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INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

IC/88/411

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

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QUALITATIVE ANALYSIS OF NONLINEAR INCIDENCE RATE UPON THE BEHAVIOUR OF AN EPIDEMIOLOGICAL MODEL *

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MIRAMARE - TRIESTE
December 1988

* Not to be submitted for publication.

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We consider an epidemiological model

$$\begin{cases} \dfrac{dI}{dt} = KI^pS^q - (b+v)I, \\ \\ \dfrac{dR}{dt} = vI - (b+\gamma)R \end{cases}$$

where $H(I,S) = KI^pS^q$ is the incidence rate per infective individual k, p, q, γ, b and v are positive constants. Let $T = (b+v)t$, then the above system becomes

$$\begin{cases} \dfrac{dI}{dT} = aI^pS^q - I, \\ \\ \dfrac{dR}{dT} = r(I-R/h) \end{cases}$$

where $a = k/(b+v)$, $r = v/(b+v)$, $h = v/(b+\gamma)$. Clearly $0 < r < 1$.

In this paper, we consider the case $p = 2$, $q = I$. We replace the variables I, R and T by x, y and t respectively, and obtain the equivalent equations:

$$\begin{cases} \dfrac{dx}{dt} = ax^2(N_0-x-y) - x = P(x,y), \\ \\ \dfrac{dy}{dt} = r(x-y/h) = Q(x,y) \quad . \end{cases} \tag{1}$$

Clearly the line $x = 0$ is a solution of the system (1). Since $\left.\dfrac{dy}{dt}\right|_{y=0} = rx \geqslant 0$, we know that the line $y = 0$ has no tangential points with the orbits of the system (1). Let $N = N_0-x-y$, we have

$$\left.\frac{dN}{dt}\right|_{N=0} = (1-r)x + ry/h \geqslant 0,$$

$N = 0$ has no tangential points with the orbits of the system (1) too. For this reason, we restrict our attention to the feasible region

$$D = \{(x,y)\colon x \geqslant 0,\ y \geqslant 0,\ N = N_0-x-y \geqslant 0\} \quad .$$

In D, we know that $O(0,0)$ is a stable node. Other singular points are $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$ where x_i is the root of the equation

$$a(1+h)x^2 - aN_0 x + 1 = 0 \quad ,$$

and $x_i = [aN_0 \pm \sqrt{a^2N_0^2 - 4a(1+h)}]/2a(1+h)$, $y_i = hx$: $(i = 1,2)$. As

$$aN_0^2 - 4(1+h) > 0 \quad ,$$

the two singular points M_1 and M_2 exist. We might as well suppose $0 < x_1 < x_2$, thus M_1 is the saddle point, since the discriminant $q(x_1,y_1)$ is negative:

$$q(x_1,y_1) = rx_1\sqrt{a^2N_0^2 - 4a(1+h)}/h < 0;$$

M_2 is a non-saddle point, since the discriminant is positive:

$$q(x_2,y_2) = rx_2\sqrt{a^2N_0^2 - 4a(1+h)}/h, > 0 \ ,$$

and since $p(x_2,y_2) = (ahx_2^2 + r-h)/h$, and $ahx_2^2 + r - h > 0$, we obtain that M_2 is stable; as $ahx_2^2 + r - h < 0$, M_2 is unstable. When

$$aN_0^2 - 4(1+h) = 0 \quad ,$$

then $M_1 = M_2$. Here the two singularities coincide. If

$$aN_0^2 - 4(1+h) < 0,$$

then M_1 and M_2 disappear.

<u>Theorem 1</u> If $r - h \geqslant 0$, the system (1) has no closed orbit and no singularities in D.

<u>Proof</u> Let B be a Dulac function:

$$B(x,y) = x^{\alpha}y^{\beta}$$

where

$$\alpha = \frac{4(r+h) - 2aN_0}{aN_0^2 - 4h} \ , \quad \beta = \frac{aN_0(h-4r)}{r(aN_0^2 - 4h)} \ ,$$

we have

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} = x^{\alpha+1}y^{\beta-1}[g(y) - a(\alpha+3)xy] \quad ,$$

where

$$g(y) = -a(\alpha+2)y^2 + aN_0(\alpha+2)y + \beta\gamma \ .$$

Since

$$\alpha + 2 = \frac{4(r-h)}{aN_0^2 - 4h} \geqslant 0 ,$$

$$\Delta = a(\alpha+2)[aN_0^2(\alpha+2) + 4\beta r] = 0 ,$$

we have $g(y) \leqslant 0$ as $y \geqslant 0$. Thus

$$\frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} < 0 \quad \text{when} \quad (x,y) \in D .$$

By Dulac's theorem there is no closed orbit and no singularity for the system (1). The proof of the theorem is complete.

<u>Theorem 2</u> When $ahx_2^2 + r - h < 0$, there is a stable limit cycle contained in D for the system (1).

<u>Proof</u> We will apply the Poincaré-Bendixon theorem. By hypothesis, we know that M_2 is unstable. We suppose that four separatrices passing through M_1 are ℓ_1^+, ℓ_1^-, ℓ_2^+ and ℓ_2^-. We will construct an annular region around M_2. We will discuss how ℓ_1^+ and ℓ_2^- are located. Let

$$\bar{x} = x - x_1, \ \bar{y} = x - y/h , \tag{2}$$

then the system (1) becomes

$$\begin{cases} \dfrac{d\bar{x}}{dt} = P(\bar{x} + x_1, h(\bar{x}-\bar{y}+x_1)) \equiv \bar{P}(\bar{x},\bar{y}) , \\ \dfrac{d\bar{y}}{dt} = \bar{P}(\bar{x},\bar{y}) - r\bar{y}/h . \end{cases} \tag{3}$$

To obtain the slope of ℓ_2^- in the lower half-plane $\{y \leqslant 0\}$, we change $\bar{y}$ into $-\bar{y}$. Then the system (3) becomes

$$\begin{cases} \dfrac{d\bar{x}}{dt} = P(\bar{x} + x_1, h(\bar{x}+\bar{y}+x_1)) \equiv \bar{P}^*(\bar{x},\bar{y}) , \\ \dfrac{d\bar{y}}{dt} = -\bar{P}^*(\bar{x},\bar{y}) - r\bar{y}/h . \end{cases} \tag{4}$$

If we compare the slopes of the orbits of the system (3) with that of the orbits of the system (4), we have

$$\left.\frac{d\bar{y}}{d\bar{x}}\right|_{(3)} - \left.\frac{d\bar{y}}{d\bar{x}}\right|_{(4)} = 2 + \frac{r}{h}\,\bar{y}\left(\frac{1}{\bar{P}^*(\bar{x},\bar{y})} - \frac{1}{\bar{P}(\bar{x},\bar{y})}\right) . \qquad (5)$$

In order to determine the sign of the difference of slopes we need to know how ℓ_1^+ and ℓ^2 are located on the plane (x,y), the curve $P(x,y) = 0$ consists of the curves: $x = 0$ and $L = ax(N_0-x-y) -1 = 0$. If we compute the derivative of L along the orbits of (1) we have:

$$\left.\frac{dL}{dt}\right|_{L=0} = -arx(x-y/h) \quad ,$$

hence $\left.\frac{dL}{dt}\right|_{L=0} > 0$ when $x-y/h < 0$, the orbit of the system (1) crosses transversely the curve $L = 0$, this shows that the separatrix ℓ_1^+ lies on the region $\{P(x,y) < 0\} \cap \{x-y/h > 0\}$. Similarly, we can prove that ℓ_2^- is contained in the region $\{P(x,y) > 0\} \cap \{x-y/h < 0\}$. Since the coordinate transformation (2) is a homeomorphism of the plane we have that ℓ_1^+ after applying (2) must be located in the region $\bar{P}(\bar{x},\bar{y}) < 0$, and ℓ_2^- will be transformed into a curve located region $\bar{P}^*(\bar{x},\bar{y}) > 0$. Therefore, on the concerned part of ℓ_1^+ and ℓ_2^-, we have by (5)

$$\left.\frac{d\bar{y}}{d\bar{x}}\right|_{(3)} - \left.\frac{d\bar{y}}{d\bar{x}}\right|_{(4)} > 0 \ .$$

This shows that on the plane (x,y), the part of ℓ_2^- after reflexion which has the line $x-y/h = 0$ as the axis of symmetry must have been contained inside of ℓ_1^+. Suppose A is the point of intersection of ℓ_2^- with $x-y/h = 0$, B is the point of intersection of ℓ_1^+ with $x-y/h = 0$ or $N = N_0-x-y = 0$. Then $M_1\ell_2^-AB\ell_1^+M_1$ is the outer boundary curve of an annular region of M_2 (see Fig.1), such that every orbit of (1) will meet transversely and M_2 is an unstable singular point. Therefore there is a stable limit cycle at least around M_2 for the system (1) by the Poincaré-Bendixon annular region theorem. The proof of Theorem 2 is completed.

ACKNOWLEDGEMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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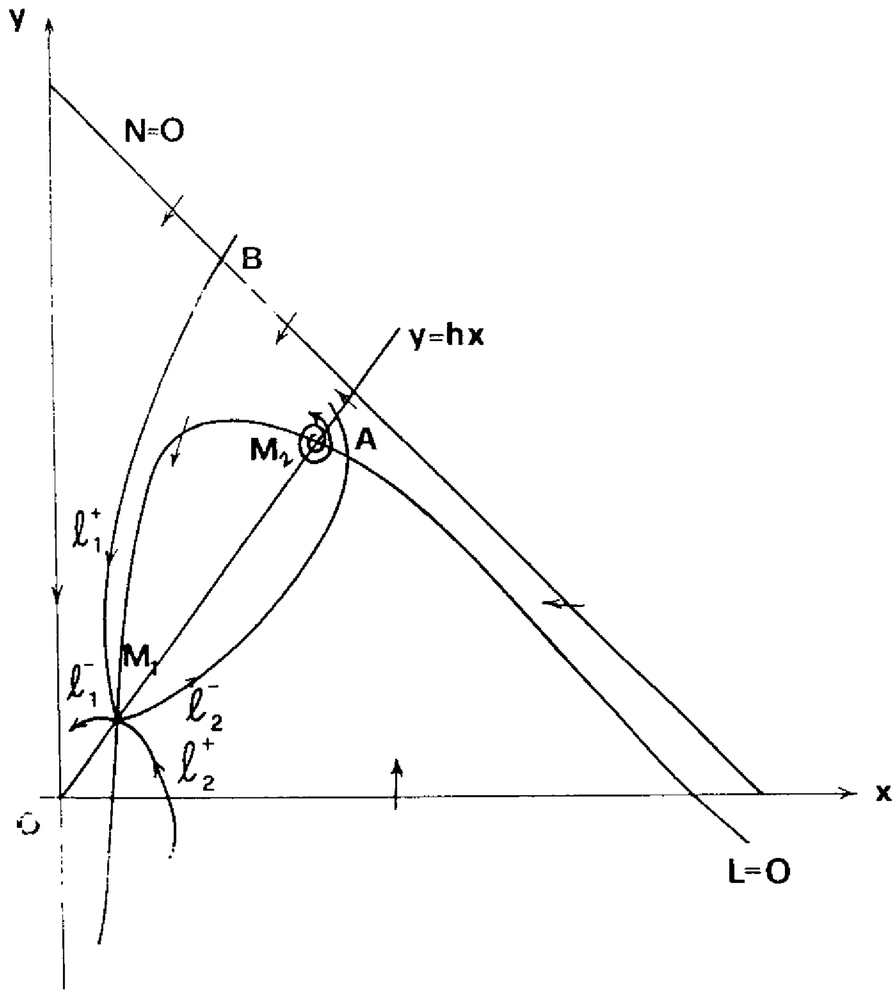

Fig.1

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