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WHAT CAN WE DO WITH ONLY A PAIR OF RUSTY COMPASSES? *

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ABSTRACT

Compasses are called rusty, if one can draw only the unit circle with them. We prove that: From two points A and B, with only rusty compasses, one can draw the points of k–section of AB, and all the vertices of a regular n–gon which has a side AB, where k is any integer greater than 1, and $n = 3, 4, 5, 6, 8, 12, 17, 257, \ldots$ etc. Generally, let A be $(0,0)$ and B be $(\lambda, 0)$, then one can draw all the points $(\lambda x, \lambda y)$ where x and y are any elements in some regular 2^m–extension of the rational field, for $m = 1, 2, 3, \ldots$

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1. INTRODUCTION

A pair of compasses is called rusty, if we can draw only the unit circle with it. What can we do with only a rusty compass? This is an old problem [1]. As we know, no one had given any solution exactly. Maybe many people think that it is hard to do some things with only a rusty compass.

Occasionally, a surprising discovery opens a new page on the problem. Several years before, Professor D. Pedoe found an interesting fact: given two points A, B but without the segment AB, if $AB < 2$, it is easy to find a point C such that $AC = BC = AB$, with only a rusty compass [2].

See Fig.1, there are five unit circles $\odot A, \odot B, \odot F, \odot D$ and $\odot E$. And the process of construction is that $F \in \odot A \cap \odot B, D \in \odot F \cap \odot A$, $E \in \odot F \cap \odot B$ and $C = (\odot D \cap \odot E) \backslash F$. Here we choose D and E such that both $A - F - D$ and $F - B - E$ are in counterclockwise order. So we have $AC = BC = AB$.

The proof is simple. It is enough to point out that

$$\frac{1}{2} \angle ACB = \angle FCB = \frac{1}{2} \angle FEB = 30^\circ .$$

So $\angle A = \angle B = \angle C = 60^\circ$. That is all.

This discovery encouraged Pedoe to ask more things. But he did not ask too much. He proposed only two simple problems carefully. The two problems are as follows:

Problem 1 Given two points A and B but without segment AB, prove or disprove that the third vertex C of the regular ΔABC can be found when $AB \geq 2$, using only a pair of rusty compasses [3].

Problem 2 Given two points A and B but without segment AB, prove or disprove that the midpoint M of the segment AB can be found, using only a pair of rusty compasses [4].

Three Chinese (L. Yang, X.R. Hou and I) have given not only positive solutions to Pedoe's two problems but also much further results [5]-[9]. An unexpected conclusion is that the effect of a rusty compass is equivalent to a ruler and a fine compass when the construction was started from only two points. But most of these works have not been published in English. Here we will sum up all of these results by a more simple method and a more clear way.

By the way, some geometers remarked that it is impossible to divide segments with rusty compasses [10].

For convenience, some notions will be used throughout the paper:

(1) $UC(X)$ – the unit circle with centre X.

(2) RC – construction with only a rusty compass.

(3) $Z = X\nabla Y$ means that ΔXYZ is a regular triangle with vertices labelled $X - Y - Z$ in counterclockwise order.

(4) $Z = \square(WXY)$ means that $WXYZ$ is a parallelogram with vertices labelled $W - X - Y - Z$ in counterclockwise order.

(5) If there exists a series of points $A = X_0, X_1, X_2, ..., X_n, X_{n+1} = B$ such that $X_kX_{k+1} = \lambda$, $0 \leq k \leq n$, then we say that this is a λ–bridge between A and B.

2. THE SOLUTION TO PROBLEM 1

The main difficulty we face first is that the distance between two given points A and B may be too large for our rusty compass. So we would build a bridge to connect the two points:

RC1 Given two points A and B, a series of points $\{X_k\}$ can be found as a 1–bridge between A, B. (Of course, using only a compass which is rusty. This remark will be omitted in all **RC** throughout the paper.)

Construction Take any one point $P_1 \in UC(A)$, let Q_1 be the intersection point of $UC(A)$ and $UC(P_1)$ such that $Q_1 = A\nabla P_1$. Then draw $Q_2 = Q_1\nabla P_1$, $P_2 = Q_2\nabla P_1$, $Q_3 = Q_2\nabla P_2$, $R_1 = Q_1\nabla Q_2$, ... So we can obtain a point–array which consists of all the vertices of these regular triangles covering whole plane as in Fig.2. These regular triangles have unit side, so the point–array will be called as a cobweb with parameter 1. It is easy to find a point Q in the cobweb such that $UC(Q) \cap UC(B) \neq \emptyset$. Then the bridge has been built.

□

RC2 Given three points A, B and C, the point D can be found such that $D = \square(ABC)$.

Construction By RC1, we can obtain $\{P_k\}$ and $\{Q_\ell\}$ as 1–bridges between B, A and B, C respectively. Then our work will consist of a series of constructions of rhombus with unit sides like Fig.3. And induction can be used to $m + n$.

□

RC3 Suppose there are two series of points $\{P_0, P_1, ..., P_n\}$ and $\{Q_1, Q_2, ..., Q_n\}$ such that

1. $\Delta Q_kP_{k-1}P_k \backsim \Delta Q_{k+1}P_kP_{k+1}$, $(k = 1, 2, ..., n-1)$.
2. $Q_k - P_{k-1} - P_k$ are in counterclockwise order along the boundary of $\Delta Q_kP_{k-1}P_k$.

Then the point Q can be found such that $\Delta QP_0P_n \backsim \Delta Q_1P_0P_1$.

Construction First see the case of $n = 2$. By **RC2** we can find a point $Q = \square(Q_1P_1Q_2)$. Then Q is the point we want. The proof could be taken as an exercise in elementary geometry. Here we give a simple proof using complex numbers. Let P_0 be the origin of the complex plane, and

Q_1, P_1, Q_2, P_2 are represented by complex numbers Z_1, Z_2, Z_3, Z_4 respectively. So the conditions 1 and 2 can be represented as:

$$\frac{Z_1}{Z_2} = \frac{Z_3 - Z_2}{Z_4 - Z_2} = Z^* .$$

Set Q as Z_5, then the fact $Q = \square(Q_1P_1Q_2)$ can be represented as $Z_5 - Z_1 = Z_3 - Z_2$. So we have

$$\frac{Z_5}{Z_4} = \frac{Z_5 - Z_1 + Z_1}{Z_4 - Z_2 + Z_2} = \frac{Z_3 - Z_2 + Z_1}{Z_4 - Z_2 + Z_2} = Z^* .$$

It means that $\Delta QP_0P_2 \backsim \Delta Q_1P_0P_1$ and the vertices are around in the same direction respectively (Fig.4).

For the case $n \geq 3$, the induction can be used as Fig.5.*

RC4 (The solution to Problem 1) Given two points A and B, one can find $C = A\nabla B$.

Construction By RC1, find $\{P_k\}$ as a 1 bridge between A, B. It is easy to find $Q_k = P_{k-1}\nabla P_k$. Then use RC3.

□

In the following we will see that the RC4 is a very useful tool, and the idea in this section could be developed to solve general problems.

3. TO ANSWER THE MIDPOINT PROBLEM

RC5 Given points X and Y, can find Z such that

$$XZ = 2XY = 2YZ$$

and the three points are collinear.

Construction $Z = ((X\nabla Y)\nabla Y)\nabla Y$. (Fig.6).

RC6 Given a point X, can find a point Y such that

$$XY < 2/\sqrt{19} .$$

Construction

1. Take an arbitrary point $P \in UC(X)$.
2. Find a point $Q = P\nabla X \in UC(X) \cap UC(P)$.
3. Take an arbitrary point $A \in \overset{\frown}{PQ}$, distinct from P, Q.

* **Remark for RC3**: By this proof it is obvious that the RC3 still be valid when $\Delta Q_kP_{k-1}P_k$ are degenerate.

4. Find $B = A\nabla P$, then $BA \perp AQ$ and $\frac{\sqrt{2}}{2} < BQ < 1$.

(Because:
(1) $BQ^2 = AB^2 + AQ^2 = AP^2 + AQ^2 < AP^2 + AQ^2 - 2AP \cdot AQ\cos\angle PAQ = PQ^2 = 1$
(2) $BQ^2 = AB^2 + AQ^2 \geq \frac{1}{2}(AB + AQ)^2 > \frac{1}{2}(AP + AQ)^2 = \frac{1}{2}$).

5. Find $E = Q\nabla B$, and take $D \in UC(B) \cap UC(Q)$ such that $D - Q - B$ are set in counterclockwise order (see Fig.7). Then:

$$DE < \sqrt{1 - \left(\frac{\sqrt{2}}{4}\right)^2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{\frac{7}{8}} - \frac{\sqrt{6}}{4} < \frac{2}{\sqrt{19}}.$$

6. Find $Y = \square(DEX)$ by RC2.

□

RC7 Given two points X and Y, can find a point Z such that $XZ = YZ$ and $XZ = \sqrt{3k^2 + 3k + 1}\, XY$. Here k may be any natural number: $k = 1, 2, 3, \ldots$

Construction It is obvious to see Fig.8. Here we gave the cases for $k = 1, 2, 3$.

In Fig.8, the pagoda was piled up by some regular triangles with a side which equals XY. So we have:

$$XZ_1 = \sqrt{\left(\left(1 + \frac{1}{2}\right)\sqrt{3}\right)^2 + \left(\frac{1}{2}\right)^2}\, XY = \sqrt{7}\, XY$$

$$XZ_2 = \sqrt{\left(\left(2 + \frac{1}{2}\right)\sqrt{3}\right)^2 + \left(\frac{1}{2}\right)^2}\, XY = \sqrt{19}\, XY$$

$$XZ_3 = \sqrt{\left(\left(3 + \frac{1}{2}\right)\sqrt{3}\right)^2 + \left(\frac{1}{2}\right)^2}\, XY = \sqrt{37}\, XY.$$

And generally,

$$XZ_k = \sqrt{\left(\left(k + \frac{1}{2}\right)\sqrt{3}\right)^2 + \left(\frac{1}{2}\right)^2}\, XY = \sqrt{3k^2 + 3k + 1}\, XY.$$

□

RC8 Given three points X, Y and Z such that

$$XZ = YZ = aXY \leq 2.$$

Then one can find a point P satisfying the condition $PX = PY = \frac{1}{a}$.

Construction Suppose $X - Y - Z$ are set in counterclockwise order, Choose $A \in UC(Z) \cap UC(X)$ and $B \in UC(Z) \cap UC(Y)$ such that both $A - X - Z$ and $B - Z - Y$ are in counterclockwise order. Let $UC(A) \cap UC(B) \backslash Z = P$, then P is the point we want (Fig.9).

The proof is easy. It is obvious that

$$\angle XPY = 2(\angle PZY + \angle PYZ) = \angle ZBY.$$

So $\Delta XPY \backsim \Delta ZBY$. Then we have $\frac{PX}{XY} = \frac{BZ}{ZY} = \frac{1}{ZY}$, which is what we want.

□

RC9 Given points X and Y with $XY = 1/\sqrt{19}$, then one can find the midpoint of the segment XY.

Construction Find points Z and Z' such that Z', X, Y and Z are collinear and $Z'X = 2XY = 2YZ$ by RC5. Let $UC(Z) \cap UC(Z') = \{P, P'\}$, then $XP = XP' = \sqrt{15/19}$ and $YP = YP' = 4/\sqrt{19}$ (Fig. 10).

Find point Q such that P, Y and Q are collinear and $PY = YQ$. Take $A \in UC(P) \cap UC(Q)$, then $YA = \sqrt{\frac{3}{19}}$ and $AY \perp PY$. Find point B such that B, Y and A are collinear and $YA = YB$. Then take $C = B\nabla A$, C must fall on the segment YP and $YC = 3/\sqrt{19}$. Find point D such that P, C, D and Y are collinear and $PD = PC + CD = 3PC = 3/\sqrt{19}$, so $YD = 1/\sqrt{19}$. Analogously, find D' on the segemnt YP' and $YD' = 1/\sqrt{19}$ (Fig.10).

Find the point $M = \square(D'YD)$, then M is the midpoint of the segment XY (Fig.10).

□

RC10 Given points A and B, one can find a series of points $\{P_k\}$ as a $\frac{1}{\sqrt{19}}$–bridge between A, B.

Construction

1. Find X_1 such that $AX_1 < \frac{2}{\sqrt{19}}$ (by RC6).
2. Make the cobweb with parameter $d = AX_1$, starting from points A and X_1.
3. Choose points $X_0 = A, X_1, X_2, \ldots, X_{\ell-1}$ from the cobweb such that $X_{k-1}X_k = d$ and $X_{\ell-1}B \leq d$. Let $X_\ell = B$.
4. Find Y_k such that $Y_kX_{k-1} = Y_kX_k = \sqrt{19}X_{k-1}X_k$ (by RC7).
5. Find Z_k such that $Z_kX_{k-1} = Z_kX_k = \frac{1}{\sqrt{19}}$ (by RC8).

Then let $P_{2m} = X_m, P_{2m+1} = Z_m, \{P_k\}$ is the set we want.

□

RC11 (The solution to Problem 2) Given points A and B, one can find the midpoint of the segment AB.

Construction By RC10 , find $\{P_k\}$ as a $\frac{1}{\sqrt{19}}$–bridge of A, B. Then find the midpoint Q_k of the segment $P_{k-1}P_k$ (by RC9). At last the degenerate case of RC3 could be used.

□

4. THE GENERAL RESULTS

Now we represent every point X in the plane by a complex number z_x. For given two points A and B it will always be assumed that $z_A = 0$ and $z_B = t$, here $t > 0$ is a real number. If the point $z = z^* t$ can be found using only a rusty compass, we say $z^* \in L(t)$. If $z^* \in L(t)$ for every real number $t > 0$, we say $z^* \in L$.

Our main result is the following:

The general RC Theorem Let set L as above, then:

1. $(\sqrt{-1})^k\sqrt{n} \in L$ for any natural numbers k and n,
2. $z_1 \pm z_2 \in L$ if $z_1 \in L$ and $z_2 \in L$,
3. $z_1 \cdot z_2 \in L$ if $z_1 \in L$ and $z_2 \in L$,
4. If $z \in L$, then $\overline{z} \in L$,
5. If $z \in L$ and $z \neq 0$, then $\frac{1}{z} \in L$,
6. If $z \in L$, then $\sqrt{z} \in L$.

We need a series of lemmas.

Lemma 1 If $z_1 \in L(t)$, $z_2 \in L(t|z_1|)$, then $z_1 \cdot z_2 \in L(t)$.

Lemma 2 If $z_1 \in L(t)$ and $z_2 \in L(t)$, then $z_1 \pm z_2 \in L(t)$.

Lemma 3 If $z \in L(t)$, then $\overline{z} \in L(t)$.

These lemmas are obvious: Lemma 1 is true by the definition of $L(t)$, Lemma 2 is by RC2 and Lemma 3 is by symmetry.

To obtain some non–trivial results, we introduce a new set S: If a d–bridge between any two points A and B can always be built with only a rusty compass for some fixed real number $d > 0$, then we say $d \in S$. By RC3 we have

Lemma 4 If $d \in S$ and $z \in L(d)$, then $z \in L$.

Lemma 5 For any integers m and k

$$\left(m + \frac{1}{2}\right) + i\left(k + \frac{1}{2}\right)\sqrt{3} \in L, \quad m \in L, \quad ik\sqrt{3} \in L$$

where $i = \sqrt{-1}$ throughout the paper.

It is obvious that $1 \in S$ (RC1) and $e^{i\pi/3} \in L(1)$, so $e^{i\pi/3} \in L$ by Lemma 4. (This is the solution to Problem 11) Notice that $e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, we can obtain Lemma 5 by Lemma 2 and Lemma 3.

Lemma 6 Let $f(d, \lambda) = \sqrt{\frac{1}{d^2} - \lambda^2}$. If $0 < \lambda \in L$, $d \in S$ and $0 < \lambda d < 1$, then $if(d, \lambda) \in L$.

Proof: The points $z_A = \lambda d$ and $z_B = -\lambda d$ can be found by $\lambda \in L$. So $z_P = \pm i\sqrt{1 - \lambda^2 d^2}$ (by $UC(A) \cap UC(B)$) can be found. Then $z_P/d = \pm i\sqrt{d^{-2} - \lambda^2} \in L(d)$. By Lemma 4, $i\sqrt{d^{-2} - \lambda^2} \in L$.

□

Lemma 6 can be shown by Fig.11.

Lemma 7 $\frac{1}{2} \in L$. (This is RC11.)

Lemma 8 If $\lambda > 0$ and $i\lambda \in L$, then $|\frac{1}{2} + i\lambda|^{-1} \in S$.

To prove Lemma 8, notice $1 \in S$, and by RC8, RC11.

Lemma 9 If $d \in S$ and $i\lambda \in L$ with $\lambda > 0$, then $|(\frac{1}{2} + i\lambda)d| \in S$.

Lemma 9 is obvious by RC8 and RC11.

Lemma 10 $(3k^2 + 3k + 1)^{1/2} \in S$ for every natural number k.

By RC7, RC8 and RC11 we have Lemma 10.

Lemma 11 If $d \in S$ and $0 < d < \frac{2}{3}$, then $\left(\frac{1}{d^2} - 2\right)^{-1/2} \in S$.

Proof: By Lemma 7 and Lemma 6

$$\pm i\sqrt{\frac{1}{d^2} - \left(1 + \frac{1}{2}\right)^2} = \pm i f\left(d, 1 + \frac{1}{2}\right) \in L.$$

Then

$$\left(\frac{1}{d^2} - 2\right)^{-1/2} = \left|\frac{1}{2} \pm i\sqrt{\frac{1}{d^2} - (1 + \frac{1}{2})^2}\right|^{-1} \in S$$

by Lemma 8.

□

Lemma 12 $(2m + 1)^{-1/2} \in S$ for every natural number m.

To prove Lemma 12, start with $d = (3k^2 + 3k + 1)^{-1/2} \in S$ (Lemma 10) then use Lemma 11 again and again.

Lemma 13 $(i)^k\sqrt{n} \in L$ for any natural number k and n.

Proof: Take natural numbers m and ℓ such that $\ell < \sqrt{2m+1}$, then $(2m+1)^{-1/2} \in S$ by Lemma 12 and $\ell \in L$ by Lemma 5. And by $\ell(2m+1)^{-1/2} < 1$ we can use Lemma 6, so

$$i\sqrt{2m+1-\ell^2} = i\, f\Big((2m+1)^{-1/2}, \ell\Big) \in L\,.$$

We can choose m and ℓ such that $2m+1-\ell^2 = n$ for any natural number n. And we have $i \in L$ while $m = \ell = 2$.

□

Lemma 14 If $\lambda \in L$ and $\frac{1}{2} \le \lambda < 2$, then $\frac{1}{\lambda} \in S \cap L$.

Proof: By $\lambda \in L$, and Lemma 8, Lemma 13, Lemma 6:

$$\sqrt{\frac{1}{2} - \lambda^2 + m^2} = f\left(\frac{1}{|\frac{1}{2} + \frac{i}{2}\sqrt{4m^2+1}|}, \lambda\right) \in L\,.$$

Here m is some natural number greater than λ. For instance we can take $m = [\lambda] + 2$.

So by Lemma 6 and Lemma 8:

$$\sqrt{\lambda^2 - \frac{1}{4}} = f\left(\frac{1}{|\frac{1}{2} + im|}, \sqrt{\frac{1}{2} - \lambda^2 + m^2}\right) \in L$$

so by Lemma 8:

$$\left|\frac{1}{2} + i\sqrt{\lambda^2 - \frac{1}{4}}\right|^{-1} \frac{1}{\lambda} \in S$$

so by Lemma 6:

$$\sqrt{\frac{1}{\lambda^2} - \frac{1}{4}} = f\Big(\frac{1}{\lambda}, \frac{1}{2}\Big) \in L\,.$$

So

$$\frac{1}{2} \pm i\sqrt{\frac{1}{\lambda^2} - \frac{1}{4}} \in L$$

and

$$\frac{1}{\lambda} = \lambda \cdot \frac{1}{\lambda^2} = \lambda \cdot \left(\frac{1}{2} + i\sqrt{\frac{1}{\lambda^2} - \frac{1}{4}}\right)\left(\frac{1}{2} - i\sqrt{\frac{1}{\lambda^2} - \frac{1}{4}}\right) \in L\,.$$

□

Lemma 15 If a real number $\lambda \in L$ and $\lambda \neq 0$, then $\frac{1}{\lambda} \in L$.

Proof: We can suppose $\lambda > 0$ because $-1 \in L$.

Take some integer k such that $\frac{1}{2} \le 2^k\lambda < 2$, then $2^k\lambda \in L$ for $2 \in L$ and $\frac{1}{2} \in L$. By Lemma 14 we have $\frac{1}{2^k\lambda} \in L$, then $\frac{1}{\lambda} = 2^k \cdot \frac{1}{2^k\lambda} \in L$.

□

Lemma 16 If $0 < \lambda \in L$, then $\sqrt{\lambda} \in L$.

Proof: Suppose $\lambda < 1$ and $\lambda \neq \frac{1}{2}$. (When $\lambda = \frac{1}{2}$ we know $\frac{1}{\sqrt{2}} \in L$ by Lemma 13 and Lemma 15. When $\lambda > 1$ we can also use Lemma 15 and replace λ by $\frac{1}{\lambda} < 1$.)

Because $\lambda \pm \frac{1}{2} \in L$ and $\frac{1}{2} < \lambda + \frac{1}{2} < 2$ so $\frac{1}{\lambda + \frac{1}{2}} \in S$ by Lemma 14. Then using Lemma 6 we have

$$i\sqrt{2\lambda} = i\, f\Big(\frac{1}{\lambda + \frac{1}{2}}, \lambda - \frac{1}{2}\Big) \in L$$

and $\sqrt{\lambda} \in L$ because $i \in L$, $\frac{1}{\sqrt{2}} \in L$ and $-1 \in L$.

□

Lemma 17 If $z \in L$ and $z \neq 0$, then $\frac{1}{z} \in L$.

Proof: For $z \in L$ we have $z \cdot \bar{z} = |z|^2 \in L$, so $|z|^{-2} \in L$ by Lemma 15. Then $\frac{1}{z} = \bar{z} \cdot |z|^{-2} \in L$.

□

Lemma 18 If $z \in L$, then $\sqrt{z} \in L$.

Proof: Let $z = \lambda e^{i\theta} = \lambda(\cos\theta + i\sin\theta)$ and $\lambda > 0$. So

$$\sqrt{z} = \sqrt{\lambda}\Big(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\Big)\,.$$

Since $z \in L$ we have $|z|^2 = z \cdot \bar{z} \in L$ and so $\lambda = \sqrt{|z|^2} \in L$ by Lemma 16, and $\frac{1}{\lambda} \in L$ by Lemma 15 so $(\cos\theta + i\sin\theta) \in L$. Then $\cos\theta \in L$, $\sin\theta \in L$. And so

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} \in L, \quad \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \in L\,.$$

Now it is obvious that $\sqrt{z} \in L$.

□

The general RC Theorem has been proved.

Problem Given three points A, B and C, prove or disprove that the point C^* can be found such that C^* and C are symmetric with respect to the straight line AB, using only a rusty compass.

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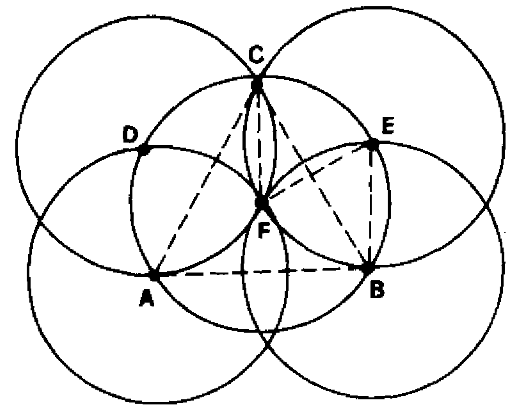

Fig.1

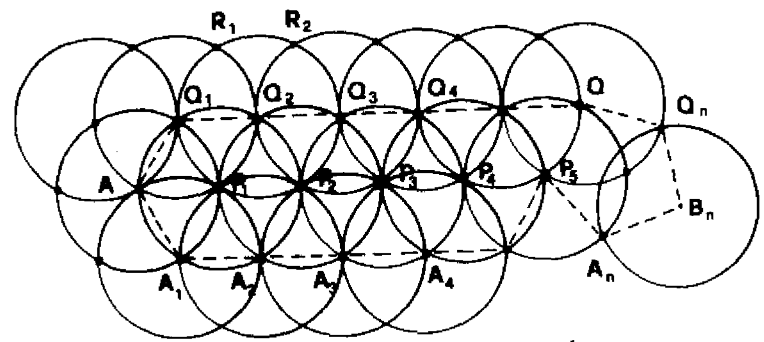

Fig. 2

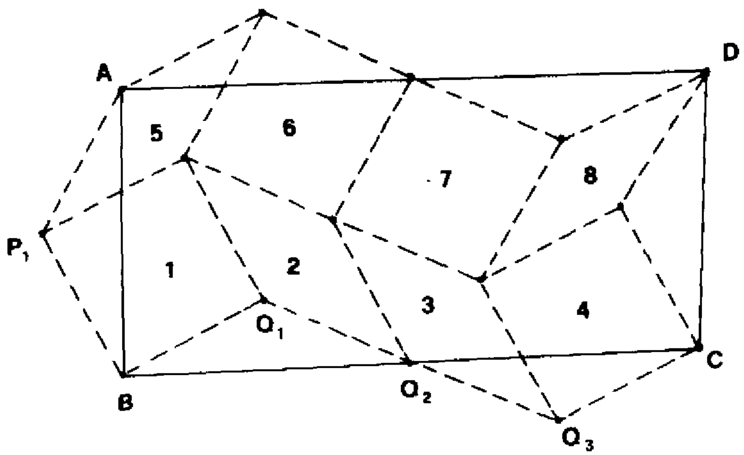

Fig.3

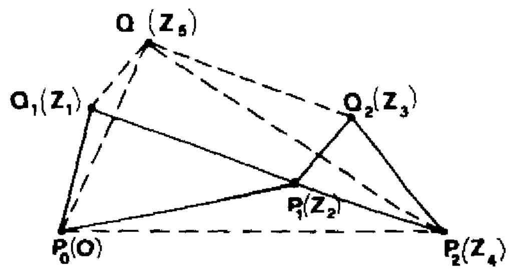

Fig.4

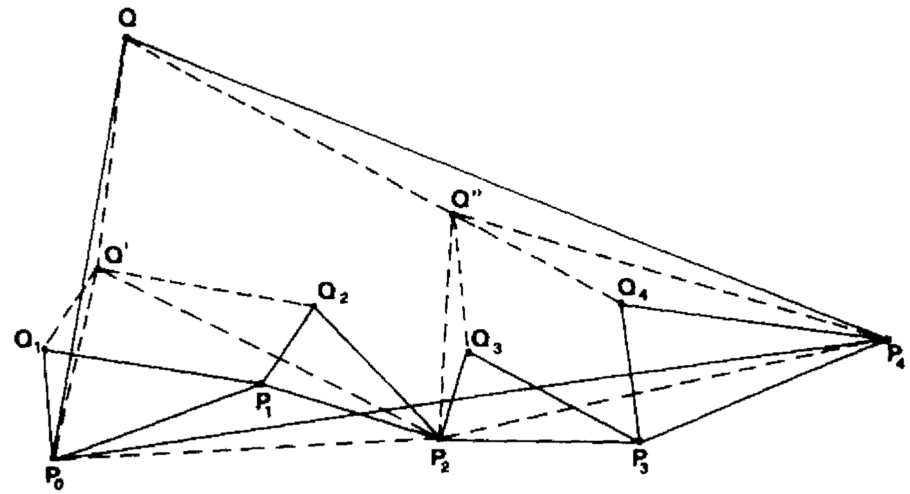

Fig.5

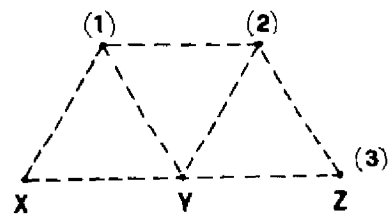

Fig.6

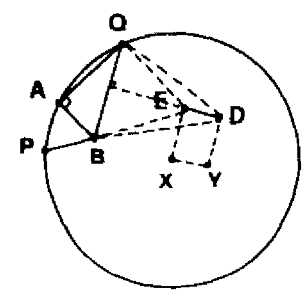

Fig.7

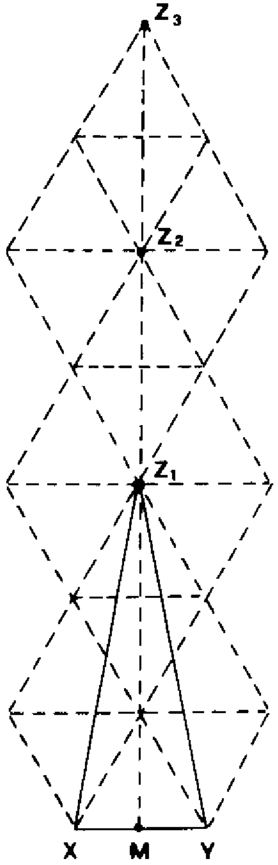

Fig.8

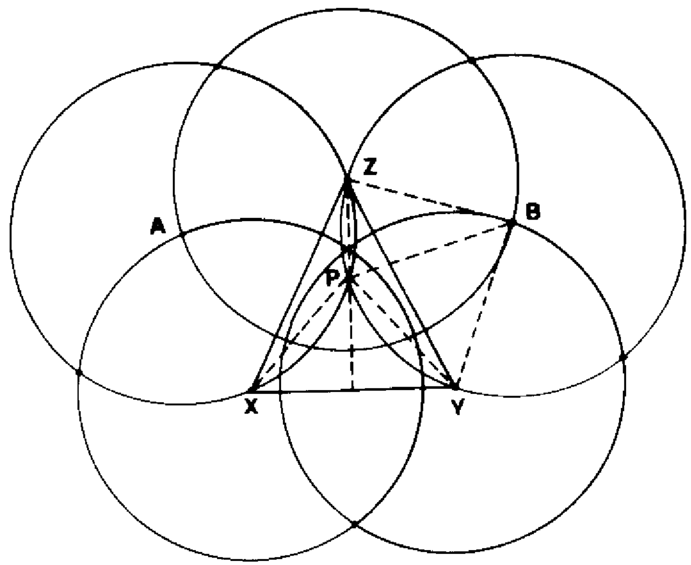

Fig.9

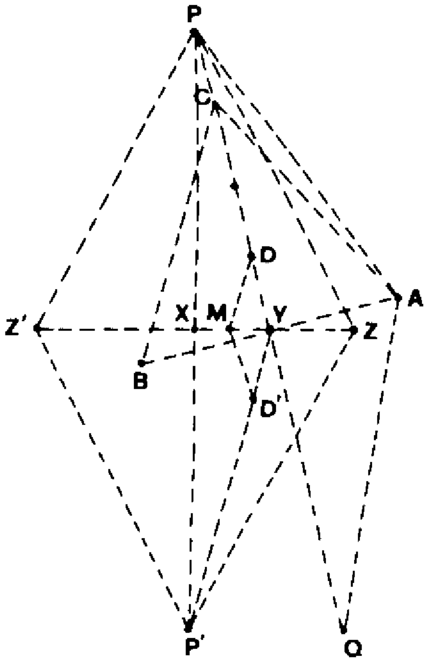
P
C
D
A
Z'
X
M
Y
Z
B
D'
P'
Q

Fig.10

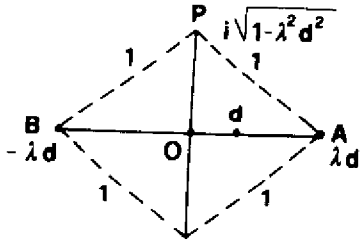
P
$i\sqrt{1-\lambda^2 d^2}$
1
1
B
$-\lambda d$
d
O
A
λd
1
1

Fig.11

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