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In the case of complex Clifford algebras a basis is constructed whose elements satisfy projector relations. The relations are sufficient conditions for the elements to span minimal ideals and hence to define algebraic spinors.

I. INTRODUCTION

Clifford algebras have been used in theoretical physics in a number of applications[1-3] (Ref. 1 contains a list of references on the subject). In the geometric algebra type of application, Clifford algebra is used to express geometric transformations on a linear space. Letting a be an invertible element of the algebra, the group of invertible elements acts on the algebra by means of the transformation $x \to axa^{-1}$, where x is an arbitrary element. Clifford algebra is a graded algebra, and the underlying vector space is the grade 1 part. Invertible elements that leave the grade 1 part invariant form the Clifford group, which contains physically important groups, such as the orthogonal group, as subgroups. Elements belonging to even-grade parts of the algebra form a subalgebra, and in Ref. 4 spinors are defined as members of this subalgebra. The geometric interpretation of those spinors is that they act on the vector space as dilations rotations. Algebraic spinors are defined as elements of minimal left ideals; as shown in Sec. III this is equivalent to the original definition of the concept.[5] Minimal ideals of an algebra may be viewed as representation spaces for irreducible representations, and therefore algebraic spinors belong to an irreducible representation of the Clifford algebra.

Let C_n denote the Clifford algebra based on a complex n-dimensional vector space. Complexifying vector spaces is not in the spirit of geometrical algebra, but the basis used as the starting point in Sec. II is obtained from a basis with arbitrary signature by means of Witt's decomposition. The decomposition of C_n algebras into minimal ideals described below is constructive. Projectors are not postulated but expressed as Clifford products of isotropic basis vectors, a construction due to Schönberg.[6] The results concerning irreducible representations obtained below are not new. Deriving them in the framework of projector bases shows that the latter span minimal ideals. As all minimal left ideals yield equivalent representations, a fact which becomes apparent in the present approach is usually overlooked: any element of the algebra may be decomposed into a sum of algebraic spinors belonging to a set of minimal ideals. Left multiplication leaves left ideals invariant. The situation is reminiscent of irreducible subspaces under the action of a group. In the case of groups this leads to conservation laws; whether the same is true for algebras is a question left for further investigations. As an example testing the validity of the expansions obtained, it is shown that Cartan's matrix representation and defining relation for spinors are recovered in this way. Section II contains the treatment of C_{2m} Clifford algebras based on even-dimensional complex vector spaces, and Sec. III extends the results to C_{2m+1} Clifford algebras based on odd-dimensional complex vector spaces. The relation to Cartan's theory of spinors is described in Sec. IV.

II. C_{2m} ALGEBRAS

A set of basis vectors may be found such that

$$e_i \cdot e_j = \delta_{ij},$$
$$e'_i \cdot e'_j = -\delta_{ij},$$
$$e_i \cdot e'_j = 0, \quad i,j = 1,...,m.$$

The set of $2m$ vectors I_i, I^i defined by

$$I_i = \tfrac{1}{2}(e_i + e'_i),$$
$$I^i = \tfrac{1}{2}(e_i - e'_i)$$

are new basis vectors satisfying the scalar product relations

$$I_i \cdot I_j = 0,$$
$$I^i \cdot I^j = 0,$$
$$I_i \cdot I^j = \tfrac{1}{2}\delta_i^j.$$

Expressed as Clifford products the relations above are

$$I_i I_j + I_j I_i = 0,$$
$$I^i I^j + I^j I^i = 0, \qquad (1)$$
$$I_i I^j + I^j I_i = \delta_i^j.$$

The vectors I_i and I^i span isotropic subspaces, and the duality of the two sets is indicated by subscripts and superscripts. From the relations above it follows that, restricted to the isotropic subspaces, the Clifford algebra reduces to a Grassman algebra. The element

$$I^{1\cdots m} = I^1 I^2 \cdots I^m$$

has the property

$$I^i I^{1\cdots m} = 0, \qquad (2)$$

for all vectors I^i. The same is valid in the dual space. As the basis I_i, I^i is not orthogonal, we have to demonstrate a preliminary proposition.

Proposition A: Letting $e_1,...,e_n$ be the basis vectors of an n-dimensional vector space, the 2^n ordered Clifford products $e_{i_1} e_{i_2} \cdots e_{i_m}$ with $i_1 < i_2 \cdots < i_m$ and $m = 0,...,n$ are a basis of the algebra.

It is well known that exterior products of basis vectors are a basis of the algebra. In the case of orthogonal vectors Clifford products of vectors coincide with exterior products, and the proposition is trivial. In the general case the proposition results from the following relations, given without proof.

Proposition B: The Clifford product of r vectors can be expressed as the sum of all possible contractions of the exterior product:

$$X_1X_2\cdots X_r = \Big(1+\sum_{(ij)} C_{ij} + \sum_{(ij)(kl)} C_{kl}C_{ij} + \cdots\Big) X_1\wedge X_2\cdots\wedge X_r.$$

The contraction C_{ij} is defined as

$$C_{ij}(X_1\wedge X_2\cdots\wedge X_r) = (-)^{\nu}(X_i\cdot X_j)X_1\wedge X_2\cdots\hat{X}_i\cdots\hat{X}_j\cdots\wedge X_r,$$

where ν is the number of vectors between X_i and X_j, and $\hat{X}_i,\hat{X}_j$ indicates omission of the vectors. A converse of Proposition B is valid.

Proposition C: The exterior product of r vectors can be expressed as the sum of all possible anticontractions of the Clifford product:

$$X_1\wedge X_2\cdots\wedge X_r = \Big(1+\sum_{(ij)} \overline{C_{ij}} + \sum_{(ij)(kl)} \overline{C_{kl}C_{ij}} + \cdots\Big) X_1X_2\cdots X_r.$$

The anticontraction is defined as a contraction with opposite sign factor.

According to Proposition A, a basis of the algebra is given by the ordered Clifford products

$$I^{i_r\cdots i_1}I_{j_1\cdots j_s} = I^{i_r}\cdots I^{i_1}I_{j_1}\cdots I_{j_s}, \quad (3)$$

where subscripts are increasing, superscripts decreasing. Since the vectors in the subset I_j are orthogonal, $I_{j_1\cdots j_s}$ is an exterior product, and the same holds for $I^{i_r\cdots i_1}$. Following Schönberg[6] introduce elements of the algebra defined by

$$P^{i_r\cdots i_1}_{j_1\cdots j_s} = I^{i_r\cdots i_1}I_{1\cdots m}I^{m\cdots 1}I_{j_1\cdots j_s}. \quad (4)$$

We call these elements projectors to describe their properties outlined below. To simplify the notation let α,β be multi-indices, i.e., an ordered subset of $\{1,2,...,m\}$. By convention the ordering in superscripts is decreasing. The following propositions is crucial or the development below.

Proposition D: The projectors P^{α}_{β} satisfy the relations

$$P^{\alpha}_{\beta}P^{\mu}_{\nu} = \delta^{\mu}_{\beta}P^{\alpha}_{\nu}, \quad (5)$$

$$\sum_{\alpha} P^{\alpha}_{\alpha} = 1. \quad (6)$$

Relation (5), given by Schönberg,[6] may be obtained as follows. Let P be the projector

$$P = I_{1\cdots m}I^{m\cdots 1} = I_1I^1I_2I^2\cdots I_mI^m.$$

Since P is the product of commuting idempotent elements I_iI^i, P is idempotent: $P^2=P$.

Letting $\mu = 1,...,m$ *denote the* full set of indices, *from* relations (1), (2), and the dual of (2) the intermediate result

$$I^{\mu}I_{\beta}I^{\gamma}I_{\mu} = \delta^{\gamma}_{\beta}I^{\mu}I_{\mu}$$

is obtained. Relation (5) *follows then from the fact that* P is idempotent.

Relation (6) may be derived as follows: from (1) it follows that

$$P = \sum_{\alpha}(-)^{\nu(\alpha)}I^{\alpha}I_{\alpha},$$

where $\nu(\alpha)$ is the number of indices in the subset α. The sum includes the null set for which $\nu(\alpha)=0$, $I^{\alpha}=I_{\alpha}=1$. We have

$$\sum_{\beta}P^{\beta}_{\beta} = \sum_{\beta}I^{\beta}PI_{\beta} = \sum_{\alpha,\beta}(-)^{\nu(\alpha)}I^{\beta}I^{\alpha}I_{\alpha}I_{\beta};$$

only terms with $\alpha\cap\beta = 0$ are nonzero. Consider the sum of terms such that $\alpha\cup\beta=\gamma$:

$$\sum_{\beta}P^{\beta}_{\beta} = \sum_{\gamma=0}^{\mu}I^{\gamma}I_{\gamma}\sum_{\alpha=0}^{\gamma}(-)^{\nu(\alpha)}.$$

The rearrangement of indices to obtain an ordered set does not produce a sign change as it is done symmetrically in subscripts and superscripts. The proof is completed by realizing that for a finite set, the number of subsets with an even number of elements is equal to the number of subsets with an *odd number, so that* $\Sigma^{\gamma}_{\alpha=0}(-)^{\nu(\alpha)}=0$ *except for* $\gamma=0$, where the result is 1. The proposition about finite sets used above may be derived by induction. The proposition is valid for one-element sets that have two improper subsets: the null set and the full set.

To demonstrate the basis property of the projectors, we express them as linear combinations of ordered products and conversely. We have

$$P^{\alpha}_{\beta} = I^{\alpha}PI_{\beta} = \sum_{\gamma}(-)^{\nu(\gamma)}I^{\alpha}I^{\gamma}I_{\gamma}I_{\beta}$$

$$= \sum_{\gamma}(-)^{\nu(\gamma)}I^{\gamma\cup\alpha}I_{\gamma\cup\beta}.$$

Nonzero terms are those with $\gamma\cap\alpha=0$ and $\gamma\cap\beta=0$. Reordering of indices in the antisymmetric exterior products $I^{\gamma\cup\alpha}$ and $I_{\gamma\cup\beta}$ may be required to obtain ordered products. The converse relations are obtained using relation (6):

$$I^{\beta}I_{\gamma} = \sum_{\alpha}I^{\beta}P^{\alpha}_{\alpha}I_{\gamma} = \sum_{\alpha}I^{\beta}I^{\alpha}PI_{\alpha}I_{\gamma} = \sum_{\alpha}P^{\alpha\cup\beta}_{\alpha\cup\gamma}. \quad (7)$$

Conditions for nonzero terms and rearrangements are as before. Let X and Y be elements of the algebra expanded in a projector basis:

$$X = X^{\alpha}_{\beta}P^{\beta}_{\alpha},\quad Y = Y^{\gamma}_{\delta}P^{\delta}_{\gamma},$$

where $X^{\alpha}_{\beta}, Y^{\gamma}_{\delta}$ are complex numbers and the summation convention is used. From relation (5)

$$XY = (X^{\alpha}_{\beta}Y^{\gamma}_{\alpha})P^{\beta}_{\gamma}.$$

Let $C(2^m)$ designate a $2^m\times 2^m$ complex matrix. We have the following proposition.

Proposition E: C_{2m} algebras admit a $C(2^m)$ matrix representation.

The second consequence of relation (5) is stated as follows.

Proposition F: The idempotent projector P^{α}_{α} generates a minimal left ideal spanned by the projectors P^{β}_{α} with α fixed.

Let α_0 denote a fixed value of the multi-index α (the summation convention does not apply to α_0). We have

$$XP^{\alpha_0}_{\alpha_0} = X^{\gamma}_{\beta}P^{\beta}_{\gamma}P^{\alpha_0}_{\alpha_0} = X^{\alpha_0}_{\beta}P^{\beta}_{\alpha_0}.$$

Letting A denote the algebra, algebraic spinor members of the minimal left ideal $AP^{\alpha_0}_{\alpha_0}$ may then be expanded as

$$\eta = \eta_\beta P^{\beta}_{\alpha_0}.$$

To $X \in A$ the mapping $\rho(X): AP^{\alpha_0}_{\alpha_0} \to AP^{\alpha_0}_{\alpha_0}$ defined by $\eta \to X\eta$ can be associated; $\rho(X)$ is a representation of the algebra $\rho(XY) = \rho(X)\rho(Y)$. The representation is irreducible since $AP^{\alpha_0}_{\alpha_0}$ is minimal. A minimal ideal is generated by each idempotent projector P^{β}_{β}. The basis property of the projectors shows that the ideals do not overlap, and relation (6) that the algebra is completely reduced. All representations defined by the minimal ideals are equivalent, since a one-to-one mapping $AP^{\alpha}_{\alpha} \to AP^{\beta}_{\beta}$ is defined by $\eta \to \eta P^{\alpha}_{\beta}$. We have thus recovered a theorem already given by Weyl,[7] which states that central simple algebras have, up to equivalence, one irreducible representation contained in the regular representation with a multiplicity equal to the dimensions of the former. [The regular representation is the mapping $\rho(a): A \to A$ defined by $X \to aX$, and the representation space is obviously identical to the algebra.]

III. C_{2m+1} ALGEBRAS

There is now an unpaired basis vector I_0 with the following scalar products:

$$I_0^2 = 1,$$
$$I_0 \cdot I_i = 0,$$
$$I_0 \cdot I^i = 0.$$

Introduce the elements

$$I_{\pm} = \tfrac{1}{2}(1 \pm I_0).$$

It is easily seen that the following relations are satisfied:

$$I_{\pm}^2 = I_{\pm},$$
$$I_+ I_- = 0 = I_- I_+.$$

The products above are Clifford products. Let ω be an index with values $+$, $-$. As linear combinations of the elements of an ordered product basis, the 2^n elements $I^\alpha I_\omega I_\beta$ are a basis of the algebra. We define a set of projectors

$$P^{\alpha}_{\omega\beta} = I^\alpha I_\mu I_\omega I^\mu I_\beta. \tag{8}$$

By straightforward computation it can be checked that the projectors satisfy the relations

$$P^{\alpha'}_{\omega'\beta'} P^{\alpha''}_{\omega''\beta''} = \delta_{\omega',\omega''} \delta^{\alpha''}_{\beta'} P^{\alpha'}_{\omega'\beta''} \tag{9}$$

and

$$\sum_{\omega,\alpha} P^{\alpha}_{\omega\alpha} = 1.$$

The basis property of the projectors derives from the relations to the elements of an ordered product basis,

$$P^{\beta}_{\omega\gamma} = \sum_{\alpha} (-)^{\nu(\alpha)} I^{\alpha\cup\beta} I_{\omega\epsilon(\alpha)} I_{\alpha\cup\gamma},$$
$$I^\beta I_\omega I_\gamma = \sum_{\alpha} P^{\alpha\cup\beta}_{\omega\epsilon(\alpha)\alpha\cup\gamma}, \tag{10}$$

where $\epsilon(\alpha) = (-)^{m+\nu(\alpha)}$ is a sign factor. From relation (9) it follows that the projectors $P^{\alpha}_{\omega\beta}$ with ω fixed span a subalgebra, and that C_{2m+1} is the direct sum of the two subalgebras. The relation among components is now

$$(X^{+\alpha}_{\beta}, X^{-\alpha}_{\beta})(Y^{+\gamma}_{\delta}, Y^{-\gamma}_{\delta}) = (X^{+\alpha}_{\beta} Y^{+\gamma}_{\alpha}, X^{-\alpha}_{\beta} Y^{-\gamma}_{\alpha}),$$

and we have the following proposition.

Proposition G: C_{2m+1} algebras admit $C(2^m) \otimes C(2^m)$ matrix representations.

In this case, the proposition about minimal ideals becomes the following.

Proposition H: The idempotent projector $P^{\alpha}_{\omega\alpha}$ generates a minimal left ideal spanned by the projectors $P^{\beta}_{\omega\alpha}$ with ω and α fixed.

The theorem about irreducible representations used for C_{2m} algebras cannot be applied here, since C_{2m+1} algebras are not central and simple. According to the proposition above we have that C_{2m+1} algebras have two irreducible representations of dimension 2^m contained in the regular representation with multiplicity 2^m.

IV. CONNECTION WITH CARTAN'S THEORY OF SPINORS

We show that Cartan's matrix representation is contained in relation (10). Going back to indices, the expansions of the basis vectors are

$$I^i = P^i + \sum_j P^{ji}_j + \sum_{jk} P^{jki}_{jk} + \cdots,$$
$$I_0 = (-)^m \Big(P' - \sum_i P'^{i}_{i} + \sum_{ij} P'^{ij}_{ij} - \cdots \Big), \tag{11}$$
$$I_i = P_i + \sum_j P^{j}_{ji} + \sum_{jk} P^{jk}_{jki} + \cdots.$$

Here $P = P_+ + P_-$ and $P' = P_+ - P_-$. Numerical results for $m = 2$, $n = 5$ are

$$I^1 = P^1 - P^{12}_{2},$$
$$I^2 = P^2 + P^{12}_{1},$$
$$I_0 = P' - P'^{1}_{1} - P'^{2}_{2} + P'^{12}_{12},$$
$$I_1 = P_1 - P^{2}_{12},$$
$$I_2 = P_2 + P^{1}_{12}.$$

If multi-indices are ordered none, 1, 2, 12, the matrices of components relative to the projectors $P^{\alpha}_{+\beta}$ are

$$H^1 = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{Bmatrix},$$

$$H^2 = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix},$$

$$H_0 = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix},$$

$$H_1 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{Bmatrix},$$

$$H_2 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}.$$

The matrices are obtained by inspection of the preceding relations and are identical with Cartan's expression[8] (subscripts are identified with Cartan's primed indices). Cartan's system of equations defining spinors[9] is obtained by working out the components of the relation

$$\eta = X\xi,$$

where X is a vector,

$$X = X_i I^i + X^0 I_0 + X^i I_i,$$

and ξ, η are algebraic spinors,

$$\eta = \sum_{p=0}^{m} \eta_{i_1 \cdots i_p} P_+^{i_1 \cdots i_p},$$

$$\xi = \sum_{p=0}^{m} \xi_{i_1 \cdots i_p} P_+^{i_1 \cdots i_p}.$$

The result is obtained using (11) and (8):

$$\eta_{i_1 \cdots i_p} = \sum_{k=1}^{p} (-)^{p-k} X_{i_k} \xi_{i_1 \cdots \hat{i}_k \cdots i_p} + (-)^p X^0 \xi_{i_1 \cdots i_p} + \sum_j X^j \xi_{i_1 \cdots i_p j}.$$

This is Cartan's relation. However, Cartan's theory goes beyond the relation given above. The concept of pure spinor is defined by nonlinear constraints on the components of algebraic spinors, and lies outside of the theory of irreducible representations.

V. CONCLUSION

The construction of projector bases leads to the results of irreducible representation theory by a simple algebraic approach. All quantities involved in the approach, projectors and spinors, are members of the algebra. A projector basis is obviously the basis appropriate to calculations involving algebraic spinors.

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