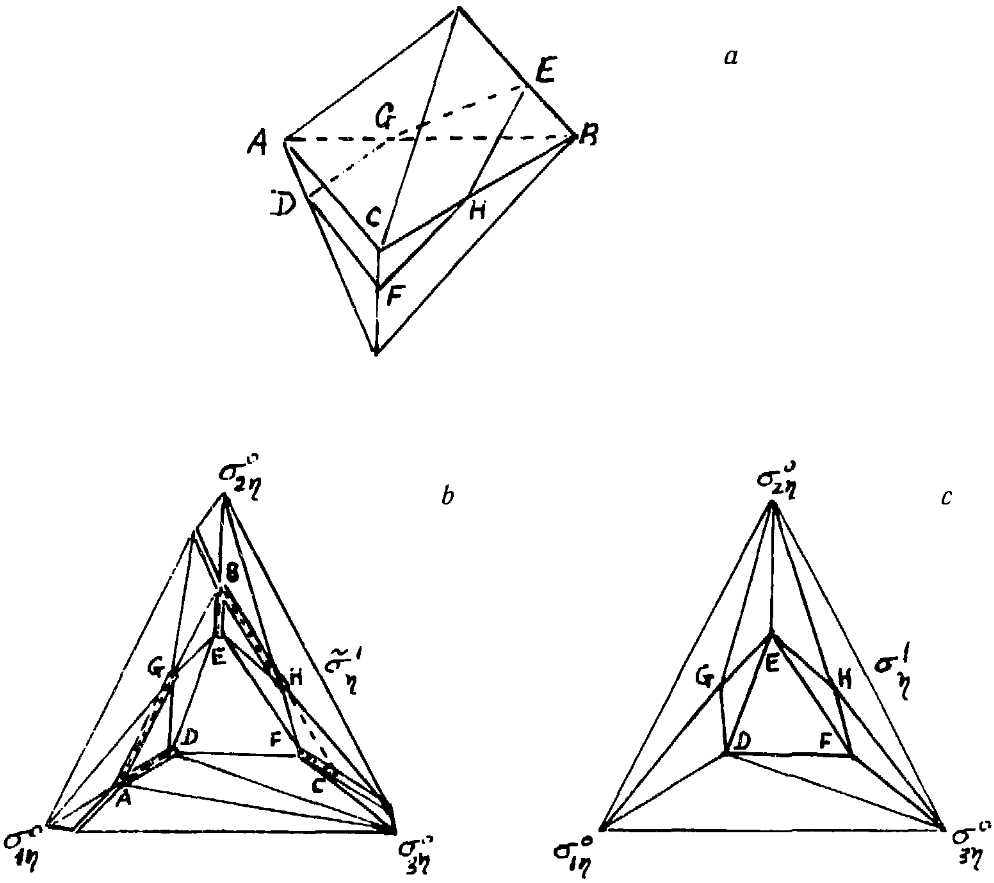

Fig. 2. Construction of the face σ^2_η.

a). *DGEHF* is a plane close to the face $\sigma^2 = ABC$ inside the union $St(\sigma^2)$ of the two tetrahedra having *ABC* as the common face. *D*, *G*, *E*, *H*, *F* are intersections of this plane with the edges $\sigma^1 \subset St(\sigma^2)$ carrying the curvature (another edges not belonging $St(\sigma^2)$ are not shown for simplicity).

b). The piecewise-flat surface $\tilde{\sigma}^2_\eta$ is a maximal continuous extension of DGEHF into the exterior of $St(\sigma^2)$ as a plane. Solid lines divide $\tilde{\sigma}^2_\eta$ into a set of the flat simplices; of these lines the double ones are the cuts symmetric with respect to the projections of curvature supports σ^1_η shown by dotted lines (see Fig. 1).

c). The piecewise-flat surface $\tilde{\sigma}^2_\eta$ bounded by straight lines is obtained by deformation of $\tilde{\sigma}^2_\eta$ near it's boundary (in the exterior of $St(\sigma^2)$). Solid lines divide σ^2_η into a set of the flat simplices s^2.

piecewise-linear mapping $\sigma^3_\eta \to \sigma^3_g$. Besides, σ^3_g and σ^3 can be connected by linear mapping. This gives the desired mapping $\sigma^3 \to \sigma^3_\eta$, to which some variation of metric corresponds,

$$\delta^{(0)}_\eta g_{\mu\nu} \equiv h_{\mu\nu}(x), \qquad x \in \sigma^3. \tag{7}$$

It can be used to calculate the effective action. By construction the result depends only on the restriction of the described mapping on $\Gamma^2(K)$.

4. THE STRUCTURE OF THE DISCRETIZED ACTION

Consider now expansion of the effective action in η and in a lattice spacing ε. Given a particular metric variation (7) we get for the variation under an additional reparametrization $x''^\mu = x'^\mu - \xi^\mu(x')$, $x' \in \sigma^3$:

$$\delta_\eta g_{\mu\nu} = h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} \,. \tag{8}$$

Substituting this into (2) and minimizing with respect to ξ^μ with boundary condition $\xi^\mu(x) = 0$, $x \in \partial\sigma^3$ we get omitting the primes

$$S_{\rm eff} = \sum_{\sigma^3} \int_{\sigma^3} [\, h_{\mu\nu} M^{\mu\nu\lambda\rho} h_{\lambda\rho} - h_{\mu\nu} M^{\mu\nu\tau\sigma} \partial_\tau K_{\sigma\varepsilon} \partial_\omega M^{\varepsilon\omega\lambda\rho} h_{\lambda\rho}] \,. \tag{9}$$

Here $K_{\sigma\varepsilon}$ is the Green function of operator $\partial M \partial$:

$$\partial_\mu M^{\mu\nu\lambda\rho} \partial_\lambda K_{\rho\tau}(x, y) = \delta^\nu_\tau \delta^{(3)}(x-y) \,,$$
$$K_{\rho\tau}(x, y) = 0 \,, \qquad x \in \partial\sigma^3 \,. \tag{10}$$

What can be said on the values of the (piecewise-constant) $h_{\mu\nu}(x)$? In the flat space $\sigma^3_\eta = \sigma^3_g$ and $h_{\mu\nu} = \mathrm{const} \equiv h^{\parallel}_{\mu\nu}$ in σ^3. As a result, the second term in (9) vanishes. Using tensor components along the edges $l \subset \sigma^3$, $h_l \equiv l^\mu l^\nu h_{\mu\nu}$ [21] we may write $h^{\parallel}_l = \delta_\eta l^2$, variation of the squared length of an edge. In a curved space $h_{\mu\nu} = h^{\parallel}_{\mu\nu} + h^{\perp}_{\mu\nu}$ where $h^{\perp}_{\mu\nu}$ is the variation of metric due to the previously considered projecting $s^0 \to s^0_{\parallel}$. $h^{\perp}_{\mu\nu}$ is constant in each s^3. What can be said on it's values? The projection onto σ^3_g, $s^0 \to s^{0\prime}$, changes the squared edge lengths l^2_i of the simplices s^3 by the orthogonal components squared $l^2_{i\perp}$. Evidently $l_{i\perp} = O(\eta)$. At the same time the deficits are linear functions of $l^2_{i\perp}$. Therefore $\Delta l^2_i = O(\eta^2) O(\varphi)$. The projecting onto σ^3_r makes the deficits vanish. For that the dihedral angles should be changed by $O(\varphi)$. So the face σ^2_η after this projecting becomes a piecewise-flat surface $\sigma^{2\prime}_\eta$ with the dihedral angles $\pi - O(\varphi)$. Therefore $\sigma^{2\prime}_\eta$ coincides with σ^2_g up to deformations $O(\eta)O(\varphi)$ and subsequent projecting $s^{0\prime} \to s^0_{\parallel}$ changes some edge lengths by $O(\eta)O(\varphi)$. This may induce metric variation $h^{\perp}_{\mu\nu} = O(\varphi)$ in the corresponding simplices s^3 if there is an edge in s^3 of the length $O(\eta)$ which has the component along x^μ and/or x^ν axes. So the simplex $s^3_{(1)}$ with the volume $O(\eta)$ may contribute the terms $O(\eta)$ to S_{eff}.

However, these terms can be shown to cancel. Let us introduce the normal x^n and tangential x^i coordinates, $i = 1, 2$, in the vicinity of σ^2. The large faces $s^2_{(1)}$ (which have the dimensions $O(1) \times O(1)$) of the flattened tetrahedra $s^3_{(1)}$ are at the distances $O(\eta)$ from σ^2_η and almost parallel to it. So in the leading order in η in our calculation of S_{eff} the coordinates x^i and x^n are tangential and normal ones respectively also with respect to the faces $s^2_{(1)}$ and only the dependence on x^n should be taken into account. Besides, the induced on $s^2_{(1)}$ metric variation $h^{\perp}_{ik}$ is continuous while it's (i, n) and (n, n) components suffer the change $O(\varphi)$ on this face. So the action in the order $O(\eta)$ takes the form

$$\sum_{s^3_{(1)}} \int_{s^3_{(1)}} [\, h^{\perp}_{\mu\nu} M^{\mu\nu\lambda\rho} h^{\perp}_{\lambda\rho} - h^{\perp}_{\mu\nu} M^{\mu\nu\lambda\rho} \partial_n (\partial^2_n)^{-1} \partial_n h^{\perp}_{\lambda\rho} \,] \tag{11}$$

where at least one of indices in each pair (μ, ν) and (λ, ρ) is n. Naively it is seen to vanish, and also a more detailed calculation using accurate definition of Green function shows that (11) is zero up to $O(\eta^2)$ terms.

Another question we would like to dwell on is the expansion over a typical lattice scale ε. The previous consideration can be regarded as that corresponding to $\varepsilon = 1$. At $\varepsilon \neq 1$ let us rescale the edge vectors $l^\mu \to \varepsilon l^\mu$ so that $l^2 = O(1)$ as before. If Regge lattice approximates smooth manifold M with a scale of curvature R the angle deficits have the scale $R\varepsilon^2$. Therefore $h^{\perp}_{\mu\nu} = O(\varepsilon^2)$ in the region which has the volume $O(\varepsilon^2)O(\eta)$. Suppose the discrete vector field $\tilde{\eta}$ has a differentiable continuum limit, i. e. the finite differences between $\tilde{\eta}$ taken at the neighbouring vertices are linear in ε. Then $h^{\parallel}_{\mu\nu} = O(1)$ in the volume $O(\varepsilon^3)$. So $h^{\perp}_{\mu\nu}$ leads only to the next-to-leading terms in S_{eff}:

$$S_{\text{eff}} = \sum_{\sigma^n} h^{\parallel}_{\mu\nu} M^{\mu\nu\lambda\rho} h^{\parallel}_{\lambda\rho} |\sigma^n| + O(\varepsilon) \tag{12}$$

where we have returned to the arbitrary dimension n.

The limit of (12) at $\varepsilon \to 0$ can be shown to reproduce finite-difference approximation for the continuum expression. This can be most easily seen on the periodic Regge lattice (taken, e. g., in the form of that used in Ref. [13]). Consider vectors $\vec\eta$ and edge vectors $\vec l_a$, $a = 1, \dots, n$ emanating from a vertex O. Suppose $O_\eta \in \sigma(\vec l_1, \dots, \vec l_n)$. Translation $T_{\vec l}$ by an edge $\vec l$ gives some another vertex O' and vectors $\vec\eta'$ and $\vec l'_a$, see Fig. 3. If the field $\vec\eta$ has a smooth continuum limit then it is sufficient to consider the case $O'_\eta \in \sigma(\vec l'_1, \dots, \vec l'_n)$ (another possibility in the configuration space of $\vec\eta$ has the measure vanishing at $\varepsilon \to 0$). Then a simple vector algebra in the basis $\{\vec l_a\}$ yields the desired expression:

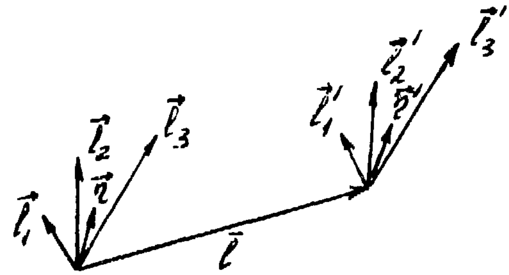

Fig. 3. Vectors $\vec\eta$, $\vec\eta'$ on periodic Regge lattice; $\vec l_a$, $\vec l'_a$, $\vec l$ are the edges.

$$\varepsilon l^a l^b h^{\parallel}_{ab} = \delta_\eta l^2 = 2(\vec l, \vec\eta' - \vec\eta) = 2(\vec l, \eta'^a \vec l'_a - \eta^a \vec l_a) =$$

$$= 2(\vec l \vec l'_a)(\eta'^a - \eta^a) + 2(\vec l \vec l'_a - \vec l \vec l_a)\eta^a =$$

$$= 2(\vec l \vec l'_a)\delta_l \eta^a + \eta^a [(\vec l + \vec l'_a - \vec l_a)^2 - l^2 - (\vec l'_a - \vec l_a)^2] =$$

$$= 2(\vec l \vec l_a)\delta_l \eta^a + \eta^a \delta_a l^2 + O(\varepsilon^2) =$$

$$= l^a l^b (2 g_{ca} \delta_b \eta^c + \eta^c \delta_c g_{ab}) + O(\varepsilon^2) \qquad (13)$$

where $g_{ab} = (\vec l_a \vec l_b)$, $\delta_a = T_a - 1$ is the finite-difference operator, T_a is the operator of translation by $\vec l_a$. This can be rewritten by defining $h^{\parallel}_{ab} = \nabla_{(a}\eta_{b)}$ with Γ^a_{bc} and $\nabla \sim \varepsilon^{-1}\delta - \Gamma$ constructed with the help of finite differences. This can be formulated in a regular coordinate system by using vector field ζ^α on the auxiliary smooth manifold M. So we get in the corresponding smooth coordinates

$$h^{\parallel}_{\alpha\beta} = \nabla_{(\alpha}\zeta_{\beta)} + O(\varepsilon),$$

$$\nabla_\alpha \zeta_\beta = \varepsilon^{-1}\delta_\alpha \zeta_\beta - \Gamma^\gamma_{\alpha\beta}\zeta_\gamma \qquad (14)$$

where $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel connection in M in the neighbourhood of the considered points.

5. CONCLUSION

We have resolved singularities of the vector field on Regge lattice by setting the modified field variables. The degree of freedom in the construction of the descrete action is determined by arbitrariness of the choice of an anzats for vector field on $(n-1)$-dimensional faces being parametrized by a discrete set of variables. It is shown that under appropriate choice of this anzats it turns possible to define a descrete action which is, first, quadratic in the field variables and, second, it possesses the true continuum limit.

We have studied the case of small deficits close to the continuum limit. In principle, one can consider the arbitrary angle deficits. For that an expression for the finite distance on the space of metrics should be used instead of it's infinitesimal form (2).

Within the suggested approach the vector ghost field action for the Polyakov string on the lattice can be discretized. In this way we reproduce the correct entropy of gravitational measure in the conformal gauge, i. e. the nonlocal term $(-26/48\pi)\int R\Delta^{-1}R$ [26]. As noted in Ref. [17] the contribution can be singled out in the considered action which can be written without using the Christoffel connection and formulated in terms of some integral variables [8—10]. The remaining term, however, is completely defined by the vector field at the singular points and brings about the problems considered in the present paper.

Also the shift vector N_i in the Hamiltonian formalism for gravity can be discretized with the help of our approach since the 3+1 gravity action is a second order polinomial in $\nabla_i N_k$ [20].

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