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GAUGE THEORIES OF INFINITE DIMENSIONAL HAMILTONIAN SUPERALGEBRAS

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Ergin Sezgin
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Symplectic diffeomorphisms of a class of supermanifolds and the associated infinite dimensional Hamiltonian superalgebras, H(2M,N), are discussed. Applications to strings, membranes and higher spin field theories are considered: The embedding of the Ramond superconformal algebra in H(2,1) is obtained. The Chern-Simons gauge theory of symplectic super-diffeomorphisms is constructed.

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1. Symplectic Super-Diffeomorphisms and the Hamiltonian Superalgebra H(2M,N)

Infinite dimensional Hamiltonian superalgebras are essentially the algebras of superfunctions defined on a superplane, relative to a super-Poisson bracket [1]. Our interest in them is due to their possible applications to the physics of relativistic membranes [2][3][4], string theories [5] and higher spin field theories [6][7][8]. Later we shall elaborate on these applications First, let us give a more precise definition of the Hamiltonian superalgebras.

Consider a superplane $\mathcal{R}^{2M,N}$ with the bosonic coordianates $x^i, y^i (i = 1, ..., M)$ and the fermionic coordinates $\theta^\alpha (\alpha = 1, ..., N)$. Let us define the following Hamiltonian 2-form [1]

$$\Omega = \sum_{i=1}^{M} dx^i \wedge dy^i + \sum_{\alpha=1}^{N} d\theta^\alpha \wedge d\theta^\alpha. \tag{1}$$

This form is evidently closed and nondegenerate. Consider, furthermore, the polynomial algebra $\Lambda[x^1, ..., x^M, y^1, ..., y^M, \theta^1, ..., \theta^N]$, which will be denoted by $\Lambda(2M, N)$, for short. The infine dimensional Hamiltonian superalgebra $H(2M, N)$ is defined by the following condition

$$\mathcal{L}_D \Omega = 0\,, \tag{2}$$

where $\mathcal{L}_D$ is the Lie derivative along the vector

$$D = \sum_{i=1}^{M} (P^i \frac{\partial}{\partial x^i} + Q^i \frac{\partial}{\partial y^i}) + \sum_{\alpha=1}^{N} R^\alpha \frac{\partial}{\partial \theta^\alpha}, \quad P, Q, R \in \Lambda(2M, N). \tag{3}$$

Using (1) and (3) in the defining relation (2) one finds that $P_i = \partial\Lambda/\partial x^i$, $Q_i = -\partial\Lambda/\partial y^i$, $R_\alpha = \partial\Lambda/\partial\theta^\alpha$ with an arbitrary polynomial Λ, and therefore the generators of $H(2M, N)$ are given by [1]

$$D_\Lambda = \sum_{i=1}^{M} (\frac{\partial\Lambda}{\partial x^i}\frac{\partial}{\partial y^i} - \frac{\partial\Lambda}{\partial y^i}\frac{\partial}{\partial x^i}) + \sum_{\alpha=1}^{N} \frac{\partial\Lambda}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\alpha}, \quad \Lambda \in \Lambda(2M, N). \tag{4}$$

The polynomial $\Lambda(x, y, \theta)$ is called the *Hamiltonian function*. The generators (4) are called the *Hamiltonian vector fields* and they obey the algebra

$$[D_{\Lambda_1}, D_{\Lambda_2}] = D_{\{\Lambda_1, \Lambda_2\}}, \tag{5}$$

where the super-Poisson bracket $\{,\}$ is defined by

$$\{\Lambda_1, \Lambda_2\} = \sum_{i=1}^{M} (\frac{\partial\Lambda_1}{\partial x^i}\frac{\partial\Lambda_2}{\partial y^i} - \frac{\partial\Lambda_1}{\partial y^i}\frac{\partial\Lambda_2}{\partial x^i}) + (-1)^{deg\Lambda_1} \sum_{\alpha=1}^{N} \frac{\partial\Lambda_1}{\partial\theta^\alpha}\frac{\partial\Lambda_2}{\partial\theta^\alpha}, \tag{6}$$

and $deg\Lambda_1$ is the grading of Λ_1. Note that the 2-form (1) has an even grade. It has been pointed out by Leites [9] that if the dimensions of the bosonic and fermionic coordinates are equal, then one can also define an odd graded Hamiltonian 2-form, $\Omega^{(1)} = \sum_{\alpha=1}^{N} d\sigma^\alpha \wedge d\theta^\alpha$. For the analogues of (4)-(6) for this case, see [9].

It is interesting to note that $H(2M,N)$ can be interpreted as the infinite dimensional extension of the superalgebra $OSp(N,2M)$ [1] and the algebra $H^{(1)}(N,N)$, associated with the symplectic super-diffemorphisms which leave $\Omega^{(1)}$ invariant, can be interpreted as the infinite dimensional extension of the superalgbera $P(N)$ [9]. Note also that $H(0,N)$ is finite dimensional. In fact, it has (2^N-1) generators, and it is known as $\tilde{H}(N)$[1].

It is straightforward to generalize (1)-(6) to (symplectic) supermanifolds other than the superplane. Consider a supermanifold $\mathcal{M}^{2M,N}$ with bosonic coordinates σ^a $(a=1,...,2M)$ and fermionic coordinates θ^α $(\alpha=1,...,N)$, collectively denoted by $z^A=(\sigma^a,\theta^\alpha)$. Assume that the manifold admits a closed non-degenerate super 2-form Ω defined by [7]

$$\Omega = dz^A\wedge dz^B\Omega_{BA}, \quad d\Omega=0, \quad sdet\,\Omega_{AB}\neq 0. \tag{7}$$

Such manifolds will be called *symplectic supermanifolds*, and the 2-form Ω will be called the *symplectic structure*. The super-diffeomorphisms of $\mathcal{M}^{2M,N}$ which preserve this symplectic structure will be called the *symplectic super-diffeomorphisms*, which will be denoted by super-$Diff_\Omega$. They are characterized by

$$\mathcal{L}_\xi\Omega=0, \tag{8}$$

where $\mathcal{L}_\xi$ is the super-Lie derivative with respect to a super-vector ξ^A. It is easy to see that the solution of (8) is (we shall assume that Λ has an even grade from now on, for simplicity):

$$\xi^A=\Omega^{AB}\partial_B\Lambda(z), \quad \Omega^{AB}\Omega_{BC}=\delta^A_C. \tag{9}$$

where $\Lambda(z)$ is an arbitrary superfunction. Thus, the super-$Diff_\Omega$ is generated by

$$D_\Lambda=\Omega^{AB}\partial_B\Lambda(z)\partial_A. \tag{10}$$

These generators obey the algebra (5) with the super-Poisson bracket $\{,\}$ now defined by

$$\{\Lambda_1,\Lambda_2\}=\Omega^{AB}\partial_B\Lambda_1\partial_A\Lambda_2. \tag{11}$$

In deriving (5), as well as checking the super-Jacobi identity for the bracket (11) one makes use of the identity

$$\Omega^{AD}\partial_D\Omega^{BC}+(-1)^{a(b+c)}\Omega^{BD}\partial_D\Omega^{CA}+(-1)^{c(a+b)}\Omega^{CD}\partial_D\Omega^{AB}=0 \tag{12}$$

$(a,b,c$ are gradings), which follows from (9) and the fact that Ω is closed. In summary, with (7)-(11) we have found the generalization of (1)-(6), as promised.

The following example [7] of symplectic supermanifolds is of particular interest. We take the even part of the supermanifold $\mathcal{M}^{2M,N}$ to be a Kahler coset manifold G/H with Kahler 2-form $J_{ab}(\sigma)$. We furthermore require that G admits a *constant* invariant tensor $\Omega_{\alpha\beta}$, which is symmetric and non-degenerate. We then take the super 2-form (8) to be

$$\Omega=d\sigma^a\wedge d\sigma^b J_{ba}+d\theta^\alpha\wedge d\theta^\beta\Omega_{\beta\alpha}. \tag{13}$$

Evidently, Ω is closed. An example of special interest is when the Kahler manifold G/H is

$$\frac{SO(q,2)}{SO(q)\times SO(2)}, \quad q=1,2,3,4,8\ mod\,8. \tag{14}$$

In this case, we can choose the matrix $\Omega_{\alpha\beta}$ as follows. We take our odd variables to be $\theta^{\alpha' i}$, where α' is a spinor index of $SO(q,2)$, and $i=1,2$ is a spinor index of $SU(2)$ or $SL(2,R)$. We then take

$$\Omega_{\alpha\beta}=C_{\alpha'\beta'}\epsilon_{ij}. \tag{15}$$

$\Omega_{\alpha\beta}$ is symmetric, since $C_{\alpha'\beta'}$ is antisymmetric for the values of q in (14) and $\epsilon_{ij}=-\epsilon_{ji}$. For some other examples of symplectic supermanifolds, see [10] where the super analogues of the complex projective spaces and the Stiefel manifolds are discussed. We shall make use of the example given above for $q=1$ in Sect. 4. We now turn to the applications of $H(2M,N)$.

2. Applications to String Theories

There exists an interesting class of conformal field theories in two dimensions based on the so called W_N algebra [11], which is a chiral operator algebra generated by conformal fields with integer spin 1,2,...,N. It contains the Virasoro algebra as a subalgebra. Using W_N one can build string theories; the W-strings [12]. Recently, Bakas [5] has argued that W_∞ provides a representation of $H(2,0)$. This connection may lead to a fruitful use of the conformal techniques in membrane physics as well as the use of membrane theoretical techniques in string theories. Here, we shall seek a relation between $H(2,1)$ and super-W_∞, which can be viewed as an infinite dimensional extension of the super-Virasoro algebra.

Let us consider a superplane $R^{2,1}$ parametrized by x,y,θ. (For an embedding of the Virasoro algebra in $Diff_\Omega(T^2)$, see [13]). Hamiltonian 2-form (1) is now simply $\Omega=2\,dx\wedge dy+d\theta\wedge d\theta$. It is convenient to switch to a complex coordinate system where $\Omega=(idz\wedge d\bar{z}+d\theta\wedge d\theta)$. Let us expand the (real) Hamiltonian function $\Lambda(z,\bar{z},\theta)$ as follows

$$\Lambda=\sum_{m,n=-\infty}^{\infty}(c^{mn}b_{mn}+\varepsilon^{mn}f_{mn}), \tag{16}$$

where c^{mn},ε^{mn} are the commuting and anticommuting expansion coefficients, respectively, and b_{mn},f_{mn} are the basis elements which we choose as follows

$$b_{mn}=2\omega^2z^{m+1}\bar{z}^{n+1}, \qquad f_{mn}=2\omega^3\theta z^{m+\frac{1}{2}}\bar{z}^{n+\frac{1}{2}}, \qquad m,n\in Z, \tag{17}$$

where $\omega\equiv\exp(-i\pi/4)$. We have used doubled valued basis elements f_{mn} to allow expansion of fermions. Further motivation for this choice of basis will become more transparent below. The ω-factors are put in as convenient normalizations. Substituting (16) into $D_\Lambda=(\frac{i}{2}\partial_z\Lambda\,\partial_{\bar{z}}-\frac{i}{2}\partial_{\bar{z}}\Lambda\,\partial_z+\partial_\theta\Lambda\,\partial_\theta)$ and expanding

$$D_\Lambda=\sum_{m,n=-\infty}^{\infty}(c^{mn}L_{mn}+\varepsilon^{mn}G_{mn}), \tag{18}$$

we find

$$L_{mn} = \frac{i}{2}\partial_z b_{mn}\partial_{\bar z} - \frac{i}{2}\partial_{\bar z} b_{mn}\partial_z$$
$$G_{mn} = \frac{i}{2}\partial_z f_{mn}\partial_{\bar z} - \frac{i}{2}\partial_{\bar z} f_{mn}\partial_z - \partial_\theta f_{mn}\partial_\theta \tag{19}$$

It is not difficult to check that the algebra of the generators L_{mn} and G_{mn} is identical to the algebra of the basis elements b_{mn} and f_{mn}, relative to the super-Poisson bracket (6). Using this fact, we obtain the superalgebra $H(2,1)$:

$$[L_{mn}, L_{pq}] = [(m+1)(q+1) - (n+1)(p+1)]L_{m+p,n+q}$$
$$[L_{mn}, G_{pq}] = [(m+1)(q+\frac{1}{2}) - (n+1)(p+\frac{1}{2})]G_{m+p,n+q}$$
$$\{G_{mn}, G_{pq}\} = 2L_{m+p,n+q}, \qquad m,n,p,q \in Z. \tag{20}$$

It is remarkable this algebra admits the following subalgebra

$$[L_{m0}, L_{p0}] = (m-p)L_{m+p,0}$$
$$[L_{m0}, G_{p0}] = (\frac{1}{2}m - p)G_{m+p,0}$$
$$\{G_{m0}, G_{p0}\} = 2L_{m+p,0}, \tag{21}$$

which is precisely the Ramond superconformal algebra. Note that, even if one chooses the basis elements f_{mn} such that m, n are half-integers, one cannot obtain the Neveu-Schwarz superconformal algebra. Note also that,

$$L_{n0} = (n+1)z^n\bar z\partial_{\bar z} - z^{n+1}\partial_z. \tag{22}$$

On holomorphic functions, only the second term operates, and that is the usual realization of the Virasoro generators. However, on non-holomorphic functions the realization (22) differs from the usual one due to the presence of the first term. In particular, $\bar L_{n0} = -L_{0n}$ and $[L_{n0}, L_{0n}] \neq 0$. Note also that $[L_{n,-1}, L_{m,-1}] = 0$. The basis elements with $m,n \geq -1$ span the algebra $H_+(2,1)$ which is, in a certain sense [5], dual to the algebra $H_-(2,1)$ spanned by the dual basis elements. Defining the spin h of a primary conformal field with components ϕ^h_n by $\frac{1}{2}(L_0 - \bar L_0)\phi^h_{-1} = h\phi^h_{-1}$, from (20) we see that L_{pq} can be interpreted as the components (labeled by p) of a primary conformal field with $spin = (q+1)$. (We use the notation, $L_{00} \equiv L_0$). To see this, note that $\bar L_0 = -L_0$ and that $[L_0, L_{-1q}] = (q+1)L_{-1q}$. Similarly G_{-1q} has $spin = (q+1)$. Bakas [5] has suggested that L_{pq} can be interpreted as generators of W_N-algebra with $q = 1, ..., N-2$. Similarly, we can interpret the algebra (20), i.e. $H(2,1)$, as the super-W_∞ algebra. It should be noted, however, that (20) does not account for the central extensions and the contributions of the lower spin fields and their derivatives which arise in the usual W-algebras. It has been suggested [5] that these are related to the cohomology of the Hamiltonian (super) algebra with non-trivial coefficients. Clearly, a lot remains to be uncovered in this subject. (See also, Sect. 4).

3. Applications to Membrane Theories

In the case of two dimensional manifolds $Diff_\Omega(\mathcal{M}^2)$ coincides with area-preserving diffeomorphisms, $SDiff(\mathcal{M}^2)$. Sometime ago, Hoppe [2] showed that $SDiff(S^2)$, which is the $SDiff$ of a 2-sphere, arises as the residual symmetry of a relativistic membrane action in a light-cone gauge. In fact, choosing a somewhat less restrictive gauge [4], one can show that the gauge-fixed membrane theory can be viewed as a one dimensional (time) gauge theory of $SDiff(S^2)$ [3][4]. To this end, consider the membrane action

$$I = \int d\tau d\sigma d\rho (\frac{1}{2}\sqrt{-g}g^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - \frac{1}{2}\sqrt{-g}), \tag{23}$$

where $i = \tau, \sigma, \rho$ labels the coordinates of the membrane world-volume with metric g_{ij}, and X^μ, where $\mu = 0, ..., d-1$, are the coordinates of a d-dimensional Minkowski spacetime with metric $\eta_{\mu\nu}$. The action is evidently invariant under the reparametrization of the world-volume, $\delta\sigma^i = \xi^i$. To fix this invariance, at least partially, we impose the following gauge conditions [4]

$$X^+ = \tau, \quad g^{00} = -h^{-1}, \qquad (24a), (24b)$$

where $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{d-1})$, $h = det g_{ab}$, and $a = \sigma, \rho$. The gauge (24a) fixes the time reparametrizations: $\xi^0 = 0$, and the gauge (24b) restricts the spatial (brane) reparametrizations to be area-preserving: $\partial_a\xi^a = 0$. Using (24) in the field equation $\frac{\delta I}{\delta X^-} = 0$, we then find that the following condition must be satisfied

$$\partial_a(hg^{0a}) = 0. \tag{25}$$

Thus, a further gauge condition is needed. We can impose the following gauge condition which is evidently consistent with (25):

$$g^{0a} = -\epsilon^{ab}h^{-1}\partial_b\omega, \tag{26}$$

where ω is the (gauge) field which must transform as

$$\delta\omega = \partial_0\Lambda + \epsilon^{ab}\partial_b\omega\partial_a\Lambda$$
$$\equiv \partial_0\Lambda + \{\omega, \Lambda\} \equiv D_0\Lambda. \tag{27}$$

Here $\Lambda(\tau,\sigma,\rho)$ is an arbitrary gauge parameter. With the gauge choices (24) and (26), one can show that the membrane action (23) reduces to

$$I = \int d\tau \int d\sigma d\rho [\frac{1}{2}D_0 X^r D_0 X^r - \frac{1}{4}\{X^r, X^s\}\{X^r, X^s\}], \quad r = 1, ..., d-2, \tag{28}$$

where $D_0 X^r = \partial_0 X^r + \{\omega, X^r\}$. The action (28) is invariant under

$$\delta\omega = D_0\Lambda, \quad \delta X^r = -\{\Lambda, X^r\}. \tag{29}$$

Thus we see that the action (28) is the gauge theory in one dimension (time) of the area-preserving diffeomorphisms of the membrane. Clearly, the role of the usual trace is played by $\int d\sigma d\rho$, and the role of the usual Yang-Mills commutator is played by the Poisson bracket $\{,\}$. A number of remarks are in order.

1. Although we considered $SDiff$ of a 2-sphere above, the underlying algebra, as one would expect, locally is the same for all 2-manifolds. An important difference arises when one considers the central extensions of the algebra. It was shown by Bars, Pope and the author [14] that for membranes the most general central extension is characterized by the space of harmonic 1-forms on the membrane, whose dimension is twice the genus of the membrane. The central extension of $SDiff(T^2)$, where T^2 is the 2-torus, had already been found by Floratos an Illiopoulos [13].

2. It was shown by Hoppe [2] that $SDiff(S^2)$ is isomorphic to $SU(\infty)$. The proof has been recently simplified by Fairlie and Zachos [15]. Furthermore, Pope and Stelle [16] have shown that $SDiff(T^2)$ is isomorphic to $SU_+(\infty)$.

3. Even for supermembranes [17], the residual symmetry in a suitable light-cone gauge is still $SDiff$ [3][4], and not its super-extension. Although a super-extension of $SDiff(T^2)$ has been found [14][3], apparently no underlying field theory is known at present. It is not ruled out that super-$SDiff(T^2)$ plays a role in the supermembrane theory, similar to the role played by the (light-cone) super-Virasoro algebra in manifestly spacetime supersymmetric string theories. Super-$SDiff(T^2)$ might also arise in a manifestly world-volume supersymmetric membrane theory, e.g. that of [18] in a suitable gauge, or in a new, yet to be discovered, "spinning membrane" theory.

4. It was shown in [4] that in a particular light-cone gauge the residual symmetries are the volume-preserving diffeomorphisms, $SDiff$, of the p-brane. In particular, in [4], the condition (25) was satisfied by imposing the gauge condition $g^{0a} = \partial_b\omega^{ab}$ with ω^{ab} transforming as a gauge field under $SDiff$. However, since $SDiff$ does not admit a Poisson bracket structure as its composition law, it may be more rewarding to impose a more resrictive gauge choice such that the residual symmetry is just the symplectic diffeomorphisms of the p-brane which does admit a Poisson bracket law. Such a gauge choice is indeed possible and is given by [19]

$$X^+ = \tau, \quad g^{00} = -h^{-1}, \quad g^{0a} = h^{-1}\Omega^{ab}\partial_b\omega, \qquad (30)$$

where Ω^{ab} is a symplectic structure on the (even) p-brane (which locally always exists), $h = det g_{ab}$, $a = 1,\ldots,p$, and ω is the gauge field for $Diff_\Omega$. It turns out that even for odd p-branes there is a natural way to break the diffeomorphism group down to $Diff_\Omega$ of an even dimensional p-brane submanifold. A detailed description will be given elsewhere [19].

5. One utility of $H(2,0)$ residual symmetry of the membrane is that the study of its anomalies (a highly interesting subject on which not much seems to be known at present) might shed light to the all important question of which super p-branes [20][17][21] are quantum consistent [22][23]. In this context, the quantum deformation of $H(2,0)$ may be of great interest [5].

6. The ultimate utility of $H(2,0)$ would arise if one can relate it to a spectrum generating algebra of the membrane, in much the same way the Virasoro algebra with a central extension arises as a spectrum generating algebra in the conformal gauge, where the Virasoro symmetry is also the residual symmetry. If such a connection indeed exists, the representations of $H(2,0)$ would play an important role in understanding the spectral properties of membranes.

4. Applications to Higher Spin Field Theories

Construction of consistent interacting field theories of higher spin fields, i.e fields of spin >2, is an old and fascinating problem. Whether the W-algebras mentioned above may lead to such theories [12] remains to be seen. Concrete progress was made few years ago by Fradkin and Vasiliev [24] who constructed a higher spin field theory in four dimensional anti-de Sitter (AdS) spacetime as a gauge theory of a higher spin algebra which is an infinite dimensional extension of the AdS algebra $SO(3,2) \approx Sp(4)$. It is obtained as an algebra of polynomials in oscillators q_α and $\tau_{\dot\alpha}$ which obey the relations: $[q_\alpha, q_\beta] = 2i\epsilon_{\alpha\beta}$, $[\tau_{\dot\alpha}, \tau_{\dot\beta}] = 2i\epsilon_{\dot\alpha\dot\beta}$, $[q_\alpha, \tau_{\dot\beta}] = 0$, $\tau_{\dot\alpha} = (q_\alpha)^\dagger$. The indices $\alpha, \beta, \ldots, \dot\alpha, \dot\beta = 1, 2$ are the spinor indices of $SO(3,1) \subset SO(3,2)$. Given an (even) ploynomial $P(q,\tau) = \sum_{k,n=0}^{\infty} P^{\alpha_1\cdots\alpha_k\dot\beta_{k+1}\cdots\dot\beta_{2n}} q_{\alpha_1}\cdots q_{\alpha_k}\tau_{\dot\beta_{k+1}}\cdots\tau_{\dot\beta_{2n}}$, the algebra of the polynomials is defined by the composition law $[P_1, P_2](q,\tau) = [P_1(q,\tau)P_2(q,\tau) - P_2(q,\tau)P_1(q,\tau)]$ [25]. In a similar expansion of the master gauge field, the components correspond to fields of $spin = (n+1)$. Fradkin and Vasiliev [24] have constructed an action in four dimensional AdS spacetime which is quadratic in the master curvature.

It is interesting to note that the Hamiltonian algebra $H(4,0)$ also provides an infinite dimensional extension of $Sp(4)$. This one is presumably not equivalent to the Fradkin-Vasiliev algebra, and it would be interesting to see whether it leads a more geometric, and simplified formulation of an alternative higher spin field theory. Another interesting possibility is to embed the AdS algebra in the symplectic *super*-diffeomorphisms of a symplectic *supermanifold* characterized by (13)-(15) with $q = 3$.

In the rest of this paper, we shall focus our attention on higher spin field theories in three dimensional (d=3) spacetimes. The problem is much more tractable there, and it has already led to interesting results due to the existence of the Chern-Simons (CS) form. Moreover, higher spin field theories based on CS actions may have interesting consequences for conformal field theories in d=2.

Few years ago, Blencowe [26] constructed a CS action for the analogue of the Fradkin-Vasiliev algebra in d=3. Recently, Bergshoeff, Blencowe and Stelle [6], in an attempt to provide a geometric formulation of this theory by using the area-preserving diffeomorphisms of a 2-hyperboloid, $SDiff(H^2)$, actually constructed an alternative, inequivalent theory. Sokatchev and the author [7] generalized this construction for symplectic super-diffeomorphisms of an arbitrary symplectic supermanifold. We conclude this paper by giving a brief description of the result.

Consider a symplectic supermanifold $\mathcal{M}^{2M,N}$ with a symplectic super 2-form Ω satisfying (7). Consider furthermore the infinite dimensional algebra corresponding to the symplectic super-diffeomorphisms characterized by (9). This algebra is given by (5) with the super-Poisson bracket defined in (11). The Hamiltonian function Λ can be viewed as a gauge parameter. We can gauge algebra of super-$Diff_\Omega$ by making this parameter local, $\Lambda(x,\sigma,\theta)$, in a spacetime with coordinates x^μ. We will consider a (2+1)-dimensional spacetime M^3. We can introduce a gauge field $\Gamma_\mu(x,\sigma,\theta)$ on $M^3 \times \mathcal{M}^{2M,N}$ which transforms as

$$\delta\Gamma_\mu = \partial_\mu\Lambda + \{\Gamma_\mu,\Lambda\}. \qquad (31)$$

The CS action on M^3 with super-$Diff_\Omega(\mathcal{M}^{2M,N})$ gauge symmetry is [7]

$$I_{CS} = \int d^3x \int d^{2M}\sigma d\theta^N \sqrt{sdetg}\,\epsilon^{\mu\nu\rho}(\Gamma_\mu\partial_\nu\Gamma_\rho + \frac{1}{3}\{\Gamma_\mu,\Gamma_\nu\}\Gamma_\rho), \qquad (32)$$

where $(sdetg)$ is the superdeterminant of the metric g_{AB} on the supermanifold. The action is invariant under local super-$Diff_\Omega$ transformations (31) provided that super-$Diff_\Omega$ is also supervolume preserving, i.e. leaves $d^{2M}\sigma d\theta^N\sqrt{sdetg}$ invariant. Noting that $\delta_\xi g_{AB} = 2\nabla_{(A}\xi_{B)}$ and $\delta_\xi(d^{2M}\sigma d\theta^N) = 0$, the volume-preservation condition means $(-1)^a\nabla_A\xi^A = 0$ (a is the grading), with ξ^A given in (9). This condition arises because of the need to partially integrate in showing the invariance of the action. The field equation which follows from (32) is

$$F_{\mu\nu}(x,\sigma,\theta) \equiv \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu + \{\Gamma_\mu,\Gamma_\nu\} = 0. \qquad (33)$$

The higher spin field theory based on an infinite dimensional extension of the $N=2, d=3$ AdS algebra $OSp(2,2)\oplus OSp(2,2)$ can be obtained as follows. We choose the supermanifold to be $\mathcal{M}^{2,4}$ with the bosonic submanifold a 2-hyperboloid, H^2, with the isometry group $SO(2,1)$. The four dimensional fermionic space is parametrized by Majorana spinors $\theta^{\alpha i}$, where $\alpha = 1,2, i = 1,2$ are the spinor indices of an $SO(2,1)$ and $SL(2,R)$, respectively. To obtain a higher spin field theory for an infinite tower of higher spin fields it is crucial to identify the this $SO(2,1)$ symmetry with the diagonal $SO(2,1)$ subgroup of the AdS group $O(2,2)\approx SO(2,1)\otimes SO(2,1)$. (See [7] for the details of the harmonic expansion of the gauge parameter.) To obtain the full AdS model we have to combine appropriately two copies of the CS actions. To see how this works, let us consider the finite dimensional supersymmetric truncation of the model, and see how the $N=2, d=3$ AdS supergravity emerges.

A finite dimensional supersymmetric truncation is given by the condition $\partial_a\Lambda(x,\sigma,\theta) = 0$, which is compatible with the composition law (5). In other words, the generators

$$D_\Lambda = C^{\alpha\beta}\epsilon^{ij}\partial_{\beta j}\Lambda(\theta)\partial_{\alpha i} \qquad (34)$$

form a closed superalgebra. The parameter $\Lambda(\theta)$ can be expanded as follows:

$$\Lambda = \Lambda_0 + \theta^{\alpha i}\epsilon_{\alpha i} + \frac{1}{2}\theta^{\alpha i}\theta^{\beta j}(\epsilon_{ij}\Lambda_{\alpha\beta} + \epsilon_{\alpha\beta}\Lambda_{ij}) + \frac{1}{3}\theta^{\alpha i}\theta^{\beta j}\theta_{\alpha j}\eta_{\beta i} + \frac{1}{4}(\theta^{\alpha i}\theta_{\alpha i})^2\Lambda_4 \qquad (35)$$

where $\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha}$ and $\Lambda_{ij} = \Lambda_{ji}$. The generators (34) with Λ given in (35), obey the Hamiltonian superalgbera $\bar{H}(N)$ which has $(2^4 - 1)$ generators [1]. Although the parameter Λ_0 drops out in (34), it does occur on the left-hand side of the composition law $\Lambda_3 = \Omega^{\alpha\beta}\partial_\beta\Lambda_1\partial_\alpha\Lambda_2$. We can add a generator corresponding to Λ_0 and obtain a 2^4 dimensional superalgebra, $C(4)$, associated with the Clifford algebra [7]. To obtain the $OSp(2,2)$, the diagonal subalgebra of the $N=2, d=3$ AdS superalgebra, we must truncate $C(4)$. In particular we must break the $SL(2,R)$ symmetry down to $SO(2)$. The correct truncation turns out to be [7]: $\Lambda_{ij} = \delta_{ij}\Lambda_0$, $\eta_{\alpha i} = \epsilon_{ij}\varepsilon_{\alpha j}$, $\Lambda_4 = 0$. With this truncation, using (34), one recovers precisely the $OSp(2,2)$ algebra. The corresponding gauge field is

$$\Gamma_\mu(OSp(2|2)) = A_\mu + \theta^{\alpha i}\psi_{\mu\alpha i} + \frac{1}{2}\theta^{\alpha i}\theta^{\beta j}(\epsilon_{ij}\Omega_{\mu\alpha\beta} + \delta_{ij}\epsilon_{\alpha\beta}A_\mu) + \frac{1}{3}\theta^{\alpha i}\theta^{\beta j}\theta_{\alpha j}\epsilon_{ik}\psi_{\beta k} \qquad (36)$$

Here $A_\mu(x)$, $\Omega_{\mu\alpha\beta}(x)$ and $\psi_{\mu\alpha i}(x)$ are the gauge fields for $SO(2)$, $SO(2,1)$ and supersymmetry, respectively. Inserting this into the CS action (32) and doing the θ-integral one obtains the following action

$$I_{CS} = \frac{1}{3}\int d^3x\epsilon^{\mu\nu\rho}(\Omega_\mu^{\alpha\beta}\partial_\nu\Omega_{\rho\alpha\beta} - 2A_\mu\partial_\nu A_\rho + 2\psi_{\mu i}^\alpha\partial_\nu\psi_{\rho\alpha i}$$
$$+\frac{2}{3}\Omega_{\mu\alpha}{}^\beta\Omega_{\nu\beta}{}^\gamma\Omega_{\rho\gamma}{}^\alpha - 2\psi_{\mu\alpha i}\psi_{\nu\beta i}\Omega_\rho^{\alpha\beta} + 2\epsilon^{ij}\psi_{\mu\alpha i}\psi_{\nu j}^\alpha A_\rho). \qquad (37)$$

To obtain the AdS action, one takes a second copy of the set of fields, A', Ω', ψ' [27][6][28] and write down a new action

$$I_{AdS} = I_{CS}(A,\Omega,\psi) - I_{CS}(A',\Omega',\psi'). \qquad (38)$$

After the introduction of the new variables $\Omega_\mu^{\alpha\beta} = m(\omega_\mu^{\alpha\beta} + e_\mu^{\alpha\beta})$ and $\Omega_\mu^{\prime\alpha\beta} = m(\omega_\mu^{\alpha\beta} - e_\mu^{\alpha\beta})$, where m is the inverse radius of AdS space, one can see that (38) is just the action for N=2 AdS supergravity in 2+1 dimensions based on the supergroup OSp(2|2)⊗OSp(2|2).

It would be interesting to quantize this theory and, in particular, find the solution space following from (33). This may lead to a connection with a new class of conformal field theories in d=2, á la Witten [29]. In the same vein, it is tempting to conjecture that a superconformal field theory of the super-W_∞ in d=2 is related to a Chern-Simons gauge theory, (32), based on the infinite dimensional Hamiltonian superalgebra H(2,1) given in (20).

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REFERENCES

[1] V.G. Kac, Advances in Math. **26** (1977) 8.

[2] J. Hoppe, MIT Ph. D. Thesis, 1982 and in *Proc. Int. Workshop on Constraints Theory and Relativistic Dynamics*, Eds. G. Longhi and L. Lusanna (World Scientific, 1987).

[3] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. **B305** (1988) 545.

[4] E. Bergshoeff, E. Sezgin, Y. Tanii and P. Townsend, *Super p-branes as gauge theories of volume preserving diffeomorphisms*, ICTP, Trieste, preprint, IC/88/186 (1988).

[5] I. Bakas, *The large N limit of extended conformal symmetries*, preprint (April 1989).

[6] E. Bergshoeff, M. Blencowe and K. Stelle, *Area-preserving diffeomorphisms and higher-spin algebras*, preprint, Imperial/TH/88-89/9.

[7] E. Sezgin and E. Sokatchev, *Chern-Simons theories of symplectic super-diffeomorphisms*, ICTP, Trieste, preprint, IC/89/72 (1989).

[8] C.N. Pope and P.K. Townsend, *Conformal higher spin in 2+1 dimensions*, preprint, CTP-TAMU-23/89.

[9] D.A. Leites, Soviet Math. Dokl. **18** (1977) 1277.

[10] M.A. Baranov and A.S. Schwarz, Funct. Anal. and Its Applic. **18** (1984) 130.

[11] V. Fateev and A.B. Zamolodchikov, Nucl. Phys. **B280** (1987) 644.

[12] A. Bilal and J.-L. Gervais, *Nonlinearly extended Virasoro algebras: New prospects for building string theories*, preprint, LPTENS 88/38 (December 1988).

[13] E. G. Floratos and J. Illiopoulos, Phys. Lett. **201B** (1988) 237.

[14] I. Bars, C.N. Pope and E. Sezgin, Phys. Lett. **210B** (1988) 85.

[15] D.B. Fairlie and C.K. Zachos, *Infinite-dimensional algebras, sine brackets and* $SU(\infty)$, preprint (February 1989).

[16] C.N. Pope and K.S. Stelle, $SU(\infty)$, $SU_+(\infty)$ *and area-preserving algebras*, preprint, CTP TAMU-07/89.

[17] E. Bergshoeff, E.Sezgin and P.K. Townsend, Phys. Lett. **189B** (1987) 75.

[18] U. Lindström and M. Roček, Phys. Lett. **218B** (1989) 207.

[19] E. Sezgin, in preparation.

[20] J. Hughes, J. Liu and J. Polchinski, Phys. Lett. **180B** (1986) 370.

[21] A. Achúcarro, J.M. Evans, P.K. Townsend and D.L. Wiltshire, Phys. Lett. **198B** (1987) 441.

[22] I. Bars, Nucl. Phys. **B308** (1988) 462.

[23] I. Bars and C.N. Pope, Class. & Quantum Grav. **5** (1988) 1157.

[24] E. Fradkin and M. Vasiliev, Ann. Phys. **77** (1987) 63.

[25] M. Vasiliev, Talk at Workshop on Superstrings, Trieste (April 1989).

[26] M.P. Blencowe, Class. & Quantum Grav. **6** (1989) 443.

[27] A. Achúcarro and P.K. Townsend, Phys. Lett. **180B** (1986) 89.

[28] E. Witten, Nucl. Phys. **311** (1988) 46.

[29] E. Witten, Commun. Math. Phys. **121** (1989) 351.

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