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*UNIFORM STABILITY FOR TIME-VARYING INFINITE-DIMENSIONAL DISCRETE LINEAR SYSTEMS**

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RESUMO

A estabilidade de sistemas lineares discretos variantes no tempo em espaço de Banach é investigada. Por um lado, é estabelecida uma coleção razoavelmente completa de condições necessárias e suficientes para a equi-estabilidade assintótica uniforme de sistemas livres. Esta inclui equi-estabilidade de potência forte e uniforme, e equi-estabilidade ℓ_p forte e uniforme, entre outras condições técnicas que também desempenham um papel essencial na teoria de estabilidade. Por outro lado, mostra-se que a equi-estabilidade assintótica uniforme para sistemas livres é equivalente a cada um dos seguintes conceitos de estabilidade uniforme para sistemas forçados: entrada-ℓ_p estado-ℓ_p, entrada-c_o estado-c_o, entrada-limitada estado-limitado, entrada-$\ell_{p>1}$ estado-limitado, entrada-c_o estado-limitado, e entrada-convergente estado-limitado; os quais são também equivalentes às suas respectivas contrapartidas não uniformes. Para sistemas variantes no tempo convergentes, os conceitos acima são também equivalentes a entrada-convergente estado-convergente. As provas apresentadas aqui são todas "elementares", no sentido que elas utilizam essencialmente apenas o teorema de Banach-Steinhaus.

UNIFORM STABILITY FOR TIME-VARYING
INFINITE-DIMENSIONAL DISCRETE LINEAR SYSTEMS

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Abstract. Stability for time-varying discrete linear systems in a Banach space is investigated. On the one hand, it is established a fairly complete collection of necessary and sufficient conditions for uniform asymptotic equistability for input-free systems. This includes uniform and strong power equistability, and uniform and strong ℓ_p-equistability, among other technical conditions which also play an essential role in stability theory. On the other hand, it is shown that uniform asymptotic equistability for input-free systems is equivalent to each of the following concepts of uniform stability for forced systems: ℓ_p-input ℓ_p-state, c_o-input c_o-state, bounded-input bounded-state, $\ell_{p>1}$-input bounded-state, c_o-input bounded-state, and convergent-input bounded-state; which are also equivalent to their nonuniform counterparts. For time-varying convergent systems, the above is also equivalent to convergent-input convergent-state stability. The proofs presented here are all "elementary" in the sense that they are based essentially only on the Banach-Steinhaus theorem.

Key words. Discrete-time systems, linear systems, infinite-dimensional systems, stability theory, time-varying systems.

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1. INTRODUCTION

Consider the class of infinite-dimensional discrete dynamical systems operating in a deterministic environment, whose models are described by a linear difference equation evolving in a Banach space. Such a class can be naturally split into four subclasses, according to whether the models are either homogeneous (i.e. input-free systems) or inhomogeneous (i.e. forced systems) on the one hand, and either autonomous (i.e. time-invariant systems) or nonautonomous (i.e. time-varying systems) on the other hand. These comprise the whole class of systems we shall be dealing with in this paper. Thus we omit, from now on, the qualifications infinite-dimensional, discrete, linear and deterministic, since they will be implicitly understood.

The stability problem (mainly strong and uniform asymptotic stability) for input-free time-invariant systems has been investigated by several authors (e.g. see [3],[5],[6],[8] and [12]). A few results for forced time-invariant systems have also been considered in the literature (e.g. see [3],[7] and [12]). Some interesting results on stability for time-varying systems, which comprise the central theme of this paper, have recently appreared in the literature. For input-free systems, the relationship between weak, strong and uniform power equistability, as well as between strong and uniform ℓ_p-equistability, was analised in [11]; and further results on strong and uniform asymptotic equistability were presented in [9]. For forced systems, it was shown in [10] that $\ell_{p>1}$-input bounded-state stability implies uniform power equistability under uniform equicontrollability assumption. As far as the stability problem for continuous-time systems in concerned see, for instance, the references in [11].

In this paper we shall be dealing with the uniform stability problem for each of the four subclasses described above, with emphasis in forced time-varying systems. It is presented in section 2 a fairly complete collection of necessary and sufficient conditions for uniform asymptotic equistability. Section 3 is concerned with a discussion of the results obtained in section 2. These two sections deal with input--free time-varying systems. Their purpose is fourfold. They contain the auxiliary results that will be needed in the sequel, survey the previous results, introduce further new results, and present "elementary" proofs for some known results which were originally established by other "nonelementary" means. Sections 4 and 5 investigate the stability problem for forced time-varying systems. The particular case of convergent time-varying systems is also considered.

The notation used throughout this paper is summarized as follows. X will denote a Banach space and $B[X]$ the Banach algebra of all bounded linear operators of X into itself. We shall use the same symbol $\|\ \|$ to denote both the norm in X and the induced uniform norm in $B[X]$. $\ell_p(X)$ for any real number $p \geq 1$, $c_0(X)$, $c(X)$, and $\ell_\infty(X)$ (with their standard norms $\|\ \|_p$, $\|\ \|_\infty$, $\|\ \|_\infty$, and $\|\ \|_\infty$, respectively) will stand, as usual, for the Banach spaces of all X-valued sequences $\underline{x}=\{x(i)\in X;\ i\geq 0\}$ such that $\|\underline{x}\|_p=(\sum_{i=0}^{\infty}\|x(i)\|^p)^{1/p}<\infty$, $\lim_{i\to\infty}\|x(i)\|=0$, $\lim_{i\to\infty}\|x(i)-x\|=0$ for some $x\in X$, and $\|\underline{x}\|_\infty=\sup_{i\geq 0}\|x(i)\|<\infty$, respectively, so that $\ell_p(X)\subset c_0(X)\subset c(X)\subset \ell_\infty(X)$. Given a sequence of operators $\{A(k)\in B[X];\ k\geq 0\}$, set $\Phi(k,k)=I$ (the identity in $B[X]$) for every $k\geq 0$, and

$$\Phi(k+\ell,k)=\Pi_{j=k}^{k+\ell-1}A(j)=A(k+\ell-1)\ \ldots\ A(k)$$

for every $\ell\geq 1$ and $k\geq 0$, so that

$$\Phi(k+\ell+m,k)=\Phi(k+\ell+m,k+\ell)\ \Phi(k+\ell,k)$$

for every $k,\ell,m\geq 0$. $\{\Phi(k+\ell,k)\in B[X];\ k,\ell\geq 0\}$ will be referred to as the *evolution operator process* associated with the sequence of operators $\{A(k)\in B[X];\ k\geq 0\}$.

2. INPUT-FREE SYSTEMS

Given any integer $k\geq 0$ and an arbitrary $x\in X$, consider the sequence $\{x(\ell)\in X;\ \ell\geq 0\}$ recursively defined by the following nonautonomous homogeneous difference equation.

(1-a) $\quad x(\ell+1)=A(k+\ell)\ x(\ell)\ ;\qquad x(0)=x\in X,$

whose solution is

(1-b) $\quad x(\ell)=\Phi(k+\ell,k)x\qquad \forall\ell\geq 0\quad ,$

where $\{\Phi(k+\ell,k)\in B[X];\ k,\ell\geq 0\}$ is the evolution operator process associated with the sequence of operators $\{A(k)\in B[X];\ k\geq 0\}$. The purpose of this section is to present, in an unified way, several necessary and sufficient conditions for uniform asymptotic equistability.

Definition 1. The model (1) (or equivalently, the operator sequence $\{A(k)\in B[X];\ k\geq 0\}$) is *uniformily asymptotically equistable* if, for each $\varepsilon>0$, there exists an integer $\ell_\varepsilon\geq 0$ such that

$$\ell\geq\ell_\varepsilon\implies\|\Phi(k+\ell,k)x\|\leq\varepsilon\|x\|\qquad\forall k\geq 0\qquad\forall x\in X\ ;$$

or equivalently, if $\sup_{k\geq 0}\|\Phi(k+\ell,k)\|\to 0$ as $\ell\to\infty$.

Theorem 1. Consider the model (1). The following assertions are equivalent.

(A) $\lim_{\ell\to\infty}\sup_{k\geq 0}\|\Phi(k+\ell,k)\|=0$.

(B) $\sup_{k\geq 0}\|\Phi(k+\ell,k)\|<1\quad\forall\ell\geq\ell_1$ for some $\ell_1\geq 1$.

(C) $\sup_{k\geq 0}\|A(k)\|<\infty$ and $\sup_{k\geq 0}\|\Phi(k+\ell_0,k)\|<1$ for some $\ell_0\geq 1$.

(D) $\limsup_{\ell\to\infty}\sup_{k\geq 0}\|\Phi(k+\ell,k)\|^{1/\ell}<1.$

(E) There exist real constants $\gamma\geq 1$ and $\alpha\in(0,1)$ such that

$$\|\Phi(k+\ell,k)\|\leq\gamma\alpha^{\ell}\qquad\forall k,\ell\geq 0.$$

(F) For every $p>0$ there exists a positive number σ_p such that

$$\sum_{\ell=0}^{\infty}\|\Phi(k+\ell,k)\|^{p}\leq\sigma_p\qquad\forall k\geq 0.$$

(G) For every $p>0$ there exists a positive number σ_p such that

$$\sum_{\ell=0}^{\infty}\|\Phi(k+\ell,k)x\|^{p}\leq\sigma_p\|x\|^{p}\qquad\forall k\geq 0,\qquad\forall x\in X.$$

(H) For some $q>0$ there exists a positive number σ_q such that

$$\sum_{\ell=0}^{\infty}\|\Phi(k+\ell,k)\|^{q}\leq\sigma_q\qquad\forall k\geq 0.$$

(I) For some $q>0$ there exists a positive number σ_q such that

$$\sum_{\ell=0}^{\infty}\|\Phi(k+\ell,k)x\|^{q}\leq\sigma_q\|x\|^{q}\qquad\forall k\geq 0,\qquad\forall x\in X.$$

(J) For every $p>0$ there exists a positive number μ_p such that

$$\|\Phi(k+\ell,k)\|^{p}\leq(\ell+1)^{-1}\mu_p\qquad\forall k,\ell\geq 0.$$

(K) For some $q>0$ there exists a positive number μ_q such that

$$\|\Phi(k+\ell,k)\|^{q}\leq(\ell+1)^{-1}\mu_q\qquad\forall k,\ell\geq 0.$$

(L) For every $p>0$ there exists a positive number ρ_p such that

$$\sum_{m=0}^{\ell}\|\Phi(k+\ell,k+m)\|^{p}\leq\rho_p\qquad\forall k,\ell\geq 0.$$

(M) For some $q>0$ there exists a positive number ρ_q such that

$$\sum_{m=0}^{\ell}\|\Phi(k+\ell,k+m)\|^{q}\leq\rho_q\qquad\forall k,\ell\geq 0.$$

(N) $\lim_{n\to\infty}\sup_{k\geq 0}\sup_{\nu\geq 0}\|\sum_{\ell=n}^{n+\nu}\Phi(k+\ell,k)\|=0$.

<u>Proof</u>. To begin with let us remark that it is implicitly assumed in (A),(B) and (D) that the underlying nonnegative sequence $\{\sup_{k\geq 0}\|\Phi(k+\ell,k)\|:\ \ell\geq 0\}$ is well-defined (i.e. $\sup_{k\geq 0}\|\Phi(k+\ell,k)\|<\infty$ for every $\ell\geq 0$, so that (for $\ell=1$) $\sup_{k\geq 0}\|A(k)\|<\infty$). Now consider the following auxiliary assertions:

(O) There exist real functionals $\gamma(\cdot),\alpha(\cdot):X\to\mathbb{R}$ such that $\gamma(x)\geq\|x\|$, $\alpha(x)\in(0,1)$,

$$\|\Phi(k+\ell,k)x\|\leq\gamma(x)\alpha(x)^{\ell}\qquad\forall k,\ell\geq 0;\qquad\forall x\in X.$$

(P) For every $p>0$ there exists a real functional $\sigma_p(\cdot):X\to\mathbb{R}$ such that

$$\sum_{\ell=0}^{\infty}\|\Phi(k+\ell,k)x\|^{p}\leq\sigma_p(x)\qquad\forall k\geq 0,\qquad\forall x\in X.$$

(Q) For some $q>0$ there exists a real functional $\sigma_q(\cdot):X\to\mathbb{R}$ such that
$\sum_{\ell=0}^{\infty}\|\Phi(k+\ell,k)x\|^q \le \sigma_q(x) \qquad \forall k\ge 0, \qquad \forall x\in X.$

(R) There exist real functionals $q(\cdot),\sigma(\cdot):X\to\mathbb{R}$ such that $q(x)>0$ and
$\sum_{\ell=0}^{\infty}\|\Phi(k+\ell,k)x\|^{q(x)} \le \sigma(x) \qquad \forall k\ge 0, \qquad \forall x\in X.$

(S) For every $p>0$ there exists a real functional $\mu_p(\cdot):X\to\mathbb{R}$ such that
$\|\Phi(k+\ell,k)x\|^p \le (\ell+1)^{-1}\mu_p(x) \qquad \forall k,\ell\ge 0, \qquad \forall x\in X.$

(T) For some $q>0$ there exists a real functional $\mu_q(\cdot):X\to\mathbb{R}$ such that
$\|\Phi(k+\ell,k)x\|^q \le (\ell+1)^{-1}\mu_q(x) \qquad \forall k,\ell\ge 0, \qquad \forall x\in X.$

(U) There exist real functionals $q(\cdot),\mu(\cdot):X\to\mathbb{R}$ such that $q(x)>0$ and
$\|\Phi(k+\ell,k)x\|^{q(x)} \le (\ell+1)^{-1}\mu(x) \qquad \forall k,\ell\ge 0, \qquad \forall x\in X.$

We shall verify that the implications in Diagram 1 hold true.

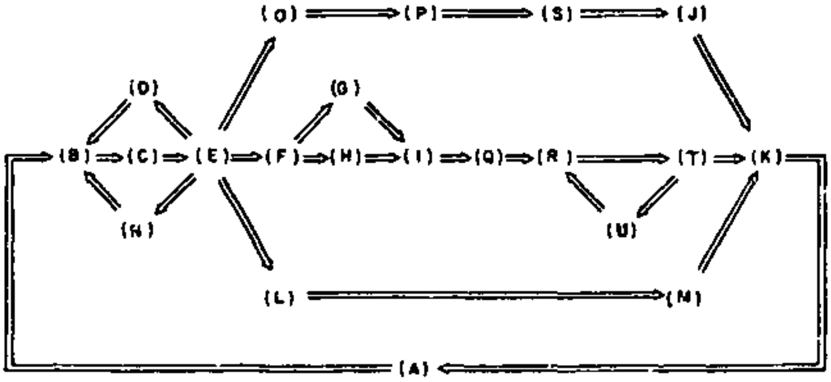

Diagram 1

First we show that (C) $\Longrightarrow$ (E). Suppose (C) holds true, and set

$$\alpha = \sup_{k\ge 0}\|\Phi(k+\ell_0,k)\|^{1/\ell_0} < 1 .$$

It is readily verified by induction (on n) that, for every $n\ge 0$,

$$\sup_{k\ge 0}\|\Phi(k+n\ell_0,k)\| \le \alpha^{n\ell_0} .$$

For each $\ell\ge 0$ let n_ℓ be the least positive integer greater than or equal to ℓ/ℓ_0 (i.e. $0\le n_\ell-1\le \ell/\ell_0\le n_\ell$), so that $0\le \ell-(n_\ell-1)\ell_0\le \ell_0$ and $\alpha^{n_\ell\ell_0}\le\alpha^{\ell}$ for every $\ell\ge 0$. Thus, for any $k,\ell\ge 0$,

$$\|\Phi(k+\ell,k)\| \le \|\Phi(k+\ell,k+(n_\ell-1)\ell_0)\|\,\|\Phi(k+(n_\ell-1)\ell_0,k)\|$$
$$\le \sup_{k\ge 0}\|\Lambda(k)\|^{\ell-(n_\ell-1)\ell_0}\,\alpha^{(n_\ell-1)\ell_0} \le \alpha^{-\ell_0}\max\{1,\sup_{k\ge 0}\|\Lambda(k)\|^{\ell_0}\}\,\alpha^{\ell} .$$

Hence (C) $\Longrightarrow$ (E). Next we verify that (P) $\Longrightarrow$ (S). Let p be an arbitrary positive real number. For any $k,\ell\geq 0$ we have

$$(\ell+1)\|\Phi(k+\ell,k)x\|^p = \sum_{m=0}^{\ell}\|\Phi(k+\ell,k)x\|^p$$

$$\leq \sum_{m=0}^{\ell}\|\Phi(k+\ell,k+m)\|^p\|\Phi(k+m,k)x\|^p$$

for all $x\in X$. Note that (P) obviously implies that, for any $p>0$, $\|\Phi(k+\ell,k)x\|\leq$ $\leq \sigma_p(x)^{1/p}<\infty$ for all $x\in X$ and every $k,\ell\geq 0$. Thus, by the Banach-Steinhaus theorem (e.g. see [2, p.66]), there exists a positive constant η such that $\|\Phi(k+\ell,k)\|\leq\eta<\infty$ for every $k,\ell\geq 0$. Hence, for any $k,\ell\geq 0$,

$$(\ell+1)\|\Phi(k+\ell,k)x\|^p \leq \eta^p\sigma_p(x) < \infty$$

for all $x\in X$, whenever (P) holds true. Hence (P) $\Longrightarrow$ (S). Proceeding in a similar fashion we get

$$(\ell+1)\|\Phi(k+\ell,k)\|^q \leq \sum_{m=0}^{\ell}\|\Phi(k+\ell,k+m)\|^q\|\Phi(k+m,k)\|^q \leq \rho_q^2 < \infty$$

for every $k,\ell\geq 0$, whenever (M) holds true. Hence (M) $\Longrightarrow$ (K). By combining the two approaches used above we finally supply a proof for (R) $\Longrightarrow$ (T). Suppose (R) holds true and, for each $x\in X$, let $n(x)$ be the least positive integer greater than or equal to $q(x)$ (i.e. $0\leq n(x)-1\leq q(x)\leq n(x)<\infty$), so that $0\leq\ell\leq(\ell+1)^{n(x)}-1$ for all $x\in X$ and every $\ell\geq 0$. Thus, for any $k,\ell\geq 0$,

$$(\ell+1)^{q(x)}\|\Phi(k+\ell,k)x\|^{q(x)} \leq (\ell+1)^{n(x)}\|\Phi(k+\ell,k)x\|^{q(x)} = \sum_{m=0}^{(\ell+1)^{n(x)}-1}\|\Phi(k+\ell,k)x\|^{q(x)}$$

$$\leq \sum_{m=0}^{(\ell+1)^{n(x)}-1}\|\Phi(k+\ell,k+m)\|^{q(x)}\|\Phi(k+m,k)x\|^{q(x)}$$

for all $x\in X$. Since (R) obviously implies that $\|\Phi(k+\ell,k)x\|\leq\sigma(x)^{1/q(x)}<\infty$ for all $x\in X$ and every $k,\ell\geq 0$, we get $\|\Phi(k+\ell,k)\|\leq\eta<\infty$ (for some positive constant η) for every $k,\ell\geq 0$, by using the Banach-Steinhaus theorem again. Therefore, for any $k,\ell\geq 0$,

$$(\ell+1)\|\Phi(k+\ell,k)x\| \leq \eta\left(\sum_{m=0}^{\infty}\|\Phi(k+m,k)x\|^{q(x)}\right)^{1/q(x)} \leq \eta\sigma(x)^{1/q(x)} < \infty$$

for all $x\in X$. Hence (R) $\Longrightarrow$ (T) with q=1. A straightforward application of the Banach-Steinhaus theorem yields (S) $\Longrightarrow$ (J) and (T) $\Longrightarrow$ (K). The remaining implications in Diagram 1 are trivial. Note that, since $B[X]$ is a Banach space, assertion (N) means (by definition) that the family of series $\{\sum_{\ell=0}^{\infty}\Phi(k+\ell,k);\ k\geq 0\}$ is uniformly equiconvergent.

□

3. REMARKS

Remark 1. (Further equivalent assertions). The set of equivalent assertions in Theorem 1 is certainly not exhaustive. For instance, "sup" (whenever it appears implicitly or explicitly in Theorem 1) can be replaced by "limsup". To illustrate this we shall show that each of the assertions below, which will be required later in this paper, is also equivalent to those equivalent assertions in Theorem 1.

(A') $\lim_{\ell\to\infty}\limsup_{k\to\infty}\|\Phi(k+\ell,k)\| = 0.$

(D') $\limsup_{\ell\to\infty}\limsup_{k\to\infty}\|\Phi(k+\ell,k)\|^{1/\ell} < 1.$

This is readily verified as follows. Consider the following auxiliary assertions: (B') and (C') obtained by changing "$\sup_{k\geq 0}$" to "$\limsup_{k\to\infty}$" in (B) and (C), respectively; and (E') obtained by changing the requirement "$\forall k\geq 0$" to "$\forall k\geq k_0$ for some $k_0\geq 0$" in (E). Note that (A) $\Longrightarrow$ (A') $\Longrightarrow$ (B') $\Longrightarrow$ (C') and (E') $\Longrightarrow$ (D') $\Longrightarrow$ (B') trivially. Moreover, it is a simple matter to show that (E') $\Longrightarrow$ (E). Finally, by applying exactly the same technique used to establish that (C) $\Longrightarrow$ (E) in the proof of Theorem 1, it can be shown that (C') $\Longrightarrow$ (E'). Therefore, each of (A') to (E') is also equivalent to those equivalent assertions in Theorem 1.

Remark 2. (Time-invariant systems). For the particular case of a constant sequence $\{\Lambda(k)=\Lambda\in\mathcal{B}[X];\ k\geq 0\}$, the associated equivalent assertions in Theorem 1 are trivially obtained by changing $\Phi(k+\ell,k)$ to Λ^ℓ in the theorem statement. Many of these equivalent assertions for time-invariant systems are well-known (e.g. see [6]). Among then, the following will be needed in the sequel (here $r_\sigma(\Lambda)$ denotes the spectral radius of $\Lambda\in\mathcal{B}[X]$): For an arbitrary $\Lambda\in\mathcal{B}[X]$ the assertions below are equivalent,

($\bar{A}$) $\lim_{\ell\to\infty}\|\Lambda^\ell\| = 0.$

($\bar{D}$) $r_\sigma(\Lambda) \overset{\text{def.}}{=} \lim_{\ell\to\infty}\|\Lambda^\ell\|^{1/\ell} < 1.$

($\bar{N}$) $\{\sum_{\ell=0}^{n}\Lambda^\ell\in\mathcal{B}[X];\ n\geq 0\}$ converges uniformly.

($\bar{N}_0$) $\sum_{\ell=0}^{n}\Lambda^\ell \to (I-\Lambda)^{-1}\in\mathcal{B}[X]$ uniformly as $n\to\infty$.

Whereas ($\bar{N}$) is the time-invariant version of (N), ($\bar{N}_0$) has clearly no time-varying interpretation in general. However, it is well known (e.g. see [2, p.567]) that ($\bar{D}$) $\Longrightarrow$ ($\bar{N}_0$), and ($\bar{N}_0$) $\Longrightarrow$ ($\bar{N}$) trivially. Note that the natural time-invariant counterpart of Definition 1 is: The autonomous version of model (1) (or equivalently, an operator $\Lambda\in\mathcal{B}[X]$) is *uniformly asymptotically stable* if $\|\Lambda^\ell\| \to 0$ as $\ell\to\infty$.

Remark 3. (A brief review). The equivalence between (A) and (E) is well known (e.g. see [9]). Assertion (E) [(O)] is usually referred to as *uniform [strong] power equistability* (cf. [9]-[11]). The expression in the left hand side of (D) was called the generalized spectral radius of the sequence $\{\Lambda(k)\in B[X];\ k\geq 0\}$ in [11], where the equivalence between (D) and (E) was analysed. Note that, according to Remarks 1 and 2, the inequality in (D') also generalizes the spectral radius condition in ($\bar{D}$). Assertion (I) [(Q) and (R)] was referred to as *ℓ_q-uniform* [*ℓ_q-(strong)* and $\ell_q(x)$, respectively] *equistability* in [11], where it has also been established the equivalence between each of (E) to (G), (I) and (O) to (R), for the case of $p,q\geq 1$, by using the Baire category theorem. If we agree that "in this context a proof is 'elementary' if it does not use the Baire category theorem " (cf. [4, p.13]) and recalling that (even in a Banach space setting) the Banach-Steinhaus theorem has an "elementary" proof (cf.[1, p.98]), we conclude that the proofs presented here are substantially simpler than those in [11], since we have used (beyond really elementary analysis) just the Banach-Steinhaus theorem for establishing the equivalence between each of (A) to (U). Now consider the following assertion which, according to Definition 1, is naturally referred to as *strong asymptotic equistability*.

$$(A_s)\quad \lim_{\ell\to\infty}\sup_{k\geq 0}\|\Phi(k+\ell,k)x\| = 0 \qquad \forall x\in X.$$

Note that (A) $\Longrightarrow$ (A_s) trivially, but (A_s) $\not\Longrightarrow$ (A) in general. Actually, (A_s) $\not\Longrightarrow$ (A) even for a constant sequence $\{\Lambda(k)=\Lambda\in B[X];\ k\geq 0\}$, although ($A_s$) $\Longrightarrow$ (A) for the particular case of an operator sequence constantly equal to a compact operator (e.g. see [6]). However, (A_s) $\Longrightarrow$ (A) whenever $\sup_{k\geq 0}\|\Phi(k+\ell_0,k)\|<1$ for some $\ell_0\geq 1$ (recall that (C) $\Longrightarrow$ (A) in Theorem 1). By using the above result, it has been proved in [9] that (A_s) $\Longrightarrow$ (A) whenever $\{\Lambda(k)\in B[X];\ k\geq 0\}$ is collectively compact (i.e. whenever the set $\cup_{k\geq 0}\{\Lambda(k)x\in X;\ \|x\|\leq 1\}$ is relatively compact in X). Note that a constant sequence $\{\Lambda(k)=\Lambda\in B[X];\ k\geq 0\}$ is collectively compact if and only if $\Lambda\in B[X]$ is compact.

Remark 4. (Time-varying convergent systems). If an operator sequence converges uniformly, then it shares the same stability properties with its limit. Precisely: Let $\{\Phi(k+\ell,k)\in B[X];\ k,\ell\geq 0\}$ be the evolution operator process associated with a sequence of operators $\{\Lambda(k)\in B[X];\ k\geq 0\}$, which is supposed to converge uniformly to $\Lambda\in B[X]$. Since $\lim_{k\to\infty}\|\Lambda(k)-\Lambda\|=0$, it is readily verified by induction in ℓ that $\lim_{k\to\infty}\|\Phi(k+\ell,k)-\Lambda^{\ell}\|=0$ for every $\ell\geq 0$. Hence

$$\lim_{k\to\infty}\|\Phi(k+\ell,k)\| = \|\Lambda^{\ell}\| \qquad \forall\ell\geq 0.$$

By combining the above result with Remark 1 we get

$$(A) \iff (A') \iff (\bar{A}) \iff \lim_{\ell\to\infty}\lim_{k\to\infty}\|\Phi(k+\ell,k)\|=0,$$

with assertions (A), (A') and ($\bar{A}$) as in Theorem 1 and Remarks 1 and 2.

4. FORCED SYSTEMS

Given a sequence $\underline{u}=\{u(i)\in X;\ i\geq 0\}$, consider another sequence $\underline{x}=\{x(i)\in X;\ i\geq 0\}$ recursively defined by the following nonautonomous inhomogeneous difference equation.

(2-a) $\quad x(i+1) = \Lambda(i)x(i) + u(i+1), \qquad x(0) = u(0),$

whose solution is

(2-b) $\quad x(i) = \sum_{j=0}^{i}\Phi(i,j)u(j) \qquad \forall i\geq 0,$

where $\{\Phi(i,j)\in B[X];\ 0\leq j\leq i\}$ is the evolution operator process associated with the sequence of operators $\{\Lambda(i)\in B[X];\ i\geq 0\}$. The purpose of this section is to investigate some aspects of (uniform) input-state stability.

Theorem 2. Consider the model (2). The following assertions are equivalent.

(a) $\{\Lambda(i)\in B[X];\ i\geq 0\}$ is uniformly asymptotically equistable.

(b) For every $p\geq 1$ there exists a positive number λ_p such that
$\|\underline{x}\|_p \leq \lambda_p\|\underline{u}\|_p \qquad \forall \underline{u}\in\ell_p(X).$

(c) For some $q\geq 1$ there exists a positive number λ_q such that
$\|\underline{x}\|_q \leq \lambda_q\|\underline{u}\|_q \qquad \forall \underline{u}\in\ell_q(X).$

(d) There exists a positive number λ_∞ such that
$\|\underline{x}\|_\infty \leq \lambda_\infty\|\underline{u}\|_\infty \qquad \forall \underline{u}\in\ell_\infty(X).$

(e) $\underline{u}\in\ell_p(X) \Longrightarrow \underline{x}\in\ell_p(X)$ for every $p\geq 1$.

(f) $\underline{u}\in\ell_q(X) \Longrightarrow \underline{x}\in\ell_q(X)$ for some $q\geq 1$.

(g) $\underline{u}\in\ell_\infty(X) \Longrightarrow \underline{x}\in\ell_\infty(X)$.

(h) $\underline{u}\in c_0(X) \Longrightarrow \underline{x}\in c_0(X)$.

Moreover, the assertion below implies the above ones.

(i) $\underline{u}\in c(X) \Longrightarrow \underline{x}\in c(X)$.

Proof. If (a) holds true then, from (E) (cf. Theorem 1) and (2-b) we get

(3) $\quad \|x(i)\| \leq \gamma_i \overset{\text{def.}}{=} \sum_{j=0}^{i}\alpha^{i-j}\beta_j$

for some pair of constants $\gamma\geq 1$ and $\alpha\in(0,1)$, with $\beta_i=\gamma\|u(i)\|$, for every $i\geq 0$. Recall that the convolution (or the Cauchy product) $c=\{\gamma_i;\ i\geq 0\}=a*b$ of a scalar sequence $a=\{\alpha^i;\ i\geq 0\}$ in ℓ_1 and a scalar sequence $b=\{\beta_i;\ i\geq 0\}$ in ℓ_p lies itself in ℓ_p, with $\|c\|_p\leq\|a\|_1\|b\|_p$, for any $p\geq 1$ (cf. [2, p.529]); which clearly also holds true if we set $p=\infty$. Hence, since $\|a\|_1=\sum_{i=0}^{\infty}\alpha^i=(1-\alpha)^{-1}$, it follows from (3) that (b) and (d) hold true with $\lambda_p=\lambda_\infty=\gamma(1-\alpha)^{-1}$ for any $p\geq 1$. Thus (a) $\Longrightarrow$ (b,d). Note that (b) $\Longrightarrow$ (e) $\Longrightarrow$ (f) and (d) $\Longrightarrow$ (g) trivially. Now we show that (g) $\Longrightarrow$ (d). For each $i\geq 0$, consider the transformation $\theta_i:\ell_\infty(X)\to X$ given, according to (2-b) by

$$x(i)=\theta_i\underline{u}=\sum_{j=0}^{i}\Phi(i,j)u(j)$$

for all $\underline{u}=\{u(j)\in X;\ j\geq 0\}\in\ell_\infty(X)$, which is clearly linear and bounded (i.e. $\theta_i\in B[\ell_\infty(X),X]$ - actually $\|\theta_i\|_{B[\ell_\infty(X),X]}\leq\sum_{j=0}^{i}\|\Phi(i,j)\|$). If (g) holds true, then

$$\sup_{i\geq 0}\|\theta_i\underline{u}\|<\infty\qquad\forall\underline{u}\in\ell_\infty(X),$$

so that $\sup_{i\geq 0}\|\theta_i\|_{B[\ell_\infty(X),X]}=\lambda_\infty$, for some positive constant λ_∞, by the Banach--Steinhaus theorem. Hence (d) holds true, since $\sup_{i\geq 0}\|\theta_i\underline{u}\|\leq\lambda_\infty\|\underline{u}\|_\infty$ for all $\underline{u}\in\ell_\infty(X)$. Thus (g) $\Longrightarrow$ (d). In a similar fashion we can show that (f) $\Longrightarrow$ (c). For each $n\geq 0$ and an arbitrary $\underline{u}=\{u(j)\in X;\ j\geq 0\}\in\ell_q(X)$, set $\underline{x}_n=\{x_n(i)\in X;\ i\geq 0\}$ with

$$x_n(i)=\begin{cases}\sum_{j=0}^{i}\Phi(i,j)u(j) & \text{if}\quad i\leq n\\ 0 & \text{if}\quad i>n,\end{cases}$$

so that $\underline{x}_n\in\ell_1(X)\subseteq\ell_q(X)$; and consider the transformation $\Psi_n:\ell_q(X)\to\ell_q(X)$ given by

$$\Psi_n\underline{u}=\underline{x}_n$$

for all $\underline{u}\in\ell_q(X)$, which is clearly linear and bounded (i.e. $\Psi_n\in B[\ell_q(X)]$ - actually $\|\Psi_n\|_{B[\ell_1(X)]}\leq\sum_{i=0}^{n}\max_{0\leq j\leq i}\|\Phi(i,j)\|$ and $\|\Psi_n\|^{q/(q-1)}_{B[\ell_q(X)]}\leq\sum_{i=0}^{n}\sum_{j=0}^{i}\|\Phi(i,j)\|^{q/(q-1)}$ for $q>1$). If (f) holds true, then

$$(4)\quad\|\underline{x}_n\|_q=\left(\sum_{i=0}^{n}\|x(i)\|^q\right)^{1/q}\leq\|\underline{x}\|_q<\infty$$

for every $n\geq 0$, whenever $\underline{u}\in\ell_q(X)$, according to (2-b). Therefore,

$$\sup_{n\geq 0}\|\Psi_n\underline{u}\|_q<\infty\qquad\forall\underline{u}\in\ell_q(X),$$

so that $\sup_{n\geq 0}\|\Psi_n\|_{B[\ell_q(X)]}=\lambda_q$, for some positive constant λ_q, by the Banach--Stainhaus theorem. Hence

$$\|\underline{x}\|_q = \sup_{n\geq 0}\|\Psi_n\underline{u}\|_q \leq \lambda_q\|\underline{u}\|_q \qquad \forall\underline{u}\in\ell_q(X),$$

according to (4). Thus (f) $\Longrightarrow$ (c). On the other hand, each of (c) and (d) implies (a). To verify this take an arbitrary $u\in X$ and set, for each $k\geq 0$, $\underline{u}_k=\{u_k(i)\in X;\ i\geq 0\}$ $\in\ell_q(X)\subset\ell_\infty(X)$ for any $q\geq 1$ as follows:

$$u_k(i) = \begin{cases} u & \text{if} \quad i=k \\ 0 & \text{if} \quad i\neq k, \end{cases}$$

so that $\|\underline{u}_k\|_q=\|\underline{u}_k\|_\infty=\|u\|$ for each $k\geq 0$ and any $q\geq 1$. Thus, for each $k\geq 0$, we get $\underline{x}_k=\{x_k(i)\in X;\ i\geq 0\}$ from (2-b), given by

$$x_k(i) = \begin{cases} \Phi(i,k)u & \text{if} \quad i\geq k \\ 0 & \text{if} \quad 0\leq i<k. \end{cases}$$

If (c) holds true, then

$$\textstyle\sum_{\ell=0}^{\infty}\|\Phi(k+\ell,k)u\|^q = \sum_{i=k}^{\infty}\|\Phi(i,k)u\|^q = \|\underline{x}_k\|_q^q \leq \lambda_q^q\|\underline{u}_k\|_q^q = \lambda_q^q\|u\|^q$$

for every $k\geq 0$ and all $u\in X$, so that (I) (cf. Theorem 1) holds true. Thus (c) $\Longrightarrow$ (a). Now, for each $k\geq 0$, let $\underline{y}_k=\{y_k(i)\in X;\ i\geq 0\}$ be recursively defined by model (2) with $\underline{u}=\{u(i)\in X;\ i\geq 0\}$ replaced by $\{\underline{x}_k=x_k(i)\in X;\ i\geq 0\}$, so that

$$y_k(i) = \begin{cases} (i-k+1)\Phi(i,k)u & \text{if} \quad i\geq k \\ 0 & \text{if} \quad 0\leq i<k, \end{cases}$$

according to (2-b). If (d) holds true, then

$$\begin{aligned}\sup_{\ell\geq 0}(\ell+1)\|\Phi(k+\ell,k)u\| &= \sup_{i\geq k}(i-k+1)\|\Phi(i,k)u\| = \|\underline{y}_k\|_\infty \\ &\leq \lambda_\infty\|\underline{x}_k\|_\infty \leq \lambda_\infty^2\|\underline{u}_k\|_\infty = \lambda_\infty^2\|u\|\end{aligned}$$

for every $k\geq 0$ and all $u\in X$, so that (K) (cf. Theorem 1) holds true. Thus (d) $\Longrightarrow$ (a). Next we show that (a) $\Longleftrightarrow$ (h). First recall that

(5) $\quad \lim_{i\to\infty}\sum_{j=0}^{i}\alpha^{i-j}\xi_j = (1-\alpha)^{-1}\lim_{i\to\infty}\xi_i$

for every convergent scalar sequence $\{\xi_i;\ i\geq 0\}$ if and only if $|\alpha|<1$ (e.g. see [7]). From (3) and (5) it follows that (a) $\Longrightarrow$ (h). On the other hand, consider the following auxiliary assertions.

(h') $\underline{u}\in c_0(X) \Longrightarrow \underline{x}\in \ell_\infty(X)$,

(h") $\|\underline{x}\|_\infty \leq \lambda_0 \|\underline{u}\|_\infty \qquad \forall \underline{u}\in c_0(X)$

for some positive constant λ_0. We claim that

$$(h) \Longrightarrow (h') \Longleftrightarrow (h'') .$$

Actually (h) $\Longrightarrow$ (h') $\Longleftarrow$ (h") trivially, and (h') $\Longrightarrow$ (h") as follows. For each $i\geq 0$, let $\theta_i^0 \in B[c_0(X),X]$ be the restriction of $\theta_i \in B[\ell_\infty(X),X]$ on the Banach space $c_0(X)\subset \ell_\infty(X)$. Proceeding as in the proof of (g) $\Longrightarrow$ (d), we get $\sup_{i\geq 0}\|\theta_i^0\|_{B[c_0(X),X]}=\lambda_0$ for some positive constant λ_0, whenever (h') holds true, so that $\sup_{i\geq 0}\|\theta_i^0\underline{u}\|\leq\lambda_0\|\underline{u}\|_\infty$ for all $\underline{u}\in c_0(X)$. Thus (h') $\Longrightarrow$ (h"). Now, for an arbitrary $u\in X$ and each $k\geq 0$, consider the sequences $\underline{u}_k$, $\underline{x}_k$ and $\underline{y}_k$ defined above. Note that

$$\|\underline{x}_k\|_\infty \leq \lambda_0 \|\underline{u}_k\|_\infty = \lambda_0\|u\|$$

for every $k\geq 0$ whenever (h) holds true, since (h) $\Longrightarrow$ (h") and $\underline{u}_k\in c_0(X)$ for every $k\geq 0$; and

$$\|\underline{y}_k\|_\infty \leq \lambda_0 \|\underline{x}_k\|_\infty$$

for every $k\geq 0$ whenever (h) holds true, since $\underline{x}_k\in c_0(X)$ for every $k\geq 0$ whenever (h) holds true (because $\underline{u}_k\in c_0(X)$ for every $k\geq 0$) and (h) $\Longrightarrow$ (h"). Therefore,

$$\sup_{\ell\geq 0}(\ell+1)\|\Phi(k+\ell,k)u\| = \|\underline{y}_k\|_\infty \leq \lambda_0^2\|u\|$$

for every $k\geq 0$ and all $u\in X$, so that (K) (cf. Theorem 1) holds true. Thus (h) $\Longrightarrow$ (a). Finally consider the following further auxiliary assertions.

(i') $\underline{u}\in c(X) \Longrightarrow \underline{x}\in \ell_\infty(X)$,

(i") $\|\underline{x}\|_\infty \leq \lambda\|\underline{u}\|_\infty \qquad \forall \underline{u}\in c(X)$

for some positive constant λ; and let $\theta_i' \in B[c(X),X]$ be the restriction of $\theta_i\in B[\ell_\infty(X),X]$ on the Banach space $c(X)\subset \ell_\infty(X)$, for each $i\geq 0$. Proceeding exactly as in the proof of (h) $\Longrightarrow$ (a) we get: (i') $\Longleftrightarrow$ (i"), so that (i) $\Longrightarrow$ (i"), which implies that (i) $\Longrightarrow$ (a). Note that (a) $\not\Longrightarrow$ (i) (e.g. take $\Lambda(i)=0$ for $i=0,2,4,\dots$ and $\Lambda(i)=(1/2)I$ for $i=1,3,5,\dots$, and $\{u(i)=u\neq 0\in X;\ i\geq 0\}$).

□

Corollary 1. Suppose $\{\Lambda(i)\in\mathcal{B}[X];\ i\geq 0\}$ converges uniformly to $\Lambda\in\mathcal{B}[X]$. Then all the assertions (a) to (i) in Theorem 2 are equivalent. Moreover, they are also equivalent to the following further one

(j) $\exists\ (I-\Lambda)^{-1}\in\mathcal{B}[X]$ and $\lim_{i\to\infty}x(i) = (I-\Lambda)^{-1}\lim_{i\to\infty}u(i)$ $\forall \underline{u}\in c(X)$.

Proof. According to Theorem 2, we just need to show that (a) $\Longrightarrow$ (j), since (j) $\Longrightarrow$ (i) trivially. First recall that (a) $\Longrightarrow$ (h). Now take the limiting operator $\Lambda\in\mathcal{B}[X]$ and set, for an arbitrary $u\in X$,

$$\tilde{x}(i) = \sum_{j=0}^{i}(\Phi(i,j)-\Lambda^{i-j})u$$

$$\tilde{u}(i+1) = (\Lambda(i)-\Lambda)\sum_{j=0}^{i}\Phi(i,j)u,$$

for each $i\geq 0$. It is readily verified that, for every $i\geq 0$,

$$\tilde{x}(i+1) = \Lambda\tilde{x}(i) + \tilde{u}(i+1), \qquad \tilde{x}(0)=\tilde{u}(0)=0.$$

If (a) holds true and $\|\Lambda(i)-\Lambda\|\to 0$ as $i\to\infty$, then the above autonomous model is uniformly asymptotically stable, according to Remark 4, so that (h) is also applicable to it in particular. Moreover, $\lim_{i\to\infty}\|\tilde{u}(i)\|=0$ whenever $\lim_{i\to\infty}\|\Lambda(i)-\Lambda\|=0$ and (a) holds true (since (a) $\Longrightarrow$ (L) in Theorem 1). Thus

$$(6)\quad \lim_{i\to\infty}\|\sum_{j=0}^{i}(\Phi(i,j)-\Lambda^{i-j})u\| = \lim_{i\to\infty}\|\tilde{x}(i)\| = 0.$$

On the other hand we have from (2-b), for any $u\in X$ and every $i\geq 0$,

$$x(i)-\sum_{j=0}^{i}\Lambda^{i-j}u = \sum_{j=0}^{i}\Phi(i,j)[u(j)-u] + \sum_{j=0}^{i}(\Phi(i,j)-\Lambda^{i-j})u,$$

so that, if (a) holds true we get from (E) (cf. Theorem 1)

$$\|x(i)-\sum_{j=0}^{i}\Lambda^{i-j}u\| \leq \sum_{j=0}^{i}\alpha^{i-j}\zeta_j + \|\sum_{j=0}^{i}(\Phi(i,j)-\Lambda^{i-j})u\|,$$

where $\zeta_i=\gamma\|u(i)-u\|$ for every $i\geq 0$, with $\gamma\geq 1$ and $\alpha\in(0,1)$ as in (3). Hence, from (5) and (6),

$$\lim_{i\to\infty}\|u(i)-u\| = 0 \Longrightarrow \lim_{i\to\infty}\|x(i)-\sum_{j=0}^{i}\Lambda^{j}u\| = 0$$

whenever (a) holds true. Moreover, since $\Lambda(i)\to\Lambda\in\mathcal{B}[X]$ uniformly as $i\to\infty$, it follows from Remarks 2 and 4 that (a) implies $(\overline{N}_o)$ in Remark 2. Thus (a) $\Longrightarrow$ (j), since for any $u\in X$ and every $i\geq 0$,

$$\|x(i)-(I-\Lambda)^{-1}u\| \leq \|x(i)-\sum_{j=0}^{i}\Lambda^{j}u\| + \|\sum_{j=0}^{i}\Lambda^{j}-(I-\Lambda)^{-1}\|\ \|u\|. \qquad \square$$

5. CONCLUDING REMARKS

Remark 5. ($\ell_{p>1}$, c_0, and c-input ℓ_∞-state stability). For $q>1$, the result (f) ⟹ (a) in Theorem 2 becomes a particular case of that presented in [10]. Indeed, each of the assertions below is also equivalent to those equivalent assertions in Theorem 2.

(b') For every $p>1$ there exists a positive number η_p such that

$$\|\underline{x}\|_\infty \leq \eta_p \|\underline{u}\|_p \qquad \forall \underline{u}\in\ell_p(X).$$

(c') For some $q>1$ there exists a positive number η_q such that

$$\|\underline{x}\|_\infty \leq \eta_q \|\underline{u}\|_q \qquad \forall \underline{u}\in\ell_q(X).$$

(e') $\underline{u}\in\ell_p(X) \implies \underline{x}\in\ell_\infty(X)$ for every $p>1$.

(f') $\underline{u}\in\ell_q(X) \implies \underline{x}\in\ell_\infty(X)$ for some $q>1$.

(h') $\underline{u}\in c_0(X) \implies \underline{x}\in\ell_\infty(X)$.

(h") $\|\underline{x}\|_\infty \leq \eta_0 \|\underline{u}\|_\infty \quad \forall \underline{u}\in c_0(X)$, for some positive constant η_0.

(i') $\underline{u}\in c(X) \implies \underline{x}\in\ell_\infty(X)$.

(i") $\|\underline{x}\|_\infty \leq \eta \|\underline{u}\|_\infty \quad \forall \underline{u}\in c(X)$, for some positive constant η.

This can be verified as follows. From (E) (cf. Theorem 1) and (2-b), we get (a) ⟹ (b') by using the Hölder inequality. Note that (b') ⟹ (e') ⟹ (f') trivially. For each $i\geq 0$, let the transformation $\theta_{i,q}:\ell_q(X)\to X$ be the restriction of $\theta_i:\ell_\infty(X)\to X$ on $\ell_q(X)$, which is clearly linear and bounded (i.e. $\theta_{i,q}\in\mathcal{B}[\ell_q(X),X]$ - actually $\|\theta_{i,q}\|^{q/(q-1)}_{\mathcal{B}[\ell_q(X),X]} \leq \sum_{j=0}^{i}\|\Phi(i,j)\|^{q/(q-1)}$ for any $q>1$). Proceeding as in the proof of (g) ⟹ (d) in Theorem 2, we get (f') ⟹ (c'). It can be shown that (c') ⟹ (a) by using the same technique proposed in [10], which is essentially the following. Let $\varepsilon\in(0,1)$ and, for an arbitrary $u\in X$ and for each $k\geq 0$, set

$$u_k'(i) = \begin{cases} \varepsilon^{i-k}\,\Phi(i,k)u & \text{if} \quad i\geq k \\ 0 & \text{if} \quad 0\leq i<k. \end{cases}$$

For each $k\geq 0$, let $\underline{x}_k'=\{x_k'(i)\in X;\ i\geq 0\}$ be the response to $\underline{u}_k'=\{u_k'(i)\in X;\ i\geq 0\}$ in (2-b), so that

$$x_k'(k+\ell) = (1-\varepsilon)^{-1}\,(1-\varepsilon^{\ell+1})\,\Phi(k+\ell,k)\,u \qquad \forall k,\ell\geq 0.$$

Note that (c') implies $\|\Phi(k+\ell,k)\| \leq \eta_q$ for every $k,\ell\geq 0$ (consider the sequences $\underline{u}_k$ and $\underline{x}_k$ defined in the proof of Theorem 2), which implies $\|\underline{u}_k'\|_q \leq \eta_q(1-\varepsilon^q)^{-1/q}\|u\|$. Hence $\|x_k'(k+\ell)\| \leq \|\underline{x}_k'\|_\infty \leq \eta_q^2(1-\varepsilon^q)^{-1/q}\|u\|$ whenever (c') holds true. Therefore

$$\sup_{k\geq 0}\|\Phi(k+\ell,k)\| \;\leq\; \eta_q^2(1-\varepsilon)(1-\varepsilon^q)^{-1/q}(1-\varepsilon^{\ell+1})^{-1} \qquad \forall \ell\geq 0.$$

Since $q>1$, $(1-\varepsilon)(1-\varepsilon^q)^{-1/q}\to 0$ as $\varepsilon\to 1$. Then, by taking $\varepsilon=\varepsilon_1\in(0,1)$ sufficiently close to 1, there exists $\ell_1=\ell_1(\varepsilon_1)$ large enough so that (B) (cf. Theorem 1) holds true. Thus (c') $\Longrightarrow$ (a). Finally note that (g) $\Longrightarrow$ (i') $\Longrightarrow$ (h') $\Longrightarrow$ (e') and (d) $\Longrightarrow$ (i") $\Longrightarrow$ (h") $\Longrightarrow$ (h') trivially.

<u>Remark 6.</u> (ℓ_1-input ℓ_∞-state stability). By setting $\Lambda(k)=I\in\mathcal{B}[X]$ for every $k\geq 0$, it is trivially verified that (f') $\not\Longrightarrow$ (a) for q=1 in Remark 5. Actually, the assertions below are equivalent.

(a_1) $\|\Phi(k+\ell,k)\| \leq \eta \quad \forall k,\ell\geq 0,$ for some positive constant η.

(c_1) $\|\underline{x}\|_\infty \leq \eta\|\underline{u}\|_1 \quad \forall \underline{u}\in\ell_1(X),$ for some positive constant η.

(f_1) $\underline{u}\in\ell_1(X) \Longrightarrow \underline{x}\in\ell_\infty(X)$.

The verification is straightforward. (a_1) $\Longrightarrow$ (c_1) $\Longrightarrow$ (f_1) trivially, (c_1) $\Longrightarrow$ (a_1) by using the sequences $\underline{u}_k$ and $\underline{x}_k$ defined in the proof of Theorem 2, and (f_1) $\Longrightarrow$ (c_1) as follows: let $\theta_{i,1}\in\mathcal{B}[\ell_1(X),X]$ be defined as in Remark 5 (so that $\|\theta_{i,1}\|_{\mathcal{B}[\ell_1(X),X]} \leq \max_{0\leq j\leq i}\|\Phi(i,j)\|$), and proceed as in the proof of (g) $\Longrightarrow$ (d) in Theorem 2.

<u>Remark 7.</u> (An illustrative application). Suppose a given operator sequence $\{\Lambda(i)\in\mathcal{B}[X];\ i\geq 0\}$ converges uniformly to $\Lambda\in\mathcal{B}[X]$, and consider the state sequences $\underline{x}=\{x(i)\in X;\ i\geq 0\}$ and $\overline{\underline{x}}=\{\overline{x}(i)\in X;\ i\geq 0\}$ generated by the models

(2-a) $\quad x(i+1) = \Lambda(i)x(i) + u(i+1), \qquad x(0) = u(0),$

(2-$\overline{a}$) $\quad \overline{x}(i+1) = \Lambda\,\overline{x}(i) + \overline{u}(i+1), \qquad \overline{x}(0) = \overline{u}(0),$

for arbitrary bounded input sequences $\underline{u}=\{u(i)\in X;\ i\geq 0\}\in\ell_\infty(X)$ and $\overline{\underline{u}}=\{\overline{u}(i)\in X;\ i\geq 0\}\in\ell_\infty(X)$, respectively. We claim that the nonautonomous uniformly convergent model (2-a) is an asymptotic state estimator for the limiting autonomous model (2-$\overline{a}$) (i.e. $\lim_{i\to\infty}\|x(i)-\overline{x}(i)\| = 0$ whenever $\lim_{i\to\infty}\|u(i)-\overline{u}(i)\| = 0$) if and only if the limiting autonomous model (2-$\bar{a}$) is uniformly asymtotically stable. In other words, by setting $\underset{\sim}{\tilde{u}}=\underset{\sim}{u}-\underset{\sim}{\bar{u}}$ and $\underset{\sim}{\tilde{x}}=\underset{\sim}{x}-\underset{\sim}{\bar{x}}$, we claim that the assertions below are equivalent.

($\bar{a}$) $\Lambda\in B[X]$ is uniformly asymptotically stable (cf. Remark 2).

($\tilde{h}$) $\underset{\sim}{\tilde{u}}\in c_0(X) \implies \underset{\sim}{\tilde{x}}\in c_0(X)$.

This can be verified as follows. First recall that ($\bar{a}$) is equivalent to (a) in Theorem 2, according to Remark 4, since $\lim_{i\to\infty}\|\Lambda(i)-\Lambda\| = 0$. By setting $\tilde{\Lambda}(i)=\Lambda(i)-\Lambda$ for each $i\geq 0$, we get from (2-a) and (2-$\bar{a}$)

(2-$\tilde{a}$) $\tilde{x}(i+1) = \Lambda\tilde{x}(i) + w(i+1), \qquad \tilde{x}(0) = \tilde{u}(0) = w(0),$

with $\underset{\sim}{w}=\{w(i)\in X;\ i\geq 0\}$ given by $w(i+1)=\tilde{\Lambda}(i)x(i)+\tilde{u}(i+1)$ for each $i\geq 0$. Now suppose ($\bar{a}$) holds true. Recall that (a) implies (g) in Theorem 2, so that $\underset{\sim}{x}\in\ell_\infty(X)$, since $\underset{\sim}{u}\in\ell_\infty(X)$. Hence, $\underset{\sim}{\tilde{u}}\in c_0(X)$ implies $\underset{\sim}{w}\in c_0(X)$, since $\lim_{i\to\infty}\|\tilde{\Lambda}(i)\| = 0$, which implies $\underset{\sim}{\tilde{x}}\in c_0(X)$ in (2-$\tilde{a}$) whenever ($\bar{a}$) holds true, according to Theorem 2 (for the particular case of a constant sequence). Thus ($\bar{a}$) $\implies$ ($\tilde{h}$). On the other hand, set $\underset{\sim}{\bar{u}}=0$ in (2-$\bar{a}$), so that $\underset{\sim}{\tilde{u}}=\underset{\sim}{u}$ and $\underset{\sim}{\tilde{x}}=\underset{\sim}{x}$. Hence ($\tilde{h}$) implies that $\underset{\sim}{x}\in c_0(X)$ whenever $\underset{\sim}{u}\in c_0(X)$, or equivalently, ($\tilde{h}$) implies (h) in Theorem 2, which implies (a) according to Theorem 2. Thus ($\tilde{h}$) $\implies$ ($\bar{a}$).

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APPENDIX

The purpose of this appendix is to display the somewhat elementary calculations required for supporting some auxiliary results used in the main body of the paper, as well as to present some further remarks.

Proof of (E') ⟹ (E) in Remark 1. Consider the assertion below.

(E') There exist real constants $\gamma_0 \geq 1$ and $\alpha_0 \in (0,1)$ such that

$$\|\Phi(k+\ell,k)\| \leq \gamma_0 \alpha_0^{\ell} \qquad \forall \ell \geq 0 \quad \forall k \geq k_0, \qquad \text{for some } k_0 \geq 0.$$

Set $\gamma_1 = \max\{1, \sup_{n\geq 0}\|\Lambda(n)\|^{k_0}\}$, and take any integer $k \in [0,k_0]$. If $\ell \in [0,k_0-k] \subset [0,k_0]$, then

$$\|\Phi(k+\ell,k)\| \leq \sup_{n\geq 0}\|\Lambda(n)\|^{\ell} \leq \gamma_1 \leq \gamma_1 \gamma_0 \alpha_0^{-k_0} \alpha^{\ell},$$

since $0<\alpha_0<1 \Longrightarrow \alpha_0^{-(k_0-\ell)} \geq 1$ for every $\ell \in [0,k_0]$. On the other hand, if $\ell \in (k_0-k,\infty)$, so that $k+\ell > k_0$, then

$$\|\Phi(k+\ell,k)\| \leq \|\Phi(k+\ell,k_0)\| \, \|\Phi(k_0,k)\| \leq \|\Phi(k_0+\ell-(k_0-k),k_0)\| \sup_{n\geq 0}\|\Lambda(n)\|^{k_0-k}$$

$$\leq \gamma_0 \gamma_0^{\ell-(k_0-k)} \gamma_1 \leq \gamma_1 \gamma_0 \alpha_0^{-k_0} \alpha_0^{\ell}.$$

Thus, (E') ⟹ (E) with $\gamma = \gamma_1 \gamma_0 \alpha_0^{-k_0} \geq \gamma_0 \geq 1$ and $\alpha = \alpha_0 \in (0,1)$.

Proof of (C') ⟹ (E') in Remark 1. If $\limsup_{k\to\infty}\|\Phi(k+\ell_0,k)\| < 1$ for some $\ell_0 \geq 1$, then there exists an integer $k_0 \geq 0$ and a positive constant $\eta_0 \in [\limsup_{k\to\infty}\|\Phi(k+\ell_0,k)\|, 1)$ such that $\|\Phi(k+\ell_0,k)\| < \eta_0$ for every $k \geq k_0$, so that $\sup_{k\geq k_0}\|\Phi(k+\ell_0,k)\| \leq \eta_0 < 1$. Therefore, by setting

$$\alpha_0 = \sup_{k\geq k_0}\|\Phi(k+\ell_0,k)\|^{1/\ell_0} \leq \eta_0^{1/\ell_0} < 1,$$

we still get (by induction in n)

$$\sup_{k\geq k_0}\|\Phi(k+n\ell_0,k)\| \leq \alpha_0^{n\ell_0} \qquad \forall n \geq 0.$$

Hence (cf. proof of (C) ⟹ (E) in Theorem 1),

$$\|\Phi(k+\ell,k)\| \leq \alpha_0^{-\ell_0} \max\{1, \sup_{k\geq 0}\|\Lambda(k)\|^{\ell_0}\} \alpha_0^{\ell} \qquad \forall \ell \geq 0 \qquad \forall k \geq k_0 .$$

Thus (C') ⟹ (E'). Note that $\sup_{k\geq 0}\|\Lambda(k)\| < \infty \iff \limsup_{k\to\infty}\|\Lambda(k)\| < \infty$.

Remark 1a. It is worth noting that the boundedness assumption is actually needed in assertion (C) of Theorem 1, since

$$\sup_{k\geq 0}\|\Phi(k+\ell_0,k)\| < 1 \text{ for some } \ell_0\geq 1 \quad \not\Longrightarrow \quad \sup_{k\geq 0}\|\Lambda(k)\| < \infty$$

(e.g. set $\Lambda(k)=kI$ for $k=0,2,4,\ldots$ and $\Lambda(k)=I/2(k+1)$ for $k=1,3,5,\ldots$, so that $\|\Phi(k+2,k)\| \leq 1/2$ for every $k\geq 0$). Also note that

$$\|\Lambda(k)\| < 1 \qquad \forall k\geq 0 \qquad \not\Longrightarrow \qquad (A)$$

(e.g. set $\Lambda(k)=kI/(k+1)$ for every $k\geq 0$, so that $\Phi(k+\ell,k)=kI/(k+\ell)$ for every $k,\ell\geq 0$), although

$$\limsup_{k\to\infty}\|\Lambda(k)\| < 1 \qquad \Longrightarrow \qquad (A),$$

since $\limsup_{k\to\infty}\|\Phi(k+\ell,k)\| \leq \limsup_{k\to\infty}\|\Lambda(k)\|^{\ell}$ for every $\ell\geq 0$, and (A) $\Longleftrightarrow$ (A').

Remark 1b. It is also worth noting that assertion (M) in Theorem 1 can be written as

(M') $\sum_{j=k}^{i}\|\Phi(i,j)\|^q \leq \rho_q < \infty \qquad \forall 0\leq k\leq i, \qquad$ for some $q>0$,

which is clearly equivalent to

(M") $\sum_{j=0}^{i}\|\Phi(i,j)\|^q \leq \rho_q < \infty \qquad \forall i\geq 0, \qquad$ for some $q>0$;

and assertion (N) in Theorem 1 can be equivalently stated as follows.

(N') For each $\varepsilon>0$ there exists an integer $n_\varepsilon\geq 1$ such that

$$n\geq n_\varepsilon \Longrightarrow \sup_{\nu\geq 0}\Big\|\sum_{\ell=0}^{n+\nu}\Phi(k+\ell,k) - \sum_{\ell=0}^{n-1}\Phi(k+\ell,k)\Big\| \leq \varepsilon \qquad \forall k\geq 0,$$

which means that the family of sequences $\{\{\sum_{\ell=0}^{n}\Phi(k+\ell,k)\in B[X];\ n\geq 0\};\ k\geq 0\}$ of partial sums is an equi-Cauchy family in the uniform norm topology. Since $B[X]$ is a Banach space, it is actually an equiconvergent family in that topology. In other words,

(N") the family of series $\{\sum_{\ell=0}^{\infty}\Phi(k+\ell,k);\ k\geq 0\}$ is uniformly equiconvergent

iff (by definition) (N') holds true, whenever $B[X]$ is a Banach space (or equivalently, whenever X is a Banach space).

Proof of Remark 4. Assume that $\|\Lambda(k)-\Lambda\| \to 0$ as $k\to\infty$. The result

(7) $\lim_{k\to\infty}\|\Phi(k+\ell,k)-\Lambda^{\ell}\| = 0 \qquad\qquad \forall \ell\geq 0$

is trivially verified for $\ell=0,1$. Suppose it holds for some $\ell\geq 1$. Then, for every $k\geq 0$,

$$\|\Phi(k+\ell+1,k)-\Lambda^{\ell+1}\| = \|\Lambda(k+\ell)[\Phi(k+\ell,k)-\Lambda^{\ell}] + [\Lambda(k+\ell)-\Lambda]\Lambda^{\ell}\|$$

$$\leq \sup_{k\geq 0}\|\Lambda(k)\|\,\|\Phi(k+\ell,k)-\Lambda^{\ell}\| + \|\Lambda^{\ell}\|\,\|\Lambda(k+\ell)-\Lambda^{\ell}\| \to 0 \quad \text{as} \quad k\to\infty,$$

so that it holds for $\ell+1$, which concludes the proof of (7) by induction in ℓ. Hence

(8) $\limsup_{k\to\infty}\|\Phi(k+\ell,k)\| = \lim_{k\to\infty}\|\Phi(k+\ell,k)\| = \|\Lambda^{\ell}\| \qquad \forall \ell\geq 0.$

Then (cf. definitions of (A') and $(\bar{A})$ in Remarks 1 and 2)

$$(\bar{A}) \Longrightarrow (A') \Longrightarrow \lim_{\ell\to\infty}\lim_{k\to\infty}\|\Phi(k+\ell,k)\| = 0 \Longrightarrow (\bar{A}).$$

By recalling that (A) $\Longleftrightarrow$ (A') (cf. Remark 1), we get the equivalence in Remark 4.

<u>Remark 4a</u>. Suppose the operator sequence $\{\Lambda(k)\in\mathcal{B}[X];\ k\geq 0\}$ converges uniformly to $\Lambda\in\mathcal{B}[X]$. According to (8) it follows that

$$\|\Phi(k+\ell,k)\| \leq \eta < \infty \qquad \forall k,\ell\geq 0 \quad \Longrightarrow \quad \sup_{\ell\geq 0}\|\Lambda^{\ell}\| < \infty$$

for some nonnegative constant η. On the other hand, the converse fails. That is, even if $\{\Lambda(k)\in\mathcal{B}[X];\ k\geq 0\}$ converges uniformly to $\Lambda\in\mathcal{B}[X]$,

$$\sup_{\ell\geq 0}\|\Lambda^{\ell}\| < \infty \quad \not\Longrightarrow \quad \|\Phi(k+\ell,k)\| \leq \eta < \infty \qquad \forall k,\ell\geq 0$$

(e.g. set $\Lambda(k)=(k+2)I/(k+1)$ for every $k\geq 0$, so that $\Lambda(k)\to\Lambda=I$ uniformly as $k\to\infty$ and $\|\Phi(k+\ell,k)\| = (\ell+k+1)/(k+1)$ for every $k,\ell\geq 0$).

<u>Remark 4b</u>. However, if uniform equiconvergence

(9) $\lim_{k\to\infty}\sup_{\ell\geq 0}\|\Phi(k+\ell,k)-\Lambda^{\ell}\| = 0$

holds, instead of just pointwise uniform convergence in (7), then

$$\sup_{\ell\geq 0}\|\Lambda^{\ell}\| < \infty \quad \Longrightarrow \quad \|\Phi(k+\ell,k)\| \leq \eta < \infty \qquad \forall k,\ell\geq 0.$$

Actually, suppose $\Lambda\in\mathcal{B}[X]$ is power bounded, and set $\omega_0=\sup_{n\geq 0}\|\Lambda^n\|$. From (9) it follows that, for each $\varepsilon>0$ there exists an integer $k_\varepsilon\geq 0$ such that

$$k\geq k_\varepsilon \Longrightarrow \|\Phi(k+\ell,k)-\Lambda^{\ell}\| \leq \varepsilon \Longrightarrow \|\Phi(k+\ell,k)\| \leq \varepsilon + \omega_0 \qquad \forall \ell\geq 0.$$

Hence, by setting $\omega_\varepsilon = \sup_{n\geq 0}\|\Lambda(n)\|^{k_\varepsilon}$, we get

$$\|\Phi(k+\ell,k)\| \leq \|\Phi(k+k_\varepsilon+\ell-k_\varepsilon,k+k_\varepsilon)\|\,\|\Phi(k+k_\varepsilon,k)\| \leq (\varepsilon+\omega_0)\omega_\varepsilon$$

for every $k\geq 0$ and $\ell\geq k_\varepsilon$. Now recall that

$$\|\Phi(k+\ell,k)\| \leq \max\{1,\omega_\varepsilon\}$$

for every $k\geq 0$ and $0\leq\ell\leq k_\varepsilon$. Therefore, for each $\varepsilon>0$ there exists a real number $\eta_\varepsilon\geq 1$ such that

$$\|\Phi(k+\ell,k)\| \leq \eta_\varepsilon \overset{\text{def.}}{=} \max\{1,\omega_\varepsilon,(\varepsilon+\omega_0)\omega_\varepsilon\} < \infty \qquad \forall k,\ell\geq 0.$$

Note that, by setting $\varepsilon=1$, we get

$$\eta_1 = \max\{1,(1+\omega_0)\omega_1\} \leq \max\{(1+\omega_0),(1+\omega_0)\omega_1\}$$

$$= (1+\omega_0)\max\{1,\omega_1\} \leq (1+\omega_0)(1+\omega_1),$$

so that

$$\|\Phi(k+\ell,k)\| \leq (1+\sup_{n\geq 0}\|\Lambda^n\|)(1+\sup_{n\geq 0}\|\Lambda(n)\|^{k_1}) < \infty \qquad \forall k,\ell\geq 0.$$

Remark 4c. Note that $\{\Phi(k+\ell,k);\, k\geq 0\}$ converges strongly to Λ^ℓ for every $\ell\geq 0$, whenever $\{\Lambda(k);\, k\geq 0\}$ converges strongly to Λ (this follows by induction in ℓ, similarly to the proof of uniform convergence in (7)). However, the result $(\bar{A}) \Rightarrow (A)$ in Remark 4 does require uniform convergence. To verify this, set $\Lambda(k)=S^{*k}$ for every $k\geq 0$, where $S^*\in B[\ell_2]$ is the unilateral left shift operator on $X=\ell_2$. Recall that $\Lambda(k)=S^{*k}\to\Lambda=0$ strongly as $k\to\infty$, and $\|\Lambda(k)\| = \|S^{*k}\| = 1$ for every $k\geq 0$. Thus $\Phi(k+\ell,k)=S^{*\ell[k+(\ell-1)/2]}$ for every $k,\ell\geq 0$, so that $\Phi(k+\ell,k)\to\Lambda^\ell=0$ strongly as $k\to\infty$ for every $\ell\geq 0$, and $\|\Phi(k+\ell,k)\| = 1$ for every $k,\ell\geq 0$. Hence $(\bar{A}) \not\Rightarrow (A)$ in Remark 4 if uniform convergence is replaced by strong convergence.

Remark 5a. Further equivalent assertions can still be obtained if "$\sup_{i\geq 0}\|x(i)\|$" in Remark 5 in replaced by "$\limsup_{i\to\infty}\|x(i)\|$". Actually, each of the assertions below is also equivalent to those equivalent assertions in Theorem 2 and Remark 5.

(b") For every $p>1$ there exists a positive number η'_p such that

$$\limsup_{i\to\infty}\|x(i)\| \leq \eta'_p\|\underline{u}\|_p \qquad \forall\underline{u}\in\ell_p(X).$$

(c") For some $q>1$ there exists a positive number η'_q such that

$$\limsup_{i\to\infty}\|x(i)\| \leq \eta'_q\|\underline{u}\|_q \qquad \forall\underline{u}\in\ell_q(X).$$

(d') $\limsup_{i\to\infty}\|x(i)\| \le \eta_\infty\|\underline{u}\|_\infty$ $\forall \underline{u}\in\ell_\infty(X)$, for some positive constant η_∞.

(d") $\limsup_{i\to\infty}\|x(i)\| \le \eta'_\infty \limsup_{i\to\infty}\|u(i)\|$ $\forall \underline{u}\in\ell_\infty(X)$, for some positive constant η'_∞.

(e") $\underline{u}\in\ell_p(X) \implies \limsup_{i\to\infty}\|x(i)\| < \infty$ for every p>1.

(f") $\underline{u}\in\ell_q(X) \implies \limsup_{i\to\infty}\|x(i)\| < \infty$ for some q>1.

(g') $\underline{u}\in\ell_\infty(X) \implies \limsup_{i\to\infty}\|x(i)\| < \infty$.

(g") $\limsup_{i\to\infty}\|u(i)\| < \infty \implies \underline{x}\in\ell_\infty(X)$.

(g‴) $\limsup_{i\to\infty}\|u(i)\| < \infty \implies \limsup_{i\to\infty}\|x(i)\| < \infty$.

(h')' $\underline{u}\in c_0(X) \implies \limsup_{i\to\infty}\|x(i)\| < \infty$.

(h")' $\limsup_{i\to\infty}\|x(i)\| \le \eta'_0\|\underline{u}\|_\infty$ $\forall \underline{u}\in c_0(X)$, for some positive constant η'_0.

(i')' $\underline{u}\in c(X) \implies \limsup_{i\to\infty}\|x(i)\| < \infty$.

(i")' $\limsup_{i\to\infty}\|x(i)\| \le \eta'\|\underline{u}\|_\infty$ $\forall \underline{u}\in c(X)$, for some positive constant η'.

(i")" $\limsup_{i\to\infty}\|x(i)\| \le \eta" \lim_{i\to\infty}\|u(i)\|$ $\forall \underline{u}\in c(X)$, for some positive constant $\eta"$.

Up to a single step, which is actually straightforward, the verification is trivial. The straightforward step goes as follows. From (3) we get

$$\|x(i)\| \le \gamma'_i \overset{\text{def.}}{=} \gamma\sum_{j=0}^{i}\alpha^{i-j}\sup_{n\ge j}\|u(n)\|$$

for every $\underline{u}\in\ell_\infty(X)$, whenever (a) holds true. Recall that, by definition, $\lim_{i\to\infty}\sup_{n\ge i}\|u(n)\| = \limsup_{i\to\infty}\|u(i)\| \le \|\underline{u}\|_\infty$. Hence, according to (5), $\lim_{i\to\infty}\gamma'_i=\gamma(1-\alpha)^{-1}\limsup_{i\to\infty}\|u(i)\|$, so that (a) $\implies$ (d"). Finally note that

(d") $\implies$ (i")" $\implies$ (i")' $\implies$ (i')' $\iff$ (i'),

(d) $\implies$ (d') $\implies$ (g') $\iff$ (g") $\iff$ (g‴) $\iff$ (g),

(c') $\implies$ (c") $\implies$ (f") $\iff$ (f'),

(b') $\implies$ (b") $\implies$ (e") $\iff$ (e'),

(h") $\implies$ (h")' $\implies$ (h')' $\iff$ (h'), trivially (since $\sup_{i\geq 0}\| x(i)\| < \infty \Longleftrightarrow \limsup_{i\geq 0}\| x(i)\| < \infty$, and $\limsup_{i\to\infty}\| x(i)\| \leq \sup_{i\geq 0}\| x(i)\|$). However, it is worth noting that the assertions

$$(d_0)\quad \|\underline{x}\|_\infty \leq \eta \limsup_{i\to\infty}\| u(i)\| \qquad \forall \underline{u}\in\ell_\infty(X),$$

$$(i_0)\quad \|\underline{x}\|_\infty \leq \eta \lim_{i\to\infty}\| u(i)\| \qquad \forall \underline{u}\in c(X),$$

for some positive constant η, are not implied by the preceding equivalent assertions. For, if (a) $\Longrightarrow$ (i_0), then $\underline{x}=0$ for all $\underline{u}\in c_0(X)$, whenever (a) holds true, which is clearly false even for time-invanriant systems. Thus (a) $\not\Longrightarrow$ (i_0), so that (a) $\not\Longrightarrow$ (d_0), since (d_0) $\Longrightarrow$ (i_0) trivially.

Remark 8. (Comments on the proof of Theorems 1 and 2). Part of the techniques behind the approaches used to show that (C) $\Longrightarrow$ (E) and (P) $\Longrightarrow$ (S) in Theorem 1 were motivated by the earlier works dealing with specific uniform stability conditions for finite-dimensional continuous-time systems (e.g. see [17, p.190-193]). The idea of combining those two approaches, as developed in the proof of (R) $\Longrightarrow$ (T) in Theorem 1, does not seem to have been considered before even for particular cases. The proof of Theorem 2 begun by showing that (a) $\Longrightarrow$ (b,d). This was a routine application of the Cauchy product technique. Such a technique had already been used before for supporting stability results for time-invariant systems (e.g. see [12] and [19]). An earlier work [18], dealing with the one-dimensional continuous-time case, has provided a key insight which motivated the idea behind the approach used to show that (g) $\Longrightarrow$ (d) in Theorem 2. The proofs of (c) $\Longrightarrow$ (a) and (d) $\Longrightarrow$ (a) in Theorem 2 were based on a rather simple technique. This consisted of using pulse inputs $\underline{u}_k$ (for supplying responces $\underline{x}_k$ which in turn were plugged in the model again), and then applying the previously established conditions of Theorem 1. The result (a) $\Longrightarrow$ (h) in Theorem 2 was shown to hold true by a straightforward application of (5), which is in fact a corollary of the Silverman-Teoplitz theorem (cf. [2, p.75]), that had also been used before to supply stability properties for time-invariant systems (cf. [7]). As far as the proofs of (f) $\Longrightarrow$ (c), (h) $\Longrightarrow$ (a) and (i) $\Longrightarrow$ (a) in Theorem 2 (as well as the proof of (a) $\Longrightarrow$ (j) in Corollary 1) are concerned, we are not aware of any previous similar approach even for particular cases. The main tool, widely used to support many of the proofs in this paper, was the Banach-Steinhaus theorem. Perhaps one of the first attempts of using it in the stability literature was that in [15] (see also [18]). The version of the Banach-Steinhaus theorem (see [14] and [13, p.50]), which is also known as the *Principle of Uniform Boundedness*, that has sufficed our needs, reads as follows (cf. [1, p.98], [2, p.66], or [16, p.16]): For Banach spaces X and Y, and for any $\Lambda \subseteq B[X,Y]$,

$$\sup_{A\in\Lambda} \| Ax \| < \infty \qquad \forall x \in X \qquad \Longrightarrow \qquad \sup_{A\in\Lambda} \| A \| < \infty \ .$$

For instance, the proofs of (g) ⟹ (d), (f) ⟹ (c), and (h) ⟹ (a) in Theorem 2 were heavily based on the above result, while the proofs of (S) ⟹ (J) and (T) ⟹ (K) in Theorem 1 are reduced to an immediate application of it.

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