IC/89/86

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

AN APPROXIMATION METHOD FOR NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE

C.E. Chidume

and

Chika Moore

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1989 MIRAMARE - TRIESTE

IC/89/86

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

AN APPROXIMATION METHOD FOR NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE *

C.E. Chidume **
International Centre for Theoretical Physics, Trieste, Italy
and
Chika Moore
School of Postgraduate Studies, University of Nigeria,
Nsukka, Nigeria.

ABSTRACT

The solution of a nonlinear integral equation of Hammerstein type in Hilbert spaces is approximated by means of a fixed point iteration method. Explicit error estimates are given and, in some cases, convergence is shown to be at least as fast as a geometric progression.

MIRAMARE – TRIESTE
May 1989

* To be submitted for publication.
** Permanent address: Department of Mathematics, University of Nigeria, Nsukka, Nigeria.

1. INTRODUCTION

Let Ω be a σ-finite measure space with measure denoted by dx. A nonlinear integral equation of Hammerstein type on Ω is an equation of the form

$$u(x) + \int_\Omega K(x,y)\, f(y,u(y))dy = f(x), \quad x \in \Omega, \tag{1}$$

where we seek a real-valued function u on Ω which satisfies the relation (1) for a given kernel $K(x,y)$, nonlinear function $f(y,u)$ and a given inhomogeneous term f. In operator theoretic forms, the problem of determining the solutions of (1) with u lying in a given Banach space of functions defined on Ω can be put in the form of a nonlinear functional equation

$$u + KNu = f, \tag{2}$$

with the linear and nonlinear mapping K and N given by,

$$Kv(x) = \int_\Omega K(x,y)\, v(y)dy, \quad Nu(x) = f(x,u(x)). \tag{3}$$

We note immediately that several problems that occur in differential equations – for instance, elliptic boundary value problems whose linear parts posses Green's functions, can, as a rule, be transformed into Hammerstein equations (see e.g., [24]). Moreover, operators of the Hammerstein type (i.e., operators of the form $I + AB$) play a crucial role in the study of feedback control systems (see e.g., [16]). Hammerstein equations have been studied extensively by various authors see e.g., [1–9], [12], [16,17], [20–22], [24,25].

If $X = H$, a Hilbert space and $K = I$, the identity operator of H, the Hammerstein Eq.(2) reduces to

$$x + Nx = f. \tag{4}$$

Eq.(4) has been studied extensively by several authors under some monotonicity and continuity conditions on N (see e.g., [10], [11], [13], [18], [19]). When a solution to Eq.(4) is known to exist, iterative methods for approximating such a solution have been studied by various authors. In this connection, W.G. Dotson [18], proved that an iteration process of the Mann–type [23], under suitable conditions, converges strongly to the unique solution of Eq.(4) when N is a Lipschitz and *monotone operator* (defined below) with Lipschitz constant 1. This result has been extended by one of the authors [11], to mappings with Lipschitz constant $L \geq 1$. Recently, R.E. Bruck Jr. [10], obtained a solution of Eq.(4), under certain conditions, as the limit of an iteratively constructed sequence in a Hilbert space when N is a multi–valued monotone operator. No continuity assumption was imposed on N. This result has also been extended by on of the authors [13], to L_p spaces, $p \geq 2$.

For the more general operator Eq.(2), although several *existence* theorems have been established under various monotonicity and continuity conditions on K and N, there are very few results in the literature on methods for approximating a solution to Eq.(2) when one exists. Recently, for $X = H$, a Hilbert space, one of the authors [12], proved *weak convergence* theorems for Eq.(2) using some well–known fixed point iteration methods.

It is our purpose in this paper to prove *strong convergence* theorems for Eq.(2) in Hilbert spaces, by using a fixed point iteration process of the Mann–type (see e.g., [23]). Explicit error estimates will be given and, in some cases, convergence will be shown to be at least as fast as a geometric progression.

2. PRELIMINARIES

For a Banach space X we shall denote by J the normalized duality map from X to 2^{X^*} given by

$$Jx = \{x^* \in X^* : \|x^*\|^2 = \|x\|^2 = < x, x^* > .\},$$

where X^* denotes the dual space of X and $<,>$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex, the J is single–valued, and if X^* is uniformly convex, then J is uniformly continuous on bounded sets.

Let H be a Hilbert space. A mapping $U : H \to H$ with domain $D(U)$ in H is called *hemicontinuous* [15, p.98] if $U(x + t_n y) \xrightarrow{w} U(x)$ as $t_n \to 0^+$, where $\xrightarrow{w}$ denotes weak convergence. It is easy to see that linear maps as well as continuous maps are hemicontinuous.

A mapping U, with domain $D(U)$ in H, into H is called *monotone* if for each x, y in $D(U)$,

$$< Ux - Uy,\ x - y > \geq 0,$$

and U is called *strongly monotone with constant* $\lambda > 0$ if, for each x, y in $D(U)$,

$$< Ux - Uy,\ x - y > \geq \lambda \|x - y\|^2.$$

In the sequel, we shall be concerned with an iteration process of the *Mann–type* described as follows:

The Mann Iteration Process (see e.g., [14], [19], [23]). For a convex subset K of a Banach space X, and a mapping T of K into itself, the sequence $\{x_n\}_{n=0}^{\infty}$ in K is defined by,

$$x_0 \in K,$$

$$x_{n+1} = (1 - C_n)x_n + C_n T x_n,\ n \geq 0,$$

where $\{C_n\}_{n=0}^{\infty}$ is a real sequence satisfying: (i) $0 \leq C_n < 1$ for all n, (ii) $\lim_{n\to\infty} C_n = 0$, and (iii) $\sum_{n=0}^{\infty} C_n = \infty$. In applications, condition (iii) is sometimes replaced by $\sum_{n=0}^{\infty} C_n(1 - C_n) = \infty$. The

Mann iteration process has been studied extensively by various authors and has been successfully employed to approximate solutions of several nonlinear operator equations in Banach spaces when these equations are already known to have solutions (see e.g., [10], [11], [12], [13], [18], [19]). The following two theorems will be required in the sequel:

Theorem D (Dunn, [19] p.41) Let β_n be recursively generated by

$$\beta_{n+1} = (1 - \delta_n)\beta_n + \sigma_n^2,$$

with $n \geq 1, \beta_1 \geq 0, \{\delta_n\} \subseteq [0, 1]$, and

$$\sum_{n=1}^{\infty} \sigma_n^2 < \infty, \quad \sum_{n=1}^{\infty} \delta_n = \infty.$$

Then $\beta_n \geq 0$ for all $n \geq 1$, and $\beta_n \to 0$ as $n \to \infty$.

For the next theorem, we shall need the following definition: Let N be a mapping of a Banach space X into its dual space, X^*. N is called *coercive* if $\left(< u, N(u) > /_{\|u\|}\right) \to \infty$ as $\|u\| \to \infty$, where $<,>$ denotes the duality pairing.

Theorem* (Browder, de Figueiredo and Gupta, [9] p.702) Let X be a reflexive Banach space, A a hemicontinuous monotone mapping of X^* into X, N a coercive hemicontinuous monotone mapping of X into X^*. Then for every ω in X, the equation $v + AN(v) = \omega$ has a solution in X.

3. MAIN RESULTS

We now prove the following theorems:

Theorem 1 Let H be a real Hilbert space and let C be a closed bounded convex nonempty subset of H. Suppose,

(a) $N : C \to C$ is a Lipschitz strongly monotone map with constant $\lambda > 0$,

(b) $K : C \to C$ is a hemicontinuous strongly monotone map with constant $\mu > 0$.

Let $M = K^{-1} + N$ and define $S : C \to H$ by

$$Sx = K^{-1}f + x - Mx, x \in C,$$

for some fixed f in the range of K.

Let $\{C_n\}_{n=0}^{\infty}$ be a real sequence satsifying: (i) $0 \leq C_n < 1$ for all n, (ii) $\sum_{n=0}^{\infty} C_n = \infty$, and (iii) $\sum_{n=0}^{\infty} C_n^2 < \infty$. Then the sequence $\{Z_n\}_{n=0}^{\infty}$ in H defined by $x_0 \in C$,

$$Z_n = (1 - C_n)x_n + C_n S x_n,\ n \geq 0, \tag{5}$$

where $\{x_n\}_{n=0}^{\infty}$ in C is such that for each $n \geq 1$,

$$||x_n - Z_{n-1}|| = \inf_{x \in C} ||x - Z_{n-1}||,$$

converges strongly to the unique solution of the Hammerstein Eq.(2).

Proof: Observe first that since N is Lipschitz, it is hemicontinuous. Also the condition $< Kx, x > \geq \mu ||x||^2$ implies K is coercive. So, the existence of a solution to Eq.(2) follows from Theorem*. Furthermore, K^{-1} exists, is Lipschitz with Lipschitz constant μ^{-1} and satisfies $< K^{-1}x, x > \geq \mu^{-1}||K^{-1}x||^2$ for each $x \in R(K)$. Moreover, S is Lipschitz with constant $L_S \leq L_N + \lambda^{-1} + 1$, where L_N denotes the Lipschitz constant of N. Observe also that,

$$< Mx - My, x - y > = < K^{-1}x - K^{-1}y, x - y > + < Nx - Ny, x - y > \geq \lambda ||x - y||^2$$

so that,

$$< Sx - Sy, x - y > = < x - y - (Mx - My), x - y > \leq (1 - \lambda)||x - y||^2. \quad (6)$$

Let q denote a solution of the Hammerstein Eq.(2). Then q is a fixed point of S. Let $R : H \to C$ denote the proximity map, i.e., R is the map which assigns to each $x \in H$ the unique point of C nearest to x. Then R is *nonexpansive* (i.e., $||Rx - Ry|| \leq ||x - y||$ for each x, y in H, see e.g. [15]). Starting with $x_0 \in C$, we obtain Sx_0 in H and so compute Z_0 in H by the relation $Z_0 = (1 - C_0)x_0 + C_0 S x_0$. Then $x_1 = R(Z_0)$ lies in C. Using this x_1 we can compute Z_1 in H by $Z_1 = (1 - C_1)x_1 + C_1 S x_1$. Let $x_2 = R(Z_1)$. Continuing this process, we generate the sequence $\{Z_n\}_{n=0}^{\infty}$ in H. Observe that for $n \geq 1$,

$$||x_n - q|| = ||R(Z_{n-1}) - R(q)|| \leq ||Z_{n-1} - q||.$$

Using (5) and (6) we obtain

$$\begin{aligned} ||Z_n - q||^2 &= ||(1 - C_n)(x_n - q) + C_n(Sx_n - Sq)||^2 \\ &= (1 - C_n)^2||x_n - q||^2 + C_n^2||Sx_n - Sq||^2 \\ &\quad + 2C_n(1 - C_n) < Sx_n - Sq, x_n - q > \\ &\leq [(1 - C_n)^2 + 2(1 - \lambda)C_n(1 - C_n)]||x_n - q||^2 + L_S^2 C_n^2 ||x_n - q||^2 \quad (7) \\ &\leq (1 - \lambda C_n)^2 ||Z_{n-1} - q||^2 + d^2 C_n^2, \end{aligned}$$

where we have added $(1 - \lambda)^2 C_n^2 ||x_n - q||^2$ to the RHS of (7), and have set $d = L_S \sup_{n \geq 0} ||x_n - q||$. Now, set $\rho_n = ||Z_{n-1} - q||^2$, $1 - \gamma_n = (1 - \lambda C_n)^2$ to obtain,

$$\rho_{n+1} \leq (1 - \gamma_n)\rho_n + d^2 C_n^2. \quad (8)$$

A standard argument (see e.g. Dunn [19], Chidume [14]) now yields that $\rho_n \to 0$ as $n \to \infty$. For completeness, however, we sketch the details of this argument here. Inequality (8) and a simple induction yield:

$$0 \leq \rho_n \leq A^2 \alpha_n, \ n \geq 1, \quad (9)$$

where $\alpha_n \geq 0$ is recursively generated by

$$\alpha_{n+1} = (1 - \gamma_n)\alpha_n + C_n^2, \ \alpha_1 = 1,$$

and

$$A^2 = \max\{\rho_1, d^2\}.$$

Moreover,

$$\gamma_n = 1 - (1 - \lambda C_n)^2 = \lambda(2 - \lambda C_n)C_n,$$

so that using conditions (ii) and (iii) we obtain:

$$\sum_{n=0}^{\infty} \gamma_n = 2\lambda \sum_{n=0}^{\infty} C_n - \lambda^2 \sum_{n=0}^{\infty} C_n^2 = \infty.$$

Furthermore, $\sum_{n=0}^{\infty} C_n^2 < \infty$ implies $\lim_{n \to \infty} C_n = 0$. So, we can choose a positive integer N_0 sufficiently large such that for all $n \geq N_0, \gamma_n \in [0, 1]$. For $t \geq 1$, set $\beta_t = \alpha_{N_0+t}, \delta_t = C_{N_0+t}$, so that $\sum_{t=1}^{\infty} \sigma_t^2 = \sum_{t=1}^{\infty} C_{N_0+t}^2 < \infty$. It then follows from Theorem D that $\alpha_n \to 0$ as $n \to \infty$. Inequality (9) now yields $\rho_n \to 0$ as $n \to \infty$ so that $\{Z_n\}_{n=0}^{\infty}$ converges strongly to q, completing the proof of the theorem

□

Remark 1 It is a consequence of the proof of Theorem 1 that under the hypothesis of the theorem the solution of the Hammerstein Eq.(2) must be *unique*. The element $q \in F(S)$, where $F(S)$ denotes the set of fixed points of S (and hence the set of solutions of the Hammerstein Eq.(2)), was arbitrarily chosen. Suppose now there is a $q^* \in F(S)$ with $q^* \neq q$. Repeating the argument of Theorem 1 relative to q^*, one sees that $\{Z_n\}_{n=0}^{\infty}$ converges to both q and q^*, showing that $F(S) = \{q\}$.

There are particular choices of the real sequence $\{C_n\}_{n=0}^{\infty}$ and an alternate method which yield the conclusion of Theorem 1 with the additional information of an explicit error estimate. Moreover, the boundedness assumption on C is not needed with this method.

Theorem 2 Let C be a closed convex nonempty subset of a real Hilbert space H and let K, N, M, S and q be as in Theorem 1. Let the sequence $\{Z_n\}_{n=0}^{\infty}$ be as in Theorem 1, where the real sequence $\{C_n\}_{n=0}^{\infty}$ satisfies

(i) $0 < C_n \leq \lambda(L_S^2 + 2\lambda - 1)^{-1}$ for all $n \geq 0$,

(ii) $\sum_{n=0}^{\infty} C_n = \infty$.

Then $\{Z_n\}_{n=0}^{\infty}$ converges strongly to the solution of Eq.(2). Moreover, if $C_n \equiv \lambda(L_S^2 + 2\lambda - 1)^{-1}$ for all $n \geq 0$, then

$$||Z_n - q|| \leq \delta^{n/2} ||Z_0 - q||,$$

where

$$\delta = \left[1 - \lambda^2 (L_S^2 + 2\lambda - 1)^{-1}\right] \in (0,1).$$

Proof The existence of a unique solution q, for Eq.(2) follows as in Theorem 1. Moreover, a computation similar to that in the proof of Theorem 1 yields,

$$\begin{aligned} ||Z_n - q||^2 &\leq \left[(1 - C_n)^2 + 2(1-\lambda)C_n(1 - C_n) + L_S^2 C_n^2\right] ||Z_{n-1} - q||^2 \\ &= \left[1 - \{2\lambda - (L_S^2 + 2\lambda - 1)C_n\}C_n\right] ||Z_{n-1} - q||^2, \end{aligned} \quad (10)$$

so that using condition (i) we obtain

$$||Z_n - q||^2 \leq (1 - \lambda C_n)||Z_{n-1} - q||^2 \leq exp(-\lambda C_n)||Z_{n-1} - q||^2,$$

i.e.,

$$||Z_n - q||^2 \leq exp(-\lambda C_n)||Z_{n-1} - q||^2.$$

Iterating this inequality from 1 to N, and using condition (ii), we obtain

$$||Z_N - q||^2 \leq exp\left(-\lambda \sum_{n=1}^{N} C_n\right)||Z_0 - q||^2 \to 0 \quad \text{as} \quad N \to \infty.$$

Hence, $\{Z_n\}_{n=0}^{\infty}$ conveys strongly to q.

If $C_n \equiv \lambda(L_S^2 + 2\lambda - 1)^{-1}$ for $n \geq 0$ then inequality (10) reduces to

$$||Z_n - q||^2 \leq \{1 - \lambda^2 (L_S^2 + 2\lambda - 1)^{-1}\}||Z_{n-1} - q||^2,$$

so that a simple induction now yields:

$$||Z_n - q|| \leq \delta^{n/2} ||Z_0 - q||,$$

where δ is as defined. This completes the proof.

□

Remark 2 If $C = H$ in Theorem 2, the approximation method reduces to the following Mann-type iteration scheme: $x_0 \in H$,

$$x_{n+1} = (1 - C_n)x_n + C_n S x_n, \quad n \geq 0.$$

In this case, we obtain, as in the proof of Theorem 2,

$$||x_{n+1} - q||^2 \leq (1 - \lambda C_n)||x_n - q||^2,$$

and the conclusions of Theorem 2 still hold.

Acknowledgments

One of the authors (C.E.C.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[1] H. Brézis, "Equations et in equations non–linéaires dans les espaces vectoriels en dualité", Ann. Inst. Fourier (Grenoble) **18** (1968) 115–176.

[2] H. Brézis and F.E. Browder, "Some new results about Hammerstein equations", Bull. Amer. Math. Soc. **80** (1974) 568–572.

[3] H. Brézis and F.E. Browder, "Existence theorems for nonlinear integral equations of Hammerstein type", Bull. Amer. Math. Soc. **81** (1975) 73–78.

[4] H. Brézis and F.E. Browder, "Nonlinear integral equations and systems of Hammerstein type", Advances in Math. **18** (1975) 115–144.

[5] H. Brézis and F.E. Browder, "Singular Hammerstein equations and maximal monotone operators", Bull. Amer. Math. Soc **82** (1976) 623–625.

[6] H. Brézis and F.E. Browder, Linear maximal monotone operators and singular nonlinear integral equations of Hammerstein type, E. Rothe Festschrift,(Academic Press, 1978) 31–42.

[7] F.E. Browder, "Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urysohn type", *Contributions to Nonlinear Functional Analysis* (Academic Press, 1971) 425–500.

[8] F.E. Browder, "Singular nonlinear integral equations of Hammerstein type", Lect. Notes Math. 446, (Springer–Verlag, 1975) 75–95.

[9] F.E. Browder,, D.G. De Figueiredo and C.P. Gupta, "Maximal monotone operators and nonlinear integral equations of Hammerstein type", Bull. Amer. Math. Soc. **76** (1970) 700–705.

[10] R.E. Bruck Jr., "The iterative solution of the equation $f \in x+Tx$ for a monotone operator T in Hilbert space", Bull. Amer. Math. Soc. **79** (1973) 1258–1262.

[11] C.E. Chidume, "An approximation method for monotone Lipschitzian operators in Hilbert spaces", J. Austral. Math. Soc. (Series A) **41** (1986) 59–63.

[12] C.E. Chidume, "Fixed point iterations for nonlinear Hammerstein equations involving nonexpansive and accretive mappings", Indian J. Pure Appl. Math. **20** (2) (1989) 129–135.

[13] C.E. Chidume, "The iterative solution of the equation $f \in x+Tx$ for a monotone operator T in L_p spaces", J. Math. Anal. Appl. **116** (1986) 531–537.

[14] C.E. Chidume, "Iterative approximation of fixed points of Lipschitz strictly pseudocontractive mappings", Proc. Amer. Math. Soc. **99** (1987) 283–288.

[15] K. Deimling, *Nonlinear Functional Analysis* (Springer–Verlag, Berlin, Heidelberg, New York, Tokyo, 1985).

[16] V. Dolezale, "Monotone operators and applications in automation and network theory", *Studies in Automation and Control, 3* (Elsevier Science Publishers, New York, 1979).

[17] C.L. Dolph and G.J. Minty, "On nonlinear integral equations of the Hammerstein type", *Nonlinear Integral Equations* (The University Wisconsin Press, Madison, 1964) 99–154.

[18] W.G. Dotson Jr., "An iterative process for nonlinear monotonic nonexpansive operators in Hilbert spaces", Math. Comp. **32** (141) (1978) 223–225.

[19] J.C. Dunn, "Iterative construction of fixed points for multi–valued operators of the monotone type", J. Funct. Anal. No.1 **27** (1978) 30–50.

[20] A. Hammerstein, "Nichtlineare integralgleichungen nebst anwendungen", Acta. Math. **54** (1930) 117–176.

[21] I.I. Kolodner, "Equations of Hammerstein type in Hilbert space", J. Math. Mech. **13** (1964) 701–750.

[22] J. Kolomy, "Applications of some existence theorems for the solutions of Hammerstein integral equations", Comment. Math. Univ. Caroline **7** (1966) 461–478.

[23] W.R. Mann, "Mean value methods in iteration", Proc. Amer. Math. Soc. **4** (1953) 506–510.

[24] D. Pascali and S. Sburian, *Nonlinear Mappings of Monotone Type* (Editura Academiei Bucaresti, Romania, 1978).

[25] R. Sh. Chepanovich, "Nonlinear Hammerstein equations and fixed points", (Russian), Publ. Inst. Math. (Beograd), N.S. **35** (49) (1984) 119–123.

Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica