Laboratório Nacional de Computação Científica

ISSN 0101 6113

LABORATÓRIO NACIONAL DE COMPUTAÇÃO CIENTÍFICA - LNCC

RELATÓRIOS DE PESQUISA E DESENVOLVIMENTO

AGOSTO DE 1988

LNCC--

Nº 028/88 .

HIDDEN REGULARITY FOR A STRONGLY NONLINEAR WAVE EQUATION

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ABSTRACT

In this paper we consider the nonlinear wave equation,

$$\begin{cases} u'' - \Delta u + f(u) = v & \text{in} \quad Q = \Omega \times]0,T[; \\ u(0) = u_0, u'(0) = u_1 & \text{in} \quad \Omega; \\ u(x,t) = 0 & \text{on} \quad \Sigma = \Gamma \times]0,T[. \end{cases} \tag{$*$}$$

where f is a continuous function satisfying,

$$lim_{|s|} sup_{\rightarrow +\infty} \frac{f(s)}{s} > -\infty. \tag{$**$}$$

and Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary Γ. We prove that there exist a solution for (*) that satisfies the regularity condition: $\frac{\partial u}{\partial \eta} \epsilon L^2(\Sigma)$. Moreover we have that there exist a constant $C > 0$ such that,

$$\left|\frac{\partial u}{\partial \eta}\right| \leq C\{E(0) + |v|_Q^2\}. \tag{$***$}$$

RESUMO

Considera-se neste trabalho o sistema (*) onde f é uma função continua satisfazendo (**) e Ω um dominio limitado do $\mathbb{R}^n$ com fronteira Γ bem regular. Provamos que existe uma solução u do sistema, que satisfaz $\frac{\partial u}{\partial \eta} \epsilon L^2(\Sigma)$ mais ainda que existe uma constante $C > 0$ tal que (***) é válida.

1. INTRODUCTION

Let Ω be an open bounded set of $\mathbb{R}^n$, with boundary Γ of class C^2. Set $Q = \Omega \times]0,T[$ and $\Sigma = \Gamma \times]0,T[$. We will denote by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_Q$ the inner product of $L^2(\Omega)$ and $L^2(Q)$ respectively and by $|\cdot|_\Omega$, $|\cdot|_Q$ and $\|\cdot\|$, the norms in $L^2(\Omega)$, $L^2(Q)$, and $H_0^1(\Omega)$ respectively. We consider the nonlinear problem:

$$\begin{cases} u'' - \Delta u + f(u) = v & \text{in} \quad Q; \\ u(0) = u_0, u'(0) = u_1 & \text{in} \quad \Omega; \\ u(x,t) = 0 & \text{on} \quad \Sigma. \end{cases} \tag{1.1}$$

In J. L. Lions [1] was study the hidden regularity for system (1.1) when $f(s) = s^3$ and more generality for a $f(s) = s|s|^\epsilon$, where $\epsilon \geq 0$. In this work we are going to study the hidden regularity for the solution of the problem (1) when f is a continuous function satisfying,

$$\lim_{|s| \to +\infty} \inf \frac{f(s)}{s} > -\infty \tag{1.2}$$

That is, we will show that there exist a solution u of the above problem such that the normal derivative of u belongs to $L^2(\Sigma)$. Moreover we will prove that there exist a constant $C > 0$ such that:

$$\left|\frac{\partial u}{\partial \eta}\right|_\Sigma \leq C E_0, \tag{1.3}$$

where E_0 is the initial emergy of the system (1.1).

$$(E_0 = \frac{1}{2}|u_1|_\Omega^2 + \int_\Omega G(u_0)dx),$$

where $G(s) = \int_0^s f(\eta)d\eta$.

2. EXISTENCE AND HIDDEN REGULARITY OF PROBLEM (1.1)

First of all we are going to construct a sequence of real numbers $(s_\nu)_{\nu\in\mathbb{N}}$ and $(s_{-\nu})_{\nu\in\mathbb{N}}$ satisfying the following conditions,

$$s_\nu \geq \nu \qquad \forall \nu \in \mathbb{N}, \quad |f(s_\nu)| \leq C + |f(s)| \quad \forall s \geq \nu \tag{2.1}$$

$$s_{-\nu} \leq -\nu \qquad \forall \nu \in \mathbb{N}, \quad |f(s_\nu)| \leq C + |f(s)| \quad \forall s \leq -\nu \tag{2.2}$$

This sequences are going to play an important role in the sequel.

LEMMA 2.1 - Let f be a continuous function defined in $\mathbb{R}$, then there exists a sequence of real numbers,$(s_\nu)_{\nu\in\mathbb{N}}$, and a positive constant C, independing of ν, satisfying condition (2.1) and (2.2).

PROOF - Let's consider the following problem.

$$I_\nu = inf\{|f(s)|;\ s \geq \nu\} \tag{2.3}$$

If for all $\nu \in \mathbb{N}$, there exists $s_\nu \geq \nu$ such that $f(s_\nu) = I_\nu$, then this sequence satisfies condition (2.1). Now we can suppose that there exist a ν_0 such that,

$$f(s) > I_{\nu_0} \qquad for\ all\ s \geq \nu_0$$

This relation emply that $I_\nu = I_{\nu_0}$ for all $\nu \geq \nu_0$. Let us put $I_0 = I_{\nu_0}$. Since $I_0 = inf|f(s)|; s \geq \nu_0$, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ such that:

$$f(t_k) \longrightarrow I_0 \tag{2.4}$$

from the continuity of f we conclude that t_k is not bounded, then there exist a subsequence $(t_{k_\nu})_{\nu\in\mathbb{N}}$ satisfying:

$$t_{k_\nu} \geq \nu \qquad \nu \in \mathbb{N}. \tag{2.5}$$

Let us put $s_\nu = t_{k_\nu}$, from (2.4) we obtain that there exist a constant C (independing of ν) such that $|f(s_\nu)| = |f(t_{k_\nu})| \leq C$, finally from (2.5) we conclude that $(s_\nu)_{\nu\epsilon\mathbb{N}}$ satisfies condition (2.1). By the same arguments we can prove the existence of a sequence $(s_{-\nu})_{\nu\epsilon\mathbb{N}}$ satisfying condition (2.2), only consider the problem

$$I_{-\nu} = inf|f(s)|; s \leq -\nu,$$

and the result follows.■

With the sequences $(s_\nu)_{\nu\epsilon\mathbb{N}}$ and $(s_{-\nu})_{\nu\epsilon\mathbb{N}}$ constructed in Lemma 2.1 we define a sequence $(f_\nu)_{\nu\epsilon\mathbb{N}}$ of continuous function in the following way:

$$f_\nu(s) = \begin{cases} f(s) & \text{if} \quad s_\nu \leq s \leq s_\nu; \\ f(s_\nu) & \text{if} \quad s \geq s_\nu; \\ f(s_{-\nu}) & \text{if} \quad s \leq s_{-\nu}. \end{cases} \tag{2.6}$$

As a consequence of Lemma 2.1 we have that the sequence $(f_\nu)_{\nu\epsilon\mathbb{N}}$ satisfies the following properties:

$$|f_\nu(s)| \leq c + |f(s)| \qquad for \ all \quad \nu \tag{2.7}$$

$$f_\nu \longrightarrow f \quad uniformly \ on \ bounded \ sets \tag{2.8}$$

LEMMA 2.2 - Let f be a continuous function satisfying condition (1.2), and $(f_\nu)_{\nu\epsilon\mathbb{N}}$ the sequence defined in (2.6). Then there exist a positive constant C_0 such that

$$sf_\nu(s) \geq -c_0(s^2+1) \qquad \forall s\epsilon\mathbb{R} \qquad \forall \nu \geq \nu_0. \tag{2.9}$$

$$\int_0^t f_\nu(s)ds \geq -c_0(t^2 + 2|t|) \qquad \forall t\epsilon\mathbb{R}, \qquad \forall \nu\epsilon\mathbb{N}. \tag{2.10}$$

$$|\int_0^t f_\nu(s)ds| \leq \frac{1}{2}C_0|t(t+3)| + \int_0^t f(s)ds. \tag{2.11}$$

PROOF - First of all we are going to prove that there exists a positive constant C_0, such that

$$f(s) \geq -C_0 s \quad \forall s \geq N \qquad and \qquad f(s) \leq -C_0 s \quad \forall s \leq -N. \tag{2.12}$$

In fact, if $liminf\, s^{-1} f(s) = +\infty$ the expression (2.12) is valid. Now we can suppose that $liminf\, s^{-1} f(s) = \chi < +\infty$; then for $\varepsilon > 0$, there exist $N > 0$ such that $s^{-1} f(s) > \chi - \varepsilon$, for $|s| \geq N$. Let us take $C = sup\{|f(s)| \cdot |s| \leq N\}$ $C_2 = sup\{|sf(s)|; |s| \leq N\}$, and put $C_0 = max\{C, C_1, C_2, |\chi - \varepsilon|\}$ where C is the constant in (2.7), certainly for this C_0, condition (2.12) is valid. Finally multiplying the relations in (2.12) by s ($|s| \geq N$), we have by the definition of C_0, that the first part of (2.9) is valid. The second part of (2.9) follows from (2.1), (2.2), (2.6) and also, the definition of C_0, for $\nu_0 = N$.

In order to prove (2.11), let us note that from (2.12) follows that:

$$f_\nu(s) \geq -C_0(s+1) \quad \forall s \geq 0 \qquad and \qquad f_\nu(s) \leq -C_0(s-1) \quad \forall s \leq 0. \tag{2.13}$$

Integring this expression we obtain (2.10). In order to obtain (2.11) let us consider relation (2.7) then we have:

$$|\int_0^t f_\nu(s)ds| \leq \int_0^t |f_\nu(s)|ds \leq C|t| + \int_0^t |f(s)ds| \qquad \forall t \in \mathbb{R}. \tag{2.14}$$

From (2.12) we obtain that $f(s) \geq -C_0(s+1)$ $\forall s \geq 0$, which imply that $|f(s)| \leq f(s) + 2C_0(s+1)$ $\forall s \geq 0$. Then we have:

$$\int_0^t |f(s)ds| \leq \int_0^t f(s)ds + C_0 t(t+2) \qquad \forall t \geq 0 \tag{2.15}$$

and since $f(s) \leq |f(s)|$ we obtain,

$$\int_0^t |f(s)|ds \leq \int_0^t f(s)ds \qquad \forall t \leq 0 \tag{2.16}$$

Finally from (2.15) and (2.16) we obtain (2.11).■

Let us denote by $G_\nu(t) = \int_0^t f_\nu(s)ds$, then we have that

$$G_\nu \longrightarrow G \ uniformly \ on \ bounded \ sets. \tag{2.17}$$

Before to prove the main result of this paper we will prove an identity that will be fundamental in that follows.

LEMMA 2.3 - Let h be a continuous function. Let $q = (q_k)$ a field of vectors of class $[C^1(\overline{\Omega})]^n$. Then for all w satisfying,

$$w \epsilon L^2(0,T;\ H_0^1(\Omega) \cap H^2(\Omega), \quad H(w) \epsilon L^1(Q). \tag{2.18}$$

$$w' \epsilon L^2(0,T;\ H_0^1(\Omega)), \tag{2.19}$$

$$w'' \epsilon L^2(0,T;\ L^2\ (\Omega)). \tag{2.20}$$

$$\begin{cases} w'' - \Delta w + h(w) = v & \text{in} \quad Q; \\ w(0) = w_0, w'(0) = w_1 & \text{in} \quad \Omega; \\ w(x,t) = 0 & \text{on} \quad \Sigma. \end{cases} \tag{2.21}$$

where H is the primitive of h. Then we have.

$$\frac{1}{2}\int_\Sigma q_k \eta_k \left|\frac{\partial w}{\partial \eta}\right|^2 d\Sigma = \left[(w\prime(t), q_k \frac{\partial w(t)}{\partial x_k}\right]_0^T +$$

$$+\frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\{|w\prime|^2 - |\nabla w|^2 - 2H(w)\}\,dxdt +$$

$$+\int_Q \frac{\partial q_k}{\partial x_j} \times \frac{\partial w}{\partial x_k} \times \frac{\partial w}{\partial x_j} dxdt - \int_Q v\, q_k \frac{\partial w}{\partial x_k}$$

PROOF - Let us multiply $(2.21)_1$ by $q_k \frac{\partial w}{\partial x_k}$, then we have that:

$$\int_Q \{w'' - \Delta w + h(w)\} q_k \, \frac{\partial w}{\partial x_k} \, dx dt =$$

$$\int_Q v q_k \, \frac{\partial w}{\partial x_k} \, dx dt. \tag{2.22}$$

Let us denote by:

$$I_1 = \int_Q w'' q_k \, \frac{\partial w}{\partial x_k} \, dx dt.$$

$$I_2 = \int_Q \Delta w q_k \, \frac{\partial w}{\partial x_k}.$$

then we have:

$$I_1 = \Big[\Big(w'(t), q_k \, \frac{\partial w}{\partial x_k}(t)\Big)\Big]_0^T - \int_Q w' q_k \, \frac{\partial w\prime}{\partial x_k} \, dx dt.$$

$$= \Big[\Big(w'(t), q_k \, \frac{\partial w(t)}{\partial x_k}\Big)\Big]_0^T - \frac{1}{2} \int_Q q_k \, \frac{\partial |w'|^2}{\partial x_k} \, dx dt.$$

from where we obtain that:

$$I_1 = \Big[\Big(w'(t), q_k \, \frac{\partial w(t)}{\partial x_k}\Big)_\Omega\Big]_0^T + \frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} \bullet |w'|^2 dx dt. \tag{2.23}$$

On the other hand we have that:

$$I_2 = -\int_Q \frac{\partial w}{\partial x_j} \times \frac{\partial}{\partial x_j}\{q_k \frac{\partial w}{\partial x_k}\}\ dxdt + \int_\Sigma q_k \frac{\partial w}{\partial x_k} \times \frac{\partial w}{\partial \eta}\ d\Sigma.$$

$$= -\int_Q \frac{\partial w}{\partial x_j}\frac{\partial q_k}{\partial x_j} \times \frac{\partial w}{\partial x_k}\ dxdt - \int_Q \frac{\partial w}{\partial x_j} \times \frac{\partial^2 w}{\partial x_k \partial x_j} \times q_k\ dxdt.$$

$$+ \int_\Sigma q_k \frac{\partial w}{\partial x_k} \times \frac{\partial w}{\partial \eta}\ d\Sigma.$$

$$= -\int_Q \frac{\partial w}{\partial x_j} \times \frac{\partial q_k}{\partial w_j} \times \frac{\partial w}{\partial x_k} dxdt - \frac{1}{2}\int_Q q_k \frac{\partial}{\partial x_k}|\nabla w|^2 dxdt + \int_\Sigma q_k \frac{\partial w}{\partial x_k} \times \frac{\partial w}{\partial \eta} d\Sigma.$$

But since $w = 0$ on Σ we have that:

$$\frac{\partial w}{\partial x_k} = \eta_k \frac{\partial w}{\partial \eta} \qquad on \quad \Sigma.$$

and

$$|\nabla w|^2 = \left|\frac{\partial w}{\partial \eta}\right|^2 \qquad on \quad \Sigma.$$

we have:

$$I_2 = \int_Q \frac{\partial w}{\partial x_j} \times \frac{\partial w}{\partial x_k} \times \frac{\partial q_k}{\partial x_j} dxdt + \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}|\nabla w|^2 dxdt + \frac{1}{2}\int_\Sigma q_k \eta_k \left|\frac{\partial w}{\partial \eta}\right|^2 d\Sigma. \quad (2.24)$$

Finally since

$$\int_Q h(w) q_k \frac{\partial w}{\partial x_k} dxdt = \int_Q \frac{\partial}{\partial x_k} H(w) q_k = -\int_Q H(w) \frac{\partial}{\partial x_k} q_k. \qquad (2.25)$$

we have from (2.22), (2.23), (2.24) and (2.25) that the result follows.■

REMARK 2.4 - From Lemma 2.3 taking $q = (q_k)$ a field of vectors of class $[C^1(\overline{\Omega})]^n$, such that

$$q_k = \eta_k \quad on \quad \Sigma.$$

and putting:

$$C = sup\left\{|q_k(x)|, \left|\frac{\partial q_k}{\partial x_h}(x)\right|;\ k, j = 1, \cdots, n \ and\ x \epsilon \overline{\Omega}\right\}.$$

we have:

$$\frac{1}{2}\int_{\Sigma}\left|\frac{\partial w}{\partial \eta}\right|^2 d\Sigma \leq 2n\ sup_{[0,T]}\, J(t) +$$

$$+ cnT\ sup_{[0,T]}\, J(t) + cn\int_Q |H(w)|\, dxdt + 2c(sup\, J(t)) + \frac{cn}{2}|v|_Q^2 + CT\ sup\, J(t).$$

From where we have:

$$\frac{1}{2}\int_{\Sigma}\left|\frac{\partial w}{\partial \eta}\right| d\Sigma \leq C(n+1)(2+T)\, sup_{[0,T]}\, J(t) +$$

$$+ cn\frac{1}{2}|v|_Q^2 + \int_Q |H(w)| dxdt. \tag{2.26}$$

REMARK 2.5 - From (2.21) we have that

$$w'' - \Delta w + h(w) + bw = v + bw$$

multiplying this expression by w' and integrating in Ω we have:

$$\frac{d}{dt}\left\{J(t) + \int_{\Omega} H(w) + b|w|^2 dx\right\} = (v, w')_{\Omega} + b(w, w')_{\Omega}.$$

where $J(t) = \frac{1}{2}\{|w'(t)|^2 + \|w(t)\|^2\}$. If we put $C_0 = max\{1, b+1, bc^2\}$ (where c is such that $|\cdot|_{\Omega} \leq C\|\cdot\|$) we obtain, after integrate from 0 to t, that:

$$J(t)+\int_{\Omega}\{H(w)+b(w)^2\}\,dx \leq \frac{1}{2}|v|_Q^2+2c_0\,E(0)+c_0\int_0^t J(s)ds. \qquad (2.27)$$

where $E(t)$ is the energy associate with system (2.21), that is:

$$E(t)=J(t)+\int_{\Omega} H(w(x,t))dx.$$

REMARK 2.6 - Multiplying $(2.21)_1$ by w, integrating in Q, and applying Green's formula, we have that:

$$\int_Q wh(w)dxdt = \int_Q wvdxdt + \int_0^T |w\prime(t)|_\Omega^2\,dt -$$

$$-\int_0^T \|w(t)\|^2 dt - (w\prime(t), w(t))_0^T,$$

from where we have that:

$$\int_Q wh(w)dxdt \leq \frac{1}{2}|v|_Q^2 + (3T+2C)sup_{[0,T]}\,J(t). \qquad (2.28)$$

(where C is such that $|\cdot|_\Omega \leq C\|\cdot\|$).

Now we are in condition to prove the main result of this paper:

THEOREM 2.7 - Let (u_0,u_1,v) be an element of the space $H_0^1(\Omega)\times L^2(\Omega)\times L^2(Q)$, and let f be a continuous function such that $G(u_0\epsilon L^1(Q)$. Then there exist a function $u:Q\rightarrow \mathbb{R}$ satisfying,

$$u\epsilon L^\infty(0,T:H_0^1(\Omega)),\, u'\epsilon L^\infty(0,T;L^2(\Omega)). \qquad (2.29)$$

$$\begin{cases} u''-\Delta u+f(u)=v & \text{in} \quad Q; \\ u(0)=u_0, u'(0)=u_1 & \text{in} \quad \Omega; \\ u(x,t)=0 & \text{on} \quad \Sigma. \end{cases} \qquad (2.30)$$

$$G(u)\epsilon L^1(Q), \tag{2.31}$$

$$\frac{\partial u}{\partial \eta} \epsilon L^2(\Sigma). \tag{2.32}$$

$$\left|\frac{\partial u}{\partial \eta}\right|_{\Sigma}^2 \leq C\{E(0) + |v|_{L^2 Q}\}. \tag{2.33}$$

REMARK 2.8 - We are proving here that, there exist one solution satisfying the last two conditions. We don't know if all solution of (1.1) satisfies this regularity result. This is an open question.

PROOF OF THEOREM 2.7 - Let $(\rho_\mu)_{\mu\epsilon I\!N}$ be a regularizant sequence on $I\!R$. That is: $\rho_\mu \epsilon C^\infty(I\!R)$, $\forall \mu \epsilon I\!N$ and:

$$\rho_\mu(s) \geq 0 \qquad \forall s \epsilon I\!R, \quad and \quad sopp(\rho_\mu) C \left]-\frac{1}{\mu}, \frac{1}{\mu}\right[. \tag{2.34}$$

$$\int_{I\!R} \rho_\mu(s) ds = 1 \qquad \forall \mu \epsilon I\!N. \tag{2.35}$$

Let us denote by $(f_{\nu\mu})_{\mu\epsilon I\!N}$ the sequence of bounded function defined by:

$$f_{\nu\mu} = f_\nu * \rho_\mu \ \ for\ a\ fixed\ \nu.$$

Then we have that $f_{\nu\mu}$ is a C^∞ bounded function we now consider the following aproximated problem.

$$\begin{cases} u''_{\nu\mu} - \Delta u_{\nu\mu} + f_{\nu\mu}(u_{\nu\mu}) = v & \text{in} \quad Q; \\ u_{\nu\mu}(0) = u_0, u'_{\nu\mu} = u_1 & \text{in} \quad \Omega; \\ u_{\nu\mu}(x,t) = 0 & \text{on} \quad \Sigma. \end{cases} \tag{2.36}$$

As well known that for every $(u_0, u_1, v) \epsilon H_0^1(\Omega) \times L^2(\Omega) \times L^2(Q)$ there exists an unique solution for system (2.33). In order to obtain the existence of a solution for the system (1.1) satisfying condition (1.3), let us suppose that v, u_0, u_1 be test function, then we have that:

$$u_{\nu\mu} \epsilon L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)), \tag{2.37}$$

$$u'_{\nu\mu} \epsilon L^\infty(0,T; H_0^1(\Omega)), \tag{2.38}$$

$$u''_{\nu\mu} \epsilon L^\infty(0,T; L^2(\Omega)). \tag{2.39}$$

From Remark (2.4) we have that the normal derivative of $u_{\nu\mu}$, satisfies the following inequality:

$$\frac{1}{2}\int_\Sigma \left|\frac{\partial u_{\nu\mu}}{\partial\eta}\right|^2 d\Sigma \le C(n+1)(2+T) sup\, J_{\nu\mu}(t) +$$

$$+ cn\left\{\frac{1}{2}|v|_Q^2 + \int_Q |G_{\nu\mu}(u_{\nu\mu})\right\} dxdt. \tag{2.40}$$

where by $J_{\nu\mu}(t)$ we are denoting the quadratic term associated to system (2.36), that is:

$$J_{\nu\mu}(t) = \frac{1}{2}|u_{\nu\mu}(t)|_\Omega^2 + \frac{1}{2}\|u_{\nu\mu}(t)\|^2$$

By Remarks 2.5, we have that,

$$J_{\nu\mu}(t) + \int_\Omega G_{\nu\mu}(u_{\nu\mu}) + b|u_{\nu\mu}|^2\, dx \le \frac{1}{2}|v|_Q^2 +$$

$$+ 2C_0\, E_{\nu\mu}(0) + C_0\int_0^t J_{\nu\mu}(s)ds. \tag{2.41}$$

and since b is a positive number, and $G_{\nu\mu}$ is uniformly bounded for all $\mu \epsilon I\!N$, and a fixed ν, we have by Gronwall inequality that there exist a constant C_ν such that,

$$J_{\nu\mu}(t) = \frac{1}{2}|u'_{\nu\mu}(t)|_\Omega^2 + \frac{1}{2}\|u_{\nu\mu}(t)\|^2 \le C_\nu. \tag{2.42}$$

where $E_{\nu\mu}(t)$ is the energy associated to system (2.36) with non quadratic term $G_{\nu\mu}(u_{\nu\mu})$.

Finally from Remark 2.6 we obtain that:

$$\int_Q u_{\nu\mu} f_{\nu\mu}(u_{\nu\mu}) dx \leq +$$

$$+\frac{1}{2}|v|_Q^2 + (3T + 2C) sup_{[0,T]} J_{\nu\mu}(t). \tag{2.43}$$

Relation (2.40), (2.41), (2.42) and (2.43) are valid when v, u_0 and u_1 are test function. If we take a sequence of test function (v_m, u_{0m}, u_{1m}) satisfying,

$$(v_m, u_{0m}, u_{1m}) \rightarrow (v, u_0, u_1) \ \textit{strongly in} \ L^2(Q) \times H_0^1(\Omega) \times L_0^2(\Omega).$$

certainly we have that the corresponding solutions $u_{\nu um}$ converge to $u_{\nu u}$ solution of system (2.36), when the datas v, u_0 and u_1 are in $L^2(Q), H_0^1(\Omega)$ and $L^2(\Omega)$ respectively. Moreover we have:

$$u_{\nu um} \rightarrow u_{\nu u} \ \textit{strongly in} \ L^\infty(0, T; H_0^1(\Omega)) \tag{2.44}$$

$$u'_{\nu um} \rightarrow u_{\nu u} \ \textit{strongly in} \ L^\infty(0, T; L^2(\Omega)). \tag{2.45}$$

From (2.44) and (2.45) we conclude that relations (2.40), (2.41), (2.42) and (2.43) are valid when (v, u_0, u_1) belongs to $L^2(Q) \times H_0^1(\Omega) \times L^2(\Omega)$, and $u_{\nu u}$ is solution of (3.36).

On the other hand, by (2.42) we obtain that there exists a subsequence of $(u_{\nu\mu})_{\mu\in\mathbb{N}}$ satisfying:

$$u_{\nu\mu} \rightarrow u_\nu \ \textit{weak star in} \ L^\infty(0, T; H_0^1(\Omega)). \tag{2.46}$$

$$u'_{\nu\mu} \rightarrow u'_\nu \ \textit{weak star in} \ L^\infty(0, T; L^2(\Omega)). \tag{2.47}$$

from (2.46) and (2.47) we have that there exist another subsequence (that we still denoting in the same way) such that,

$$u_{\nu\mu} \rightarrow u_\nu \ \textit{strongly in} \ L^2(Q). \tag{2.48}$$

$$u_{\nu\mu} \rightarrow u_\nu \ a.e. \ in \ Q. \tag{2.49}$$

$$f_{\nu\mu}(u_{\nu\mu}) \rightarrow f_\nu(u_\nu) \ a.e. \ in \ Q. \tag{2.50}$$

$$G_{\nu\mu}(u_{\nu\mu}) \rightarrow G_\nu(u_\nu) \ a.e. \ in \ Q. \tag{2.51}$$

Since $f_{\nu\mu}$ is bounded for all $u\epsilon\mathbb{N}$ ($\nu fixed$), then $G_{\nu\mu}$ is a Lipschitz's function in $\mathbb{R}$, then by Lebesgue dominated convergence theorem we conclude that:

$$f_{\nu\mu}(u_{\nu\mu}) \rightarrow f_\nu(u_\nu) \ \textit{strongly in} \ L^2(Q). \tag{2.52}$$

$$G_{\nu\mu}(u_{\nu\mu}) \rightarrow G_\nu(u_\nu) \ \textit{strongly in} \ L^2(Q). \tag{2.53}$$

Now, from (2.48) and (2.52) we obtain:

$$u_{\nu\mu} \rightarrow u_\nu \ \textit{strongly in} \ L^\infty(0,T;H_0^1(\Omega)). \tag{2.54}$$

$$u'_{\nu\mu} \rightarrow u'_\nu \ \textit{strongly in} \ L^\infty(0,T;L^2(\Omega)). \tag{2.55}$$

Then by (2.40), (2.42) and (2.53) we obtain that there exist a subsequence of $\frac{\partial u}{\partial \eta}\nu\mu$ (which we still denote in the same way) and an element χ in $L^2(\Sigma)$ such that:

$$\frac{\partial u_{\nu\mu}}{\partial \eta} \rightarrow \chi_\nu \ \textit{weak in} \ L^2(\Sigma). \tag{2.56}$$

But since:

$$\frac{\partial u_{\nu\mu}}{\partial \eta} \to \frac{\partial u_\nu}{\partial \eta} \textit{ weak in } H^{-1}(0,T;H^{1/2}(\Gamma)).$$

we conclude that $\chi_\nu = \frac{\partial u_\nu}{\partial \eta}$. From (2.54) and (2.55) we have in particular that:

$$J_{\nu\mu}(t) \to J_\nu(t) \textit{ uniformly on } [0,T]. \tag{2.57}$$

$$E_{\nu\mu}(t) \to E_\nu(t) \textit{ uniformly on } [0,T]. \tag{2.58}$$

Then from (2.40), (2.53) and (2.57) we obtain that:

$$\frac{1}{2}\int_\Sigma \left|\frac{\partial u_\nu}{\partial \eta}\right|^2 d\Sigma \le c(n+1)(2+T)sup_{[0,T]}J_\nu(t)+$$

$$+cr\left\{\frac{1}{2}|v|_Q^2 + \int_Q |G_\nu(u_\nu)|\right\} dxdt. \tag{2.59}$$

Now by (2.41), (2.43), (2.48), (2.53), (2.57) and (2.58) we obtain:

$$J_\nu(t) + \int_\Omega \{G_\nu(u_\nu) + b|u_\nu|^2\} \le$$

$$\le \frac{1}{2}|v|_Q^2 + 2C_0\,E_\nu(0) + C_0\int_0^t J_\nu(s)ds. \tag{2.60}$$

$$\int_Q u_\nu f_\nu(u_\nu)dxdt \le \frac{1}{2}|v|_Q^2 + (3T+2C)sup_{[0,T]}J_\nu(t). \tag{2.61}$$

From (2.10) and (2.59) taking b such that $G_\nu(u_\nu) + b|u_\nu|^2$ be positive, we obtain:

$$J_\nu(t) \le \frac{1}{2}|v|_Q^2 + 2C_0\,E_\nu(0) + C_0\int_0^t J_\nu(s)ds. \tag{2.62}$$

Now by Gronwall's inequality we obtain that:

$$J_\nu(t) \le \left\{\frac{1}{2}|v|_Q^2 + 2C_0E_\nu(0)\right\}e^{C_0T}, \quad \forall t\epsilon[0,T]. \tag{2.63}$$

Since,

$$E_\nu(0) = \frac{1}{2}\left\{\|u_0\|^2 + |u_1|_\Omega^2\right\} + \int_\Sigma G_\nu(u_0)dx,$$

we conclude from (2.11) and the hypothesis of Theorem 2.6, that the second member of (2.63) is bounded by a constant $C_1 > 0$, independing of ν. Then from (2.63) we have:

$$sup_{[0,T]} J_\nu(t) \leq \left\{\frac{1}{2}|v|_Q^2 + 2C_0 E_\nu(0)\right\} e^{C_0 T} \leq C_1. \tag{2.64}$$

Then we have that there exist a subsequence of $(u_\nu)_{\nu \in I\!N}$, that we still denote on the same way, and an element $u \epsilon L^\infty(0,T;H_0^1(\Omega))$ such that $u' \epsilon L^\infty(0,T;L^2(\Omega))$, satisfying:

$$u_\nu \rightarrow u \ weak \ star \ in \ L^\infty(0,T;H_0^1(\Omega)). \tag{2.65}$$

$$u'_\nu \rightarrow u' \ weak \ star \ in \ L^\infty(0,T;L^2(\Omega)). \tag{2.66}$$

By (2.61) and (2.63) we obtain that:

$$\int_Q u_\nu f_\nu(u_\nu)dx \leq 3C_2 e^{C_0 T}\left\{|v|_Q^2 + E_\nu(0)\right\} \leq C_3. \tag{2.67}$$

where $C_2 = max\{n, T, C, C_0\}$. But from (2.9) we obtain that $|u_\nu f_\nu(u_\nu)| \leq u_\nu f_\nu(u_\nu) + 2C_0(u_\nu^2 + 1)$, from where we have:

$$\int_Q |u_\nu f_\nu(u_\nu)|dx \leq C_3 + 4C_0 \times C_1 + 2C_0 med(Q) = C_4. \tag{2.68}$$

Then by (2.8) and from Theorem 1.1 of W. A. Strauss [4] we have that:

$$f_\nu(u_\nu) \rightarrow f_{(u)} \ strongly \ in \ L^1(Q). \tag{2.69}$$

Then we conclude that u is a solution of problem (1.1). Finally from (2.59) and (2.63) we have that there exist a constant C_5 (independing of ν) such that:

$$\int_\Sigma \left|\frac{\partial u_\nu}{\partial \eta}\right|^2 d\Sigma \leq C_5\left\{|v|_Q^2 + E_\nu(0) + \int_Q |G_\nu(u_\nu)|\,dxdt\right\}. \tag{2.70}$$

But from (2.60) and (2.64) we have that there exist a constant C_6 such that,

$$\int_Q G_\nu(u_\nu)dx \leq C_6\{|v|_Q^2 + E_\nu(0)\}. \tag{2.71}$$

Now from (2.10) we have that $G_\nu(t) \leq -C_0(t^2+2|t|)$, from where we conclude that $|G_\nu(t)| \leq G_\nu(t) + 2C_0(t^2+2|t|)$ taking $t = u_\nu$ we have after to to integrate in Q, that:

$$\int_Q |G_\nu(u_\nu)dxdt \leq \int_Q G_\nu(u_\nu)dxdt + 2C_0\int_Q u_\nu^2 + 2|u_\nu|dxdt. \tag{2.72}$$

On the other hand, there exist a constant C_7 such that:

$$\int_Q |u_\nu|^2 + 2|u_\nu|dxdt \leq C_7\ sup_{[0,T]}J_\nu(t). \tag{2.73}$$

From (2.70), (2.71), (2.72) and (2.73) we obtain another constant, say C_∞ such that:

$$\int_\Sigma \left|\frac{\partial u_\nu}{\partial \eta}\right| d\Sigma \leq C_8\{|v|_Q^2 + E_\nu(0)\}. \tag{2.74}$$

Since the second member of (2.74) is bounded we obtain a subsequence of,

$$\left(\frac{\partial u_\nu}{\partial \eta}\right)_{\nu\in I\!N}$$

and a element χ in $L^2(\Sigma)$ such that:

$$\frac{\partial u_\nu}{\partial \eta} \rightharpoonup \chi \quad weak\ in\ L^2(\Sigma). \tag{2.75}$$

But,

$$\frac{\partial u_\nu}{\partial \eta} \rightarrow \frac{\partial u}{\partial \eta} \quad in \quad H^{-1}(0,T;W^{-1/p-2,p'}(\Gamma)),$$

where $p > \frac{3}{2}$, then we have that $\chi = \frac{\partial u}{\partial \eta}$, and letting $\nu \to \infty$ in (2.74) we have,

$$\int_\Sigma \left|\frac{\partial u}{\partial \eta}\right|^2 d\Sigma \leq C_8\{|v|_Q^2 + E(0)\}. \qquad Q.E.D.$$

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