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# ON THE DOMAIN OF STRING PERTURBATION THEORY *

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## ABSTRACT

For a large class of effectively closed surfaces, it is shown that the only divergences in string scattering amplitudes at each order in perturbation theory are those associated with the coincidence of vertex operators and the boundary of moduli space. This class includes all closed surfaces of finite genus, and infinite–genus surfaces which can be uniformized by a group of Schottky type. While the computation is done explicitly for bosonic strings in their ground states, it can also be extended to excited states and to superstrings. The properties of these amplitudes lead to a definition of the domain of perturbation theory as the set of effectively closed surfaces. The implications of the restriction to effectively closed surfaces on the behavior of the perturbation series are discussed.

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String scattering amplitudes can be calculated by summing over all possible histories between initial and final states, with the world sheets swept out by the strings being Riemann surfaces. There has been considerable interest in the properties of these surfaces that are needed in the scattering amplitudes, because they are directly related to the duality and the ultraviolet finiteness of string theory. Divergences do occur in the amplitudes, of course, but their sources can also be understood within the context of Riemann surface theory. First, there are physical singularities associated with coincidence of vertex operators on the world sheet. Second, there are infrared divergences for bosonic strings when tachyon and massless dilaton states propagate in long thin tubes, which arise from pinching a surface at or between the handles, a process that corresponds to approaching the boundary of moduli space. Recently, another potential divergence in string perturbation theory has been revealed by introducing a genus-independent cut-off in moduli space. Gross and Periwal [1] find that even with this cut-off the series may still diverge, as the $g^{th}$ order contribution to the bosonic string partition function grows as $c^g g!$

Since the basis of string theory is the perturbative expansion of the S-matrix, infinities are particularly important for the superstring models. In these models, the tachyon and infrared divergences can be eliminated by a GSO projection, so that superstring amplitudes are actually finite at each order in perturbation theory. It is possible that a divergence of the type found by Gross and Periwal also occurs for superstrings, but the explicit computation, which would be of considerable interest, has yet to be done. Moreover, even though it is widely believed that string theory requires a more complete formulation including non-perturbative effects and a mechanism for selecting the correct ground state, this would not ultimately diminish the significance of divergences in superstring amplitudes. A realistic theory containing the standard model in the low-energy limit should allow for consistent perturbative calculations, at least in the weak-coupling region. It would therefore be essential to identify all sources of divergences in such a theory, and the aim of this paper is to investigate a new class of string scattering amplitudes, which conceivably could introduce a new set of infinities, but actually leads to the same types of divergences found in other amplitudes.

The on-shell amplitudes derived from the Polyakov path integral involve an integration of the positions of the vertex operators on world sheets of arbitrary topology. While the sum over histories usually consists of evaluating diagrams with only a finite number of handles, it could also be extended to surfaces of infinite genus, as Friedan and Shenker [2] proposed, for example, when they formulated string theory in terms of the geometry of a universal moduli space. Indeed, it will be shown in this paper that there are several reasons for including a certain class of infinite- genus surfaces in the sum over histories. These are the surfaces that can be regarded as effectively closed and thus most resemble the closed finite-genus surfaces occurring at finite orders in perturbation theory. Effectively closed surfaces can be precisely characterized by their (ideal) boundaries, which must have Hausdorff dimension less than one, and they have several distinctive properties that will be investigated here. From an analysis of scattering amplitudes and physical considerations, one is naturally led to the following conclusion: string perturbation theory gives rise to a well-defined domain in the space of Riemann surfaces, and this domain is the set of effectively closed surfaces.

Let us begin by recalling that the scattering amplitude for N closed bosonic strings at finite order g in perturbation theory is given by an integral over a compact Riemann surface of genus g

$$\int \prod_i d^2 z_i \sqrt{g(z_i)} \langle V_1(z_1) ... V_N(z_N) \rangle \qquad (1)$$

followed by an integral over moduli space $M_g$ [3]. When the strings are in their tachyonic ground states, the vertex operators are $V_i(z_i) = e^{ip_i \cdot X(z_i)}, p_i^2 = -8$, and after normal ordering, the integral (1) becomes

$$\int \prod_i d^2 z_i \sqrt{g(z_i)} \prod_{i<j} e^{-\frac{p_i \cdot p_j}{2} \tilde{G}^{sym}(z_i, z_j)} \qquad (2)$$

where

$$\tilde{G}^{sym}(z_i, z_j) = G^{sym}(z_i, z_j) - \frac{1}{2}\hat{G}(z_i, z_i) - \frac{1}{2}\hat{G}(z_j, z_j) \qquad (3)$$

with $G^{sym}(z_i, z_j)$ being the Green function for the scalar Laplacian, symmetrized with respect to $z_i, z_j$

$$\hat{G}(z_i, z_j) = lim_{z_i' \to z_i} [G(z_i, z_i') - log d(z_i, z_i')] \qquad (4)$$

where $d(z_i, z_i')$ is the distance between $z_i$ and $z_i'$. The subtraction in (4) is clearly determined by the metric in the neighborhood of $z_i$, and under local rescalings of the metric, the changes in the exponential terms in (2) are cancelled by those in the area elements, so that the entire expression is conformally invariant [4]. On any closed surface, it follows from Stokes' theorem that the Green function for the scalar Laplacian does not exist when there is a single delta function source. A solution can be obtained, however, when a second source of the opposite strength is placed on the surface. Therefore, a Green function can be defined, but it will depend on the position of three, and even four, points, if the value at P is measured with respect to the value at some fixed point Q. Denoting the Green function by $G_{QS}(P, R)$, one easily sees that

$$\begin{aligned} G_{QS'}(P,R) &= G_{QS}(P,R) + G_{QS'}(P,S) \\ G_{QS'}^{sym}(P,R) &= G_{QS}^{sym}(P,R) + \frac{1}{2}G_{QS'}(P,R) + \frac{1}{2}G_{QS'}(R,S) \end{aligned} \qquad (5)$$

by adding a negative charge at $S$ and a positive charge at $S'$ to the surface in Fig.1. Equation (5) gives the dependence of the symmetrized Green function on the position of the positive charge. Upon substituting $G_{QS}(P, R)$ in (3), one finds that $\tilde{G}^{sym}$ is independent of S. Similarly, it is independent of Q, which is even more obvious, since the inclusion of Q in the Green function is convenient but not necessary. As $\tilde{G}^{sym}$ depends only on P,R, the integral (2) is well-defined.

It might appear that a single source would suffice on open surfaces. However, non-existence of the Green function with a single source defines a class of manifolds customarily



denoted as $O_G$ [5], which includes not only all closed finite-genus surfaces, but also many surfaces of infinite genus. They correspond precisely to the effectively closed manifolds with boundaries of zero linear measure, which can be best characterized by the uniformization theorem. Any surface of genus $g \geq 2$ has the unit disk U as a simply-connected covering and is homeomorphic to [U-{limit points of G}]/G, where G is the uniformizing Fuchsian group. If G is a Fuchsian group of the first kind, the fundamental domain has no border arc on the unit circle, and the ideal boundary, which is the complement of the set of limit points on the circle, factored by G, has zero linear measure.

Thus, for surfaces of type $O_G$, it is necessary to add a second source of opposite strength to obtain a Green function. By a well-known theorem on Riemann surfaces [6], there always exists a function with the correct logarithmic behavior at the singularities which is harmonic and square-integrable outside a neighborhood of these singularities. An explicit formula for the Green function on compact surfaces can be given in terms of prime forms. Alternatively, one can use the representation of the surface as $D/\Gamma$, where D is a domain in the extended complex plane and $\Gamma$ is a discontinuous subgroup of SL(2,C) leaving D invariant [7], and then apply the method of images. In addition to the uniformization by Fuchsian groups, any closed surface of genus g can be uniformized by a Schottky group, which is the free product of g infinite cyclic groups generated by linear transformations $T_1, ..., T_g$. Let $T_n z = \frac{\alpha_n z + \beta_n}{\gamma_n z + \delta_n}, n = 1, ..., g$, and define the isometric circle of $T_n$ to be $I_{T_n} = \{z \in C \cup \infty \, | \, |\gamma_n z + \delta_n| = 1\}$ [8]. The transformation $T_n$ maps the outside of $I_{T_n}$ to the inside of $I_{T_n^{-1}}$ and joins the two isometric circles to create a handle, provided the circles $\{(I_{T_n}, I_{T_n^{-1}}), n = 1, ..., g\}$ are disjoint. Since $\Gamma$ maps any point outside the 2g circles to points inside the circles, the exterior of the circles is taken to be the fundamental region of $\Gamma$ and is homeomorphic to the Riemann surface. The sources at R,S lie in the fundamental region and their images under $\Gamma$ are inside the circles. The Green function should be invariant under $\Gamma$ and it straightforward to show that [9]

$$\begin{aligned} G_{QS}(P,R) &= \sum_i ln \left| \frac{z_P - V_i z_R}{z_P - V_i z_S} \frac{z_Q - V_i z_S}{z_Q - V_i z_R} \right| \\ &\quad - \frac{1}{2\pi} \sum_{m,n=1}^{g} Re\{v_m(z_P) - v_m(z_Q)\} (Im \tau)^{-1}_{mn} Re\{v_n(z_R) - v_n(z_S)\} \\ v_n(z) &= \sum_i{}^{(n)} ln \left( \frac{z - V_i \xi_{1n}}{z - V_i \xi_{2n}} \right) \qquad v_n(z) - v_n(T_m z) = 2\pi i \tau_{mn} \end{aligned} \qquad (6)$$

where the $V_i$ are arbitrary products of the generators $T_1, ..., T_g$, with i labelling the elements of $\Gamma$, $\xi_{1n}, \xi_{2n}$ are the two fixed points of $T_n$, and $\sum_i^{(n)}$ represents the sum over all $V_i$ that do not have $T_n^{\pm 1}$ at the right-hand end of the product. The first sum in (6) is the expected contribution from the sources at R,S and their images, while the second term is required to make the Green function single- valued.

While P,Q,R,S are bounded away from the isometric circles, both terms in (6) are finite if the Poincare series $\sum_{i \neq I} |\gamma_i|^{-2}$ is convergent. The series has been proven to converge when the parameters of the transformations $T_n$ satisfy certain inequalities [10]. It has also been shown that the series does not converge for all Schottky groups. Even when the sums in (6) are not finite, however the formula for the Green function in terms of prime forms demonstrates its existence on all compact surfaces.

Now suppose that $\Gamma$ has an infinite number of generators. The domain D is usually taken to be the set of ordinary points of $\Gamma$, so that $z \in D$ if there is a neighborhood $N_z$ such that $V_i z \notin N_z$ for all $V_i \neq I$. If this domain is disconnected, $D/\Gamma$ may be the union of two or more Riemann surfaces. To obtain a single surface, it sufficient that D be connected, and this property will hold if the fundamental region is connected. The surface will have infinite genus if the fundamental region has an infinite number of boundary components. It is interesting to note here that there are examples of infinitely generated groups $\Gamma$ for which the isometric circles cover almost the entire complex plane, the fundamental region has a finite number of boundary components, and $D/\Gamma$ is a surface of finite genus [11]. These are exceptional cases, however, and, in general, an infinitely generated group gives rise to an infinite-genus surface. The precise requirement for $D/\Gamma$ to be an infinite-genus surface is that $\frac{d_n}{r_n} > 1 + \epsilon$ for all n and some $\epsilon$ with $d_n$ being the distance from the center $I_{T_n}$ or $I_{T_n^{-1}}$ to any of the other circles, and $r_n$ being the radius of $I_{T_n}$ [12]. The simplest example involves an extension of the Schottky group for which the isometric circles are non-overlapping and can be joined pairwise to create a sphere with an infinite number of handles. As the handles accumulate at some point on the sphere, this point must be removed to obtain a manifold (Fig.2).

Since the surface can be represented as $D/\Gamma$, where $\Gamma$ is infinitely generated, the method of images can be used again to obtain the Green function. The finiteness of the expression which depends on the convergence of the Poincare series $\sum_{i \neq I} |\gamma_i|^{-2}$, is more difficult to prove when $\Gamma$ is infinitely generated. Let us first recall Burnside's proof of convergence for Schottky groups with generators $T_1, ..., T_g$. As the elements of the group are products of the fundamental transformations, they may be classified according to the number of factors in each product. Thus the first set consists of 2g elements, the second set consists of 2g(2g-1) elements, and so on. The action of each $T_n$ can be expressed in terms of its two fixed points $\xi_{1n}, \xi_{2n}$

$$\frac{T_n z - \xi_{1n}}{T_n z - \xi_{2n}} = K_n \frac{z - \xi_{1n}}{z - \xi_{2n}} \tag{7}$$

where $K_n$ is the multiplier of the transformation. Burnside [10] showed that if the absolute values of the multipliers, $|K_1|, ..., |K_g|$, are sufficiently large, the ratio of $|\gamma_{(l+1)}|$ to $|\gamma_{(l)}|$ is bounded below, where $\gamma_{(l)}, \gamma_{(l+1)}$ refer to parameters af any transformation in the $l^{th}$ and $(l+1)^{th}$ sets respectively. Suppose $|\frac{\gamma_{(l+1)}}{\gamma_{(l)}}| > k$ and the lower bound for $|\gamma_1|, ..., |\gamma_n|$ is $|\bar{\gamma}|$. Then it follows that

$$\sum_{i \neq I} |\gamma_i|^{-2} < |\bar{\gamma}|^{-2} \left[ 2g + \frac{2g(2g-1)}{k^2} + \frac{2g(2g-1)^2}{k^4} + ... \right] \tag{8}$$



which is finite if $k^2 > 2g - 1$.

Although the argument cannot be applied directly when $g = \infty$, it may be modified to obtain a convergence proof in this case as well. Note that $|\gamma_i|^{-1}$ is the radius $r_i$ of the isometric circle $I_{V_i} = \{z \in C |\; |\gamma_i z + \delta_i| = 1\}$. The first term on the right-hand side of the inequality in (8) thus represents the sum of the square of the radii of the isometric circles $\{I_{T_n}, I_{T_n^{-1}}\}$. No distinction is made between these radii in Burnside's proof. However, if one chooses $r_n \to 0$ sufficiently fast as $n \to \infty$ (Fig. 3), the first term becomes finite. A similar calculation for the higher-order terms leads to the following theorem.

**Theorem 1.** Let $\Gamma$ be a discontinuous group acting on the complex sphere with an infinite number of generators $\{T_n\}$ that have non- overlapping isometric circles. If the distance between the circles is bounded below and the distance between $I_{T_n}$ and $I_{T_n^{-1}}$ is bounded above for all n the Poincare series $\sum_{i \neq I} |\gamma_i|^{-2}$ converges if the radii $r_n$ decrease to zero rapidly enough as $n \to \infty$.

**Proof.** Let $K_n$, $\xi_{1n}$, $\xi_{2n}$ be the multiplier and fixed points of $T_n$. From equation (7),

$$\gamma_n = \frac{K_n^{\frac{1}{2}} - K_n^{-\frac{1}{2}}}{\xi_{2n} - \xi_{1n}} \qquad \delta_n = \frac{\xi_{2n} K_n^{-\frac{1}{2}} - \xi_{1n} K_n^{\frac{1}{2}}}{\xi_{2n} - \xi_{1n}} \tag{9}$$

Suppose $V_i$ is some product of the generators and $V_{i+1} = T_n V_i$. Then

$$\frac{\gamma_{i+1}}{\gamma_i} = \gamma_n \frac{\alpha_i}{\gamma_i} + \delta_n = K_n^{\frac{1}{2}} \frac{\frac{\alpha_i}{\gamma_i} - \xi_{1n}}{\xi_{2n} - \xi_{1n}} + K_n^{-\frac{1}{2}} \frac{\xi_{2n} - \frac{\alpha_i}{\gamma_i}}{\xi_{2n} - \xi_{1n}} \tag{10}$$

so that

$$\left|\frac{\gamma_{i+1}}{\gamma_i}\right| > (|K_n|^{\frac{1}{2}} - |K_n|^{-\frac{1}{2}}) \left|\frac{\frac{\alpha_i}{\gamma_i} - \xi_{1n}}{\xi_{2n} - \xi_{1n}}\right| - |K_n|^{-\frac{1}{2}} \tag{11}$$

As $T_n^{-1}$ cannot be the leftmost member of $V_i$, and since the isometric circles are non-overlapping,, $I_{V_i^{-1}} \cup I_{T_n} = 0$. The point $\frac{\alpha_i}{\gamma_i}$ lies at the center of $I_{V_i^{-1}}$ and so it is not inside $I_{T_n}$. The fixed points $\xi_{1n}, \xi_{2n}$ are in $I_{T_n}, I_{T_n^{-1}}$ respectively and thus $|\frac{\alpha_i}{\gamma_i} - \xi_{1n}|$ is bounded below and $|\xi_{2n} - \xi_{1n}|$ is bounded above. Define c to be the lower bound for $\left|\frac{\frac{\alpha_i}{\gamma_i} - \xi_{1n}}{\xi_{2n} - \xi_{1n}}\right|$ and $c'$ to be the upper bound for $|\xi_{2n} - \xi_{1n}|$. If $|K_n|^{\frac{1}{2}} = c_1 n^q + c_2$ it is straightforward to show that $|\gamma_n| > \frac{c_1}{c'} n^q$ and $\left|\frac{\gamma_{i+1}}{\gamma_i}\right| > c c_1 n^q$, provided $c_2^2 > 1 + \frac{1}{c}$. The following bound can then be placed on the Poincare series

$$\sum_{i \neq I} |\gamma_i|^{-2} < c^2 c'^2 \left[ \frac{2}{c_1^2 c^2} \sum_{n=1}^{\infty} \frac{1}{n^2 q} + \left( \frac{2}{c_1^2 c^2} \sum_{n=1}^{\infty} \frac{1}{n^{2q}} \right)^2 + ... \right] \tag{12}$$

which converges if $c_1^2 > \frac{2}{c^2} \sum_{n=1}^{\infty} \frac{1}{n^{2q}}$. In terms of the radii of the isometric circles, the sum is finite if $r_n$ falls off faster than $\frac{cc'}{\sqrt{2\zeta(2q)}} n^{-q}$, with $q > \frac{1}{2}$.

While convergence of the Poincare series implies that the first term on the right-hand side of (6) is finite, the second term could lead to further conditions when $g = \infty$. In particular, the imaginary part of the period matrix, $Im\tau$, has an inverse if it is symmetric and positive-definite, properties which follow from the period relations for finite-genus surfaces [6].

$$(\omega, \sigma^*) = \sum_{k=1}^{g} \left[ \int_{A_k} \omega \int_{B_k} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{B_k} \omega \right] \tag{13}$$

where $\omega, \sigma$ are harmonic differentials and the cycles $A_k, B_k$ represent a canonical homology basis. These relations can be generalized to open manifolds [13], including infinite-genus surfaces. While the generalized relations involve an extra boundary term, this vanishes when the ideal boundary has zero linear measure, and the relations reduce to the usual form (13) for surfaces in the class $O_G$. The next theorem is needed to show that $D/\Gamma$ is in $O_G$.

**Theorem 2.** If $\Gamma$ is a group of Schottky type, with either a finite or infinite number of generators, whose isometric circles are non-overlapping, and D is the set of ordinary points in the extended complex plane then $D/\Gamma$ is a Riemann surface in the class $O_G$.

**Proof.** Let us begin by recalling that if $\Gamma$ is a finitely generated Schottky group with non-overlapping isometric circles, then $D/\Gamma$ is a closed finite-genus surface [14], which is certainly in $O_G$. The theorem, then, only has to be proven when $\Gamma$ is infinitely generated.

The first step is to note that a better understanding of the classification type of a Riemann surface can be achieved by passing to its universal covering rather than the intermediate Schottky covering. A surface is in $O_G$ if the uniformizing Fuchsian group acting on the unit disk is of the first or second kind (Fig. 4), which in turn is equivalent to the divergence or convergence of the associated Poincare series [15]. Therfore, given the Schottky uniformization of a Riemann surface, one would like to find the corresponding Fuchsian uniformization to determine whether the surface is in $O_G$. To obtain explicitly the parameters characterizing both the Schottky and Fuchsian uniformization of a surface is a difficult problem [16], but it will now be shown that the existence of a Schottky uniformization is sufficient to imply that the Poincare series for the Fuchsian group diverges.

Recall that if $A_1, B_1, ..., A_g, B_g$ represent a canonical homology basis on a finite-genus Riemann surface, the Schottky group is $\Gamma = \langle B_1, ..., B_g \rangle$ while the Fuchsian (or fundamental) group is $G = \langle A_1, ..., A_g, B_1, ..., B_g | A_1 B_1 A_1^{-1} B_1^{-1} ... A_g B_g A_g^{-1} B_g^{-1} = 1 \rangle$. While the fundamental group is not free for finite g, there is a natural projection from G to $\Gamma$ [17] by mapping the generators $A_1, ... A_g$ to the identity, and it is clear that every word $T \in \Gamma$ has an infinite number of preimages in G, $\{\tilde{T}_i\}$. When g is infinite, $\Gamma$ is again a free group generated by transformations associated with the B-cycles. The representation of the fundamental group given above also has an obvious generalization, but some care must be taken with the defining relations among the generators, as they customarily involve equating words of finite length with the identity. Nevertheless, the intuitive picture of the fundamental group is supported by the following result about open surfaces: the fundamental group of any subsurface, with compact closure, of a connected $C^\infty$ open Riemann surface is a free group generated by transformations corresponding to the A- and B-cycles contained in the subset [18]. There is then a projection from this fundamental group to a finitely-generated subgroup of the Schottky group which can presumably be extended to a projection from G to $\Gamma$, although it will be sufficient to consider the finitely-generated subgroups to prove the theorem.

Since the Riemann surface is homeomorphic to $D/\Gamma \simeq \tilde{D}/G$ ($\tilde{D}$ being the complement of the set of limit points of G in the unit disk), given a coordinate neighborhood N of a point z on the surface, there exist neighborhoods $N_S$ of $z_S$ and $N_U$ of $z_U$ in the fundamental domains for $\Gamma$ and G that are homeomorphic to N. Let $\Phi_0$ be the homeomorphism from $N_U$ to $N_S$ and define $\Phi_i$ to be the homeomorphism from $\tilde{T}_i N_U$ to $TN_S$, so that $\tilde{T}_i z_U = \Phi_i^{-1} \cdot T \cdot \Phi_0(z_U) = \Phi_i^{-1} \cdot T(z_S)$. This relates the action of the Fuchsian group on the unit disk with the action of the Schottky group on the Schottky covering surface and also leads to a relation between the Poincare series for the two groups. For a word $T \in \Gamma$, a minimal set of elements $\{\tilde{T}_i\}$ in G which project to T is obtained by adding to T transformations corresponding to the A-cycles. It is clear that the preimages of words of finite length in $\Gamma$ form disjoint sets in G, so that the Poincare series for G can be written as

$$\sum_{\tilde{T} \in G} |\tilde{T}'(z_U)| = \sum_{T \in \Gamma} \sum_{i} |\tilde{T}_i'(z_U)| = \sum_{T \in \Gamma} \sum_{i} |\Phi_i^{-1\prime}(T\Phi_0(z_U)) \Phi_0'(z_U)| |T'(z_S)| \tag{14}$$

(Recall that $\sum_{T\in\Gamma} |T'(z_S)| = \sum_{T \in \Gamma} |\gamma z_S + \delta|^{-2}$. Convergence of this series is equivalent to convergence of the series in Theorem 1 provided $z_S$ is bounded away from the centers of all of the isometric circles, which holds true when $z_S$ lies in the fundamental domain of the Schottky group.) We first wish to consider the summation over i in (14). Fix T to be a word of length $N_1$ and suppose that $\tilde{T}_i$ is a word of length $N_1 + N_2$ which projects to $N_2$, where $N_2$ is the number of generators corresponding to the A-cycles. As the limit points of the Fuchsian group G lie on the unit circle, the neighborhood $\tilde{T}_i N_U$ is near the boundary and decreases in Euclidean size as $N_2$ becomes large. It would seem then that in the limit as $N_2 \to \infty$, $\tilde{T}_i'(z_U) \to 0$ and the sum in (14) could converge. However, it easy to show that a lower bound of $c^{N_2}|T'(z_S)|$, $c < 1$, can be placed on the decrease of $\tilde{T}_i'(z_U)$ with the length of the word. Moreover, the number of irreducible words of length $N_1 + N_2$ in G which project to T is $2g(2g-1)^{N_2 - 1} \frac{(N_1+N_2)!}{N_1! N_2!}$, where g is the number of A-cycle generators that have been added to T. The simplest way to define g is to take the smallest compact subset of the Riemann surface whose fundamental group projects to the minimal subgroup of $\Gamma$ containing T. This is not required, however, as the addition of an arbitrary number of A-cycle generators to $\Gamma$ gives a free subgroup of G. It is convenient therefore, to choose g so that $2g - 1 > \frac{1}{c}$. Since $N_1 \geq 1$,

$$\sum_{\tilde{T} \in G} |\tilde{T}'(z_U)| > \left[ \sum_{N_2 = 0}^{\infty} (N_2 + 1) \right] \sum_{T \in \Gamma} |T'(z_S)| \tag{15}$$

From (15), it is clear that regardless of whether the Poincare series for the Schottky group $\Gamma$ converges, the series for the corresponding Fuchsian group G will always diverge. Consequently, the Riemann surface must be in $O_G$.

From this theorem it follows that the period relation holds on $D/\Gamma$ and the imaginary part of the period matrix has an inverse. The finiteness of the second term in the right-hand side of equation (6) can now be easily demonstrated. Let $z_P$ be a point on the isometric circle $I_{T_l}$ and $z_Q = T_l z_P$ be the corresponding point on $I_{T_l^{-1}}$. Then

$$\begin{aligned}\frac{1}{2\pi}\sum_{m,n=1}^{\infty} & Re\{v_m(T_l z_P) - v_m(z_P)\}\,(Im\tau)^{-1}_{mn}\,Re\{v_n(z_R) - v_n(z_S)\} \\ &= \sum_{m,n=1}^{\infty}(Im\tau)_{lm}\,(Im\tau)^{-1}_{mn}\,Re\{v_n(z_R) - v_n(z_S)\} \\ &= Re\{v_l(z_R) - v_l(z_S)\}\end{aligned} \tag{16}$$

is finite since the Poincare series for the Schottky group converges. Now note that (16) is of the form $\int_{z_P}^{T_l z_P} f(z)dz$ for some function f(z). If we take two paths from $z_P$ to $T_l z_P$, the result must be the same and given by (16). Then the integral $\oint_C f(z)dz$ over the contour in Fig. 5 is zero and Morera's theorem implies that f(z) has no singularities in the domain bounded by C. Since C can be an arbitrary contour connecting the points $z_P$, $T_l z_P$, f(z) has no singularities in the entire fundamental domain, and thus, for any two points $z_P$, $z_Q$ in the region exterior to the isometric circles, the second term on the right-hand side of equation (6) will be finite.

Having established the suitability of the series expansion for the Green function on these infinite-genus surfaces, we would like to determine its behavior near the isometric circles, particularly in the region where they accumulate at $\infty$, because the amplitude (2) involves an integration over the entire fundamental domain. While all the image charges are inside the isometric circles, it is conceivable that they might lie arbitrarily close to the circles so that $G_{QS}(P,R)$ could diverge as $z_P$ approaches these boundaries. In fact, this possibility does not occur because of the following proposition.

**Proposition.** There are no limit points on the isometric circles.

**Proof.** Suppose that there is a limit point $z_0$ on the isometric circles $I_{T_n}$. Then there is an infinite sequence of elements $V_i$ such that $lim_{i\to\infty} V_i z = z_0$ for some $z$ in the fundamental domain (Fig. 6). The points $V_i z$ must approach $z_0$ from the interior of $I_{T_n}$ because they cannot exist in the fundamental domain and the isometric circles are non-overlapping. It follows that $V_i$ must have the form $T_n^{-1}V_i'$ for all i. So $lim_{i\to\infty}T_n^{-1}V_i'z = z_0 \Rightarrow lim_{i\to\infty}V_i'z = T_n z_0$ which lies on $I_{T_n^{-1}}$. Since $V_i'$ cannot have $T_n$ as its left-most member, $V_i'z$ must be in any isometric circle except $I_{T_n^{-1}}$. The points $\{V_i'\}$ cannot accumulate at $T_n z_0$, and so $\{V_i z\}$ cannot accumulate at $z_0$.

It is obvious that if there are no limit points on the isometric circles, then the image charges cannot lie arbitrarily close to circles of finite radius. One may also recall that the size of the circles has been allowed to decrease to zero, and in this case, the image charges inside the isometric circles are arbitrarily close to the fundamental domain as the circles



approach the accumulation point. In this limit, however, the positive and negative charges cancel and the Green function remains finite.

Let us now consider the amplitude for the scattering of four closed bosonic strings in their tachyonic ground state. Recall from equation (1) that the amplitude is supposed to obtained by integrating over the positions of the four vertex operators followed by an integral over 3g-3 modular parameters. However, the SL(2,C) invariance of the Schottky parametrization can be alternatively be used to fix the positions of the vertex operators at $z_1^0$, $z_2^0$, $z_3^0$, for example [9]. Denoting the fundamental domain by $\Delta$, the amplitude becomes

$$\begin{aligned} f(z_1^0, z_2^0, z_3^0)\int_\Delta d^2z_4\,|z_4 - z_1^0|^{\frac{-p_1\cdot p_4}{2}}\prod_{i\neq I}{}'\left|\frac{z_4 - V_i z_1^0}{z_4 - V_i z_4}\frac{z_1^0 - V_i z_4}{z_1^0 - V_i z_1^0}\right|^{\frac{-p_1\cdot p_4}{2}} \\ \prod_{mn} exp\left[\frac{p_1\cdot p_4}{8\pi}Re(v_m(z_4) - v_m(z_1^0))(Im\tau)^{-1}Re(v_n(z_4) - v_n(z_1^0)\right] \\ \cdot(\text{similar factors with } z_1^0 \to z_2^0, z_3^0,\ p_1 \to p_2, p_3)\end{aligned} \tag{17}$$

where $\prod'_{i\neq I}$ indicates that the product is restricted to one of each pair $(V_i, V_i^{-1})$ and

$$\begin{aligned} f(z_1^0, z_2^0, z_3^0) = |z_2^0 - z_1^0|^{2-\frac{p_1\cdot p_2}{2}}|z_3^0 - z_1^0|^{2-\frac{p_1\cdot p_3}{2}}|z_3^0 - z_2^0|^{2-\frac{p_2\cdot p_3}{2}} \\ \prod_{i\neq I}{}'\left|\frac{z_2^0 - V_i z_1^0}{z_2^0 - V_i z_2^0}\frac{z_1^0 - V_i z_2^0}{z_1^0 - V_i z_1^0}\right|^{\frac{-p_1\cdot p_2}{2}}\left|\frac{z_3^0 - V_i z_1^0}{z_3^0 - V_i z_3^0}\frac{z_1^0 - V_i z_3^0}{z_1^0 - V_i z_1^0}\right|^{\frac{-p_1\cdot p_3}{2}} \\ \left|\frac{z_3^0 - V_i z_2^0}{z_3^0 - V_i z_3^0}\frac{z_2^0 - V_i z_3^0}{z_2^0 - V_i z_2^0}\right|^{\frac{-p_2\cdot p_3}{2}} \\ \prod_{m,n} exp\left[\frac{p_1\cdot p_2}{8\pi}Re(v_m(z_2^0) - v_m(z_1^0))(Im\tau)^{-1}_{mn}Re(v_n(z_2^0) - v_n(z_1^0))\right. \\ + \frac{p_1\cdot p_3}{8\pi}Re(v_m(z_3^0) - v_m(z_1^0))(Im\tau)^{-1}_{mn}Re(v_n(z_3^0) - v_n(z_1^0)) \\ + \frac{p_2\cdot p_3}{8\pi}Re(v_m(z_3^0) - v_m(z_2^0))(Im\tau)^{-1}_{mn}Re(v_n(z_3^0) - v_n(z_2^0))\end{aligned} \tag{18}$$

Finiteness of the integral (17) in the neighborhood of $z_1^0$, $z_2^0$, $z_3^0$ requires that $p_1\cdot p_4$, $p_2\cdot p_4$, $p_3\cdot p_4 < 4$. Momentum conservation then implies that $p_1\cdot p_4 + p_2\cdot p_4 > 4$. In terms of the Mandelstam variables, the allowed range is $-16 < s,t,u < -8, s+t, s+u, t+u > -16$. The existence of some range of momenta for which the integral is finite is necessary for analytic continuation of the amplitude to physical values of s, t, u. Since the Green function $G_{QS}(P,R)$ only has singularities at $z_R, z_S$ the integral (17) is similarly well-behaved and can be written as

$$f(z_1^0, z_2^0, z_3^0)\int_\Delta d^2z_4\,|z_4 - z_1^0|^{\frac{-p_1\cdot p_4}{2}}|z_4 - z_2^0|^{\frac{-p_2\cdot p_4}{2}}|z_4 - z_3^0|^{\frac{-p_3\cdot p_4}{2}}\Phi(z_4, \bar{z}_4) \tag{19}$$

where $\Phi(z_4, \bar{z}_4)$ is regular throughout the fundamental region [10]. Dividing $\Delta$ into three disks of radius $\Lambda$ about $z_1^0$, $z_2^0$, $z_3^0$ and the remainder of the fundamental domain, the integral (19) is

$$
\begin{aligned}
&\int_0^{2\pi} \int_{|z_4 - z_1^0| \le \Lambda} d\theta d|z_4 - z_1^0| |z_4 - z_1^0|^{1-\frac{p_1 \cdot p_4}{2}} \\
&\qquad \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(z_4 - z_1^0)^l (\bar{z}_4 - \bar{z}_1^0)^m}{l!\, m!} \partial^l \bar{\partial}^m \{ |z_4 - z_2^0|^{\frac{-p_2 \cdot p_4}{2}} |z_4 - z_3^0|^{\frac{-p_3 \cdot p_4}{2}} \Phi(z_4, \bar{z}_4) \}_{z_4 = z_1^0} \\
&\qquad + \ (similar\ terms\ with\ z_1^0 \to z_2^0, z_3^0,\ p_1 \to p_2, p_3) \ + \ finite \\
&= 2\pi \sum_{n=0}^{\infty} \frac{\Lambda^{\frac{-p_1 \cdot p_4}{2} + 2n + 2}}{\frac{-p_1 \cdot p_4}{2} + 2n + 2} \frac{1}{(n!)^2} (\partial \bar{\partial})^n \{ |z_4 - z_2^0|^{\frac{-p_2 \cdot p_4}{2}} |z_4 - z_3^0|^{\frac{-p_3 \cdot p_4}{2}} \Phi(z_4, \bar{z}_4) \}_{z_4 = z_1^0} \\
&\qquad + \ (similar\ terms\ with\ z_1^0 \to z_2^0, z_3^0,\ p_1 \to p_3, p_3) \ + \ finite
\end{aligned} \tag{20}
$$

From (20), we see that poles in the amplitude occur at s, t, u=8(n-1) , n=0,1,2,... corresponding to the tachyon and the excited states.

As mentioned earlier, the full scattering amplitude also involves an integration over moduli space. Although a precise characterization of the moduli space at infinite genus has not yet been obtained, it is useful to note that the Polyakov measure can be expressed completely in terms of the Schottky parametrization. The N-tachyon g-loop amplitude is

$$
\begin{aligned}
A(1, ..., N) \ = \ & \frac{2\pi}{(4\pi(8\pi^2)^{13})^g} \left(\frac{\kappa}{\pi}\right)^{N+2g-2} \\
& \int \prod_{m=1}^{g} d^2\xi_{1m} d^2\xi_{2m} d^2 K_m |K_m(\xi_{1m} - \xi_{2m})|^{-4} |1 - K_m|^4 \\
& \qquad \prod_i{}'' \, |1 - K_i^n|^{-4} \prod_{n \ge 1} |1 - K_i^n|^{-48} (det\ Im\tau)^{-13} \\
& \int \prod_{s=1}^{N} \frac{d^2 z_s}{Vol(SL(2,C))} \prod_{s,t} e^{\frac{-p_s \cdot p_t}{2}} \tilde{G}^{sym}(z_s, z_t)
\end{aligned} \tag{21}
$$

where $\prod_i''$ is the product over all primitive elements of the Schottky group and $\kappa$ is the string coupling constant [20]. Equation (21) leads to a natural generalization of the Polyakov measure for surfaces that can be uniformized by infinitely generated groups of Schottky type. The moduli space integral would then be infinite-dimensional, of course, and a regularization procedure would have to be developed so that it can be properly defined. One of the most obvious features of the formula for the amplitude given in (21) is the singularity in the coupling constant factor as $g \to \infty$. The elimination of this singularity should constrain the type of regularization that can be used. The regularization is also intimately connected with the divergence found by Gross and Periwal. The cut-off near the boundary of moduli space that they introduce excludes very small handles. However, the size of the handles must decrease sufficiently fast to zero to place an infinite number on the sphere. It follows that their cut-off leads to a large-genus cut-off in the perturbation series, which is significant since the bosonic string partition function is expected to grow



as $g!$. This analysis is based, of course, on the construction of infinite-genus surfaces by the placement of an infinite number of handles, accumulating at a point, on a sphere of finite size. By Theorem 2 these manifolds are in the class $O_G$, which has also been defined in this paper as the set of effectively closed surfaces. Conversely, all effectively closed surfaces can probably be obtained by placing handles on spheres, although this remains to be proven. The point that is being emphasized here, however, is that the restriction of the domain of perturbation theory to the effectively closed surfaces may be of crucial importance in making the perturbative expansion of the S-matrix well-defined.

To conclude, the path integral approach to calculating string scattering amplitudes involves summing over all possible histories between initial and final states, with the joining and splitting of strings in the interaction region describing handles on a Riemann surface. Since the interactions take place in a region of finite size, one is interested in summing over those surfaces which can be placed in a finite box. It is clear that certain types of infinite-genus surfaces, such as spheres with an infinite number of handles may be put in such a box, and they can be naturally included in the perturbative expansion of the S-matrix. This intuitive picture has been confirmed here by the explicit calculation of a bosonic string scattering amplitude, which has revealed no new types of divergences arising from the integration of the positions of the vertex operators over these surfaces. On the other hand, there are other surfaces which do not appear to be part of the perturbation series, because they have an ideal boundary with positive linear measure. If this boundary is not confined within the finite interaction region, then it will be observed as an extra string. Thus, although we may begin by describing the scattering of $n_1$ strings into $n_2$ strings, the diagrams containing surfaces with a boundaries of positive linear measure would correspond to the scattering of $n_1$ strings into $n_2 + 1$ strings. Whether these diagrams should be associated with non-perturbative effects, along the lines suggested by Friedan and Shenker, has yet to be demonstrated, but they cannot be included in the perturbative expansion of the S-matrix which maps $n_1$ initial states into $n_2$ final states. The discussion above motivates the problem of precisely defining string perturbation theory. From the computations done in this paper and general physical considerations, one is led to conjecture that non-compact infinite-genus surfaces, as well as closed finite-genus surfaces, should be included in the series, but the domain of string perturbation theory must be restricted to surfaces that are effectively closed, identified here as belonging to the class $O_G$. Not only does this definition have impact on the the behavior of the series, both for bosonic strings and superstrings, but it also provides a starting point for going beyond the perturbation expansion to obtain a more complete formulation of these theories.

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## Figure Captions

Fig. 1: Sources of opposite strength at R and S on a closed surface of finite genus.

Fig. 2: A sphere with an infinite number of handles with size decreasing to zero. The accumulation point of the handles is removed from the sphere.

Fig. 3: Fundamental region for a Schottky group is the exterior of the isometric circles.

Fig. 4: Fundamental regions of Fuchsian groups of the first and second kinds have border arcs of zero measure and positive measure on the unit disk respectively.

Fig. 5: When the integral of a function around the contour C vanishes, it has no singularities in the shaded domain bounded by C.

Fig. 6: There cannot be any limit points of the Schottky group lying on the isometric circles.

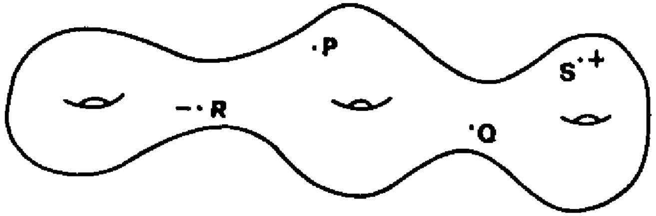



Figure 1

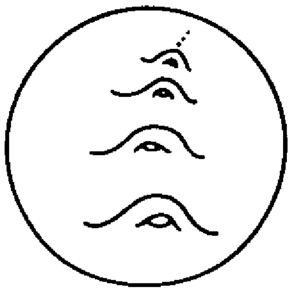


Figure 2



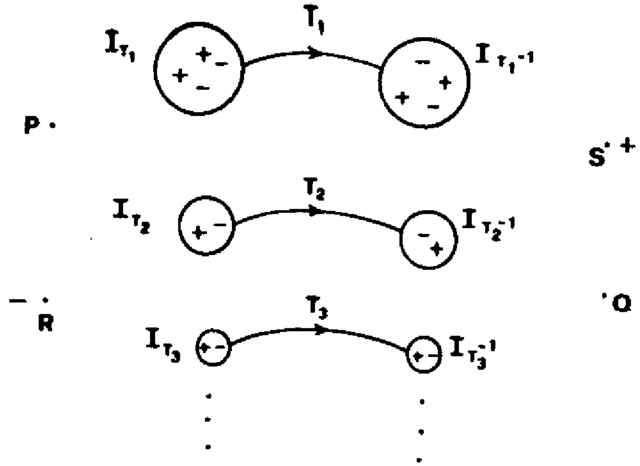
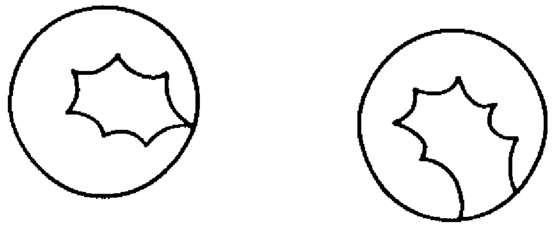



Figure 3



$O_G$ $P_G$

Figure 4

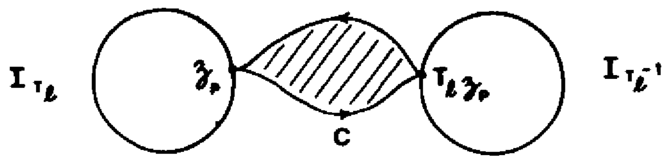



Figure 5

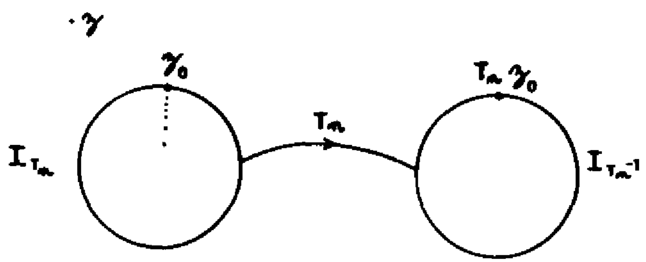



Figure 6

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