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NON NEWTONIAN GRAVITY CREEPING FLOW *

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Abstract

We derive the governing equations for creeping gravity currents of non newtonian liquids having a power law rheology, using a lubrication approximation. We consider unidirectional and axisymmetric currents. The equations differ from those for newtonian liquids, being nonlinear in the spatial derivative of the thickness of the current. However, many solutions are closely analogous to those for newtonian rheology; in particular the spreading relations can also be expressed as power laws of time, with exponents that depend on the rheological index. Similarity solutions for currents whose volume varies as a power of time are obtained. For the spread of a constant volume of liquid, analytic solutions are found. We also derive solutions of the waiting-time type, as well as the ones describing steady flows from a constant source to a sink. General travelling wave solutions are given, and analytic formulae for a simple case are derived. A phase plane formalism, that allows the systematic derivation of self similar solutions, is introduced. The application of the Boltzmann transform is briefly discussed. Present results are closely analogous to those for newtonian liquids; all the solutions obtained here have their counterparts in newtonian flows. This happens because the power law rheology, like the newtonian constitutive relation, involves a single dimensional parameter. Thus one finds similarity solutions whenever the analogous newtonian problem is self similar. Although the spreading relations are rheology-dependent, in most cases the dependence is rather weak. The present results may be of interest for geophysics since the lithosphere deforms according to an average power law rheology.

1. Introduction

Gravity currents are ubiquitous phenomena that occur in many situations of scientific and engineering interest [Simpson, 1982]. Many regimes are possible, according to the relative magnitude of the forces acting on a typical fluid element. Of considerable interest for geophysics is the viscosity dominated regime called the "creeping flow", which is characterized by a nearly horizontal motion that is so slow that inertia effects are negligible, and the flow is governed by a balance between gravity and the viscous forces. Certain magma flows [see Huppert, 1986] belong to this class. The deformations of the lithosphere associated with orogeny can also be described in terms of creeping gravity currents. Scaling laws that describe the time evolution of mountain belts have been obtained by Gratton [1988], who considered the combined effect of crustal shortening, isostasy, and creeping gravity flow at the root of the belt. However, the theory of these flows has been limited so far to newtonian fluids [Huppert, 1982; Gratton and Minotti, 1988], and has been based on the "lubrication theory" approximation [Buckmaster, 1977] which is valid when the length of the current greatly exceeds its thickness [this is usually true, except perhaps at the beginning of the phenomenon]. It should be noticed that a newtonian constitutive relation is not adequate to describe the rheology of the lithosphere [see for example the recent review by Kirby and Kronemberg, 1987], and more general stress-strain rate relationships are required. In addition, many fluids of practical interest are non newtonian, which makes the proposed extension of the theory of considerable interest.

One of the simplest non newtonian models is based on the so called power-law constitutive relation of the form [Byrd, 1976]:

$$\tau_{ij} = A\, E^{1/\iota - 1}\, \varepsilon_{ij} \qquad (1)$$

in which τ_{ij} is the deviatoric stress tensor, $\varepsilon_{ij} = [\partial v_i/\partial x_j + \partial v_j/\partial x_i]/2$ is the strain rate, and

$$E = [\varepsilon_{ij}\varepsilon_{ij}]^{1/2} \qquad (2)$$

is the second invariant of the strain rate tensor; A, and the

rheological index ι are constants. The formula (1) is sometimes known as Ostwald's or Norton's constitutive relation. A power law rheology such as (1) is usually accepted as a good description of the vertically averaged mechanical properties of the lithospheric rocks [see for example Sonder and England, 1983].

In this paper we investigate the creeping gravity currents of a non newtonian liquid that obeys the power law constitutive relation (1). By invoking the lubrication approximation we derive the equations for a current flowing on a rigid horizontal surface, and show that the thickness of the current satisfies a nonlinear parabolic differential equation which is a generalization of the nonlinear diffusion equation. Next we derive some similarity solutions that describe the flows corresponding to various initial and boundary conditions; these include: 1) the spread of a constant volume of the liquid, 2) the currents produced by sources located at the origin, 3) the steady flow from a source to a sink, and 4) solutions of the waiting-time type [like those arising in nonlinear diffusion, see for example Aronson, 1970, Knerr, 1977, Kamen, 1980, also Lacey et al., 1982, and Kath and Cohen, 1982]. In cartesian geometry, the system allows propagating wave solutions also. Finally we set up a phase plane formalism that allows us to systematically investigate the entire family of self similar solutions of the governing equations. We also discuss the Boltzmann transform method for solving them. The detailed investigation of the solutions corresponding to the integral curves in the phase plane is left for future work.

2. The lubrication approximation for a non newtonian liquid

The governing equations of the creeping gravity flow of a power law liquid on a rigid horizontal surface are obtained assuming that: 1) the motion is essentially horizontal [so that the vertical component of the velocity is negligibly small], 2) that the inertia effects are negligible, and 3) that the length of the current is much larger than its depth. These assumptions imply a purely hydrostatic pressure. We shall consider only planar and axisymmetric flows, i.e. flows that depend on a single horizontal coordinate [cartesian for planar symmetry, radial for axial symmetry]. The horizontal coordinate is denoted by x, the vertical coordinate is z, and t denotes the time. The acceleration of gravity is g, and the constitutive relation (1) is assumed.

With these assumptions, it can be easily shown that the x- and z-components of the momentum equation can be approximated as:

$$\frac{\partial p}{\partial x} = 2^{(\iota-1)/2\iota}\,\sigma A\,\frac{\partial}{\partial z}\left(\frac{\partial|v_x|}{\partial z}\right)^{1/\iota}\;,\quad \sigma = \mathrm{Sign}(v_x)\;, \tag{3}$$

and:

$$\frac{\partial p}{\partial z} = -\,\rho g\;. \tag{4}$$

In (3), (4), ρ is the density, p the pressure, and $v_x(x,z,t)$ is the horizontal component of the velocity. Notice that strictly speaking $\sigma = \mathrm{Sign}(\partial v_x/\partial z)$, but in our system it coincides with $\mathrm{Sign}(v_x)$. In deriving (3) and (4) we have assumed that the strongest variation of v_x is in z, and the x variation of v_x [which is crucial for the continuity equation] is neglected. In fact we shall sneak in the x dependence through the boundary condition at the free surface $z = h(x,t)$, where $h(x,t)$ denotes the thickness of the current.

Integrating (4) with the condition that at $z = h$, $p = 0$, we obtain

$$p = \rho g\,(h - z) \tag{5}$$

which gives the required slow x dependence to the left hand side of Eq. (3), i.e.,

$$\frac{\partial p}{\partial x} = \rho g\,\frac{\partial h}{\partial x} \tag{6}$$

Within the context of the preceding discussion, and with the boundary conditions: 1) no slip at the bottom $z = 0$, and 2) no tangential stress at $z = h$, we can readily integrate (3) to find

$$v_x = \frac{\iota+2}{\iota+1}\,v\left[1 - \left(1 - \frac{z}{h}\right)^{\iota+1}\right]\;, \tag{7}$$

where

$$v = \langle v_x\rangle = \frac{1}{h}\int_0^h v_x\,dz = 2^{(\iota-1)/2\iota}\sigma\,(\rho g/A)^{\iota}\,\frac{h^{\iota+1}}{\iota+2}\left(-\,\sigma\,\frac{\partial h}{\partial x}\right)^{\iota} \tag{8}$$

is the vertically averaged speed. From (8), it is obvious that the sign of v, as expected, is always opposite to that of $\partial h/\partial x$. The explicit appearance of σ is a necessary consequence of the non newtonian rheology.

Equation (8) represents the essence of the momentum transfer equation for this paper. We now take the vertically averaged continuity equation

$$\frac{\partial h}{\partial t} + \nabla\cdot(\mathbf{e}_x\,\langle v_x h\rangle) = \frac{\partial h}{\partial t} + \nabla\cdot(\mathbf{e}_x\,vh) = 0 \tag{9}$$

and combine it with (8) to give a single equation:

$$\frac{\partial w}{\partial t} + \sigma x^{-n}\,\frac{\partial}{\partial x}\left[x^n\,w^{\iota+2}\left(-\sigma\,\frac{\partial w}{\partial x}\right)^{\iota}\right] = 0 \tag{10}$$

where

$$w = a_\iota h\;,\quad a_\iota = \left(A\,2^{(\iota-1)/2\iota}(\iota+2)^{1/\iota}/\rho g\right)^{1/(2\iota+1)}. \tag{11}$$

Notice that the index $n = 0$ [$n = 1$] signifies cartesian [axial] symmetry.

Equation (10), or equivalently the set of equations (8) and (9), rewritten in terms of w, i.e.:

$$v = \sigma w^{\iota+1}\left(-\sigma\,\frac{\partial w}{\partial x}\right)^{\iota}\;, \tag{12a}$$

$$\frac{\partial w}{\partial t} + x^{-n}\,\frac{\partial}{\partial x}\left(x^n w v\right) = 0\;, \tag{12b}$$

are the governing equations for creeping gravity flows in the lubrication approximation; for $\iota = 1$ they reduce to the usual formulae for a newtonian fluid.

Equation (10) is a nonlinear parabolic equation of diffusive type which is, however, different from the usual equations of nonlinear diffusion, or nonlinear heat conduction [see for example Seshadri and Na, 1985]. It can be observed that the assumption of non newtonian rheology [$\iota \neq 1$] introduces a nonlinearity in the spatial derivative of w that was not present in the previously studied [newtonian] cases, so that our generalization is not at all trivial. However, we shall show in the following sections that many solutions of (10) are closely analogous to solutions pertaining to newtonian liquids. In particular, it will be shown that in many instances, the currents are characterized [as is the case for a newtonian liquid] by a sharp, well defined front. The current extends up to a certain value $x = x_f$, and as $x \to x_f$, the thickness h vanishes, there being no fluid ahead of the front. In this connection, it will be seen that the present lubrication approximation predicts profiles of the form $h \propto X^{\nu}$, with $X = |x-x_f|$ and $0 < \nu < 1$. Naturally these vertical profiles are incorrect near the front, where the approximation breaks down. However, it can be expected that the model will correctly describe the general shape and dynamics of the currents, regardless of the fact that the vertical fronts are surely not realistic. The matter has been discussed at length by Huppert [1982], and again by Gratton and Minotti [1988]. While these discussions were in the context of newtonian liquids, we see no reason why the situation should be different in the present case. Accordingly we shall accept solutions with sharp fronts, subject to the qualification that there exists a certain small region near the front where they differ markedly from the "true" solutions.

3. Similarity solutions

By making a judicious choice for w, we are left with no parameters in the governing equations. The variables w and v, in fact, have dimensions which can be completely specified in terms of length [L] and time [T]. We can take advantage of this fact and express w and v in terms of dimensionless variables Z and V,

$$w = (x^{\iota+1}t^{-1}Z)^{1/(2\iota+1)} \quad , \quad v = xt^{-1}V \ , \tag{13}$$

which, in general, depend on x, t, and the parameters of the problem that enter into its specific initial and boundary conditions. Substituting (13) into (12a) and (12b), one finds:

$$\sigma \left(\sigma \frac{V}{Z} \right)^{1/\iota} + \frac{1}{1+2\iota} \left(\iota + 1 + \frac{x}{Z}\frac{\partial Z}{\partial x} \right) = 0 \ , \tag{14}$$

and

$$\frac{t}{Z}\frac{\partial Z}{\partial t} + [(2\iota+1)(n+1) \ +\iota+1]V + V\frac{x}{Z}\frac{\partial Z}{\partial x} + (2\iota+1)x\frac{\partial V}{\partial x} - 1 = 0 \ . \tag{15}$$

Let us now assume that the problem involves only one parameter, b, with independent dimensions. Clearly, it can be assumed without loss of generality that

$$[b] = LT^{-\delta} \ , \tag{16}$$

where δ is a numerical constant. Then there will be a single dimensionless combination of x, t, and b, which we can take as

$$\zeta = x/bt^{\delta} \ . \tag{17}$$

In this case $Z = Z(\zeta)$, $V = V(\zeta)$, and the motion is self similar, ζ being the similarity variable.

For self similar flows the phase variables Z and V satisfy the following ordinary differential equations:

$$\sigma \left(\sigma \frac{V}{Z} \right)^{1/\iota} + \frac{1}{1+2\iota} \left(\iota + 1 + \frac{\zeta}{Z}\frac{\partial Z}{\partial \zeta} \right) = 0 \ , \tag{18}$$

$$[(2\iota+1)(n+1) \ +\iota+1]V + (V - \delta)\frac{\zeta}{Z}\frac{\partial Z}{\partial \zeta} + (2\iota+1)\zeta\frac{\partial V}{\partial \zeta} - 1 = 0 \ . \tag{19}$$

Later on we shall indicate how to obtain from (18) and (19) a general formalism that allows us to derive [in a systematic way] the entire family of solutions corresponding to the similarity variable ζ. In the rest of the present section, we shall discuss some special solutions of particular interest.

3.1 Creeping gravity currents whose volume varies with time according to a power law

These flows obey the global continuity equation

$$\int_0^{x_f(t)} (2\pi x)^n \, h(x,t)dx = q_\alpha t^\alpha \ , \tag{20}$$

where q_α = const., and x_f denotes the position of the front. Clearly $\alpha = 0$ corresponds to a volume conserving current, $\alpha = 1$ to a source of constant flux at $x = 0$, etc.. For the newtonian case, these flows have been studied by Huppert [1982].

Using (9) and (13) in (20) one finds

$$\delta = \frac{1+(2\iota+1)\alpha}{(2\iota+1)(n+1)+\iota+1} \ , \tag{21}$$

and

$$b = (q_\alpha/a_1)^\beta \quad , \quad \beta = \frac{2\iota+1}{(2\iota+1)(n+1)+\iota+1} \,, \tag{22}$$

$$\zeta_f = \left((2\pi)^n \int_0^1 \eta^{1/\beta - 1} \, Z^{1/(2\iota+1)} d\eta \right)^{-\beta} \,, \quad \eta = \zeta/\zeta_f \,. \tag{23}$$

From these results we can determine the spreading relations for these currents: the equation of motion of the front is given by

$$x_f(t) = \zeta_f b t^\delta \,, \tag{24}$$

and for fixed $\eta = x/x_f$, the thickness of the current varies as

$$h \propto t^\gamma \,, \quad \gamma = \delta \, [\, \alpha(\iota+1) - (n+1) \,] \,, \tag{25}$$

and the average flow velocity as

$$v \propto t^{\delta-1} \,. \tag{26}$$

To determine the profile of the current, and the dependence of the average flow velocity on η, it is of course necessary to solve the equations (18)-(19). Barring a few special cases, it is not possible to obtain close form solutions. For a very important case, i. e., that of a volume conserving current [$\alpha = 0$, implying the spreading of a constant volume of liquid], Eqs. (18) and (19) admit a special close form solution given by [$\sigma = 1$]:

$$Z = \delta \left[\left(\frac{2\iota+1}{\iota+1} \right) \left(\eta^{-(\iota+1)/\iota} - 1 \right) \right]^\iota , \quad V = \delta \,, \tag{27}$$

from which one derives,

$$h = \left[a^{n+1} q_o^{(\iota+1)/(2\iota+1)} \right]^\beta \left[\delta \, \zeta_f^{\iota+1} \left(\frac{2\iota+1}{\iota+1} \right)^\iota \right]^{1/(2\iota+1)} \times$$

$$t^{-\delta(n+1)} \left(1 - \eta^{(\iota+1)/\iota} \right)^{1/(2\iota+1)} \,, \tag{28}$$

$$v = \delta \frac{x}{t} \,, \tag{29}$$

with

$$\zeta_f = \left\{ (2\pi)^n \left[\delta \left(\frac{2\iota+1}{\iota+1} \right)^\iota \right]^{1/(2\iota+1)} \ldots \right.$$

$$\left. \ldots \int_0^1 \left(1 - \eta^{(\iota+1)/\iota} \right)^{1/(2\iota+1)} \eta^n \, d\eta \right\}^{-\beta} . \tag{30}$$

It can be verified that for $\iota = 1$ these solutions reduce to those previously known [see for instance Huppert, 1982].

3.2 Waiting time solutions

The governing equations admit what are called "waiting time solutions". These solutions represent flows whose front does not move during a finite interval of time, despite the fact that there is liquid movement behind the front. Solutions of this type appear in problems of nonlinear diffusion, nonlinear heat conduction and other related problems [see for example Aronson, 1970, Knerr, 1977, and Kamen, 1980]. For newtonian creeping gravity currents they have been studied by Gratton and Minotti [1988], who called them "fixed front" solutions. They are related to singular solutions of (18), (19) of the form Z = const., V = const.. It can be easily verified that, in the present case of non newtonian rheology, one also finds solutions of this kind.

There is, in effect, a single constant solution of (18), (19), given by :

$$V = V_0 = [(n+1)(2\iota+1) + \iota+1]^{-1}, \quad Z = Z_0 = - V_0[(2\iota+1)/(\iota+1)]^\iota \tag{31}$$

for $\sigma = -1$. The corresponding flow is given by

$$h = a_1 (x^{\iota+1} t^{-1} Z_0)^{1/(2\iota+1)} \,, \quad v = \frac{x}{t} V_0 \,, \quad t < 0 \,. \tag{32}$$

This solution is only valid for negative t, and blows up at t = 0. It represents a current whose front is stationary for a finite time. In the analogous case of the usual nonlinear diffusion equation, Lacey et al. [1982], and Kath and Cohen [1982] have shown how to construct solutions of this type that can be extended to positive time [when the front starts to move].

4. Steady flows

It is easy to verify that the governing equation (12) admits a time independent solution. It yields a current given by :

$$h = h_0 \left(1 - \eta^{(\iota - n)/\iota} \right)^{\iota/(2\iota+2)} , \qquad (33)$$

$$v = (h_0/a_1)^{2\iota+1} x_r^{-\iota} \eta^{-n} \left(1 - \eta^{(\iota-n)/\iota} \right)^{-\iota/(2\iota+2)} , \qquad (34)$$

$$n \neq \iota \;, \quad \eta = x/x_0 \;, \quad x_0 = \text{const.} \;, \quad h_0 = \text{const.} \;, \qquad (35)$$

which represents the flow over a horizontal surface of finite extent, having an edge at $x = x_0$. The liquid flows from a constant source at $x = 0$ to the edge, and spills over it. It can be verified that

$$q = (2\pi x)^n vh = (2\pi)^n x_r^{n-\iota} h_0^{2\iota+2} a_1^{-(2\iota+1)} = \text{const.}. \qquad (36)$$

These solutions are entirely analogous to those corresponding to the newtonian case studied by Gratton and Minotti [1988]; in this reference the connection between the steady state solutions and the self similar ones is discussed. In the case $n = 1$, $\iota = 1$ [newtonian] (33) and (34) are not valid and w, v depend logarithmically on x.

5. Travelling wave solutions

For the special case of cartesian geometry [$n = 0$] one can find travelling wave solutions of (12). Let

$$w = w(\xi), \quad \xi = ct - x, \quad c = \text{const.}. \qquad (37)$$

In the present case, of course, c does not depend on the properties of the fluid, but is a parameter determined by the boundary conditions, for instance a piston moving at a constant speed that pushes the liquid. As a consequence, it may assume any value. Using (37), (19) can be integrated obtaining

$$\xi - \xi_0 = \sigma \int \left[\frac{\sigma \, w^{\iota+2}}{c(w-K)} \right]^{1/\iota} dw , \qquad (38)$$

where K and ξ_0 are arbitrary constants. Thus the problem has been reduced to a quadrature.

As a special case, consider $K = 0$. Let us further assume $c > 0$; then $\sigma = 1$. Evaluating (38), one obtains:

$$w = \left[\frac{2\iota+1}{\iota} c^{1/\iota} (\xi - \xi_0) \right]^{\iota/(2\iota+1)} . \qquad (39)$$

This solution is the generalization for non newtonian liquids of the travelling wave solution found by Gratton and Minotti [1988]. It represents a current that advances with constant speed c on an infinite horizontal supporting plate; its front is located at $x = ct - \xi_0$.

If one assumes $c < 0$, which requires $\sigma = -1$ then a wave travelling in the opposite sense is obtained; we omit details for brevity.

These currents describe the flow produced by a plane piston, or by a spatula, that advances steadily, pushing a constant volume of liquid in front of it. Actually, the present approximation ceases to be valid inmediately in front of the piston, so that our formula is a good description of the profile of the current as long as one considers only the flow far from the piston.

6. Phase plane formalism

The entire family of self similar solutions of the type (13), (17) can be systematically investigated by means of a phase plane formalism developed in analogy to that of Sedov [1959] and Courant and Friedrichs [1948] for gas dynamics. In this paper, we closely follow the treatment of Gratton and Minotti [1988]. Starting from the coupled equations (18) and (19), it is possible to eliminate ζ to obtain an autonomous first order ordinary differential equation for $V(Z)$:

$$\frac{dV}{dZ} = \frac{(\delta - V)\left[(\iota+1)\sigma(\sigma V/Z)^{1/\iota} + \iota + 1\right] - 1 + \left[(2\iota+1)(n+1)+\iota+1\right]V}{(2\iota+1)\; Z \left[(2\iota+1)\sigma(\sigma V/Z)^{1/\iota} + \iota + 1\right]} . \qquad (40)$$

Once this equation has been solved, and $V(Z)$ is known, $\zeta(Z)$ can be obtained from

$$\frac{d(\ln\zeta)}{dZ} = - \frac{1}{Z \left[(2\iota+1)\sigma(\sigma V/Z)^{1/\iota} + \iota + 1\right]} . \qquad (41)$$

by means of a simple quadrature.

Thus the essential step in the solution of a self similar problem is the integration [numerical, in general] of (40). The Z-V plane is usually called the "phase plane"; a solution of (40) is represented by a curve in the phase plane, which is called an "integral curve". A single integral curve passes through any regular point of the phase plane. Any integral curve represents a self similar current of a certain sort. A self similar problem characterized by specific boundary conditions is represented [in the phase plane] by one or more pieces of the appropriate integral curves. As an example, the integral curve corresponding to the self similar current that describes the spread of a constant volume of liquid is given by the straigth line $V = \delta$ [see section 3.1].

In order to determine which integral curve corresponds to the given problem it is necessary to investigate the behavior of V(Z) in the neighbourhood of the singular points of (40). The whole Z-V plane need to be considered, and according to (13) solutions with $Z > 0$ correspond to $t > 0$, while solutions in the $Z < 0$ half plane are meaningful only for $t < 0$. It can also be observed that $\sigma = 1$ in the first [Z, V > 0] and third [Z, V < 0] quadrants, and $\sigma = -1$ in the second [Z < 0, V > 0] and fourth [Z > 0, V < 0]. The singular points themselves also represent solutions: the waiting time solution, described in section 3.2., is an example. The detailed investigation of the entire family of self similar flows is left for future work.

7. Boltzmann transform formalism

An alternative way of deriving solutions of the governing equation (10) is based on seeking solutions of the form:

$$w = t^{\mu} f(\vartheta), \quad \vartheta = x/t^{\delta} \ , \tag{42}$$

Substituting (42) into (10), one finds that the condition:

$$(\iota+1)\delta = \mu(2\iota+1) + 1 \ , \tag{43}$$

must be satisfied for (42) to be a solution. The resulting ordinary differential equation takes the form:

$$\vartheta^{n}(\mu f - \delta\vartheta f') = -\sigma\ [\vartheta^{n} f^{\iota+2}(-\sigma f')^{\iota}]' \ , \tag{44}$$

where the prime denotes the derivative with respect to ϑ. If we choose

$$\mu = -\ \delta(n+1) \ , \tag{45}$$

Eq. (43) can be integrated once to give:

$$\sigma\ \vartheta^{n} f^{\iota+2}(-\sigma f')^{\iota} = \delta\vartheta^{n+1} f + \text{const.}, \tag{46}$$

along with an *expression for* δ *in terms of the parameters of the* system,

$$\delta = [(2\iota+1)(n+1)+\iota+1]^{-1} \ . \tag{47}$$

For const. = 0, Eq. (46) becomes

$$f' f^{(\iota+1)/\iota} = -\sigma(\sigma\delta\vartheta)^{1/\iota} \ , \tag{48}$$

and can be readily integrated to yield,

$$f = [\ \lambda - \sigma(\sigma\delta)^{1/\iota}(2\iota+1)/(\iota+1)\vartheta^{(\iota+1)/\iota}\ ]^{1/(2\iota+1)} \ , \tag{49}$$

where λ is a constant of integration.

Remembering the fact that our original system is invariant under time translation [and also under space translation for the n = 0 case], we realize that

$$w = \tau^{-\delta(n+1)} \left[\lambda - \sigma(\sigma\delta)^{1/\iota}\ \frac{2\iota+1}{\iota+1}\ (x/\tau^{\delta})^{(\iota+1)/\iota} \right]^{1/(2\iota+1)}, \tag{50}$$

where $\tau = t + \Delta$ [and Δ denotes some arbitrary time], is also a solution. Equation (50) can be viewed as giving the time evolution of the system starting from a well defined initial condition

$$w_0 = w(t = 0) =$$

$$= \Delta^{-\delta(n+1)} \left[\lambda - \sigma(\sigma\delta)^{1/\iota}\ \frac{2\iota+1}{\iota+1}\ (x/\Delta^{\delta})^{(\iota+1)/\iota} \right]^{1/(2\iota+1)}. \tag{51}$$

The constant of integration λ could be determined by imposing a boundary condition. Without any loss of generality, we assume that at $x = 1$, $w_0 = 0$ giving

$$\lambda = \sigma(\sigma\delta)^{1/\imath} \frac{2\imath+1}{\imath+1} \Delta^{-\delta(\imath+1)/\imath} \tag{52}$$

which leads to

$$w = \tau^{-\delta(n+1)} \left[\sigma^{\imath+1}\delta \left(\frac{2\imath+1}{\imath+1} \right)^{\imath} \right]^{1/(2\imath+1)} \times$$

$$\left[\Delta^{-\delta(\imath+1)/\imath} - (x/\tau^{\delta})^{(\imath+1)/\imath} \right]^{1/(2\imath+1)} \tag{53}$$

a result which can be of practical importance. It can be easily verified that (53), with $\sigma = 1$, is equivalent to the solution (28)-(29) [obtained in section 3.1.] corresponding to the spread of a constant volume of liquid.

It is of interest that the solution displayed in Eq. (50) can be transformed into the waiting time solution already discussed in section 3.2. For $\tau \to 0$ [or $t \to -\Delta$], λ can be easily neglected in Eq. (50) yielding:

$$w_{\tau \to 0} \to \left[\left(\frac{2\imath+1}{\imath+1} \right)^{\imath} \delta \left(- \frac{x^{\imath+1}}{\tau} \right) \right]^{1/(2\imath+1)} , \tag{54}$$

which is precisely the solution given by Eq. (32). Notice that in this derivation the solution is valid for either positive [$\Delta = -|\Delta|$] or negative t, but it blows up at $\tau = 0$ [$t = \pm |\Delta|$]. Of course, the waiting time solution can be seen as the special solution with the constant of integration $\lambda = 0$.

8. Summary and final remarks

Based on a generalization of the lubrication approximation of Buckmaster [1977], we have derived the governing equations that describe creeping gravity currents of non newtonian liquids having a power law rheology of the Ostwald type. Currents that depend on a single horizontal coordinate have been considered, both in cartesian and in axial symmetry. The equations are of a nonlinear parabolic type, and differ from those for newtonian liquids in that they are nonlinear in the spatial derivative of the thickness of the current. However, many properties of the solutions are closely analogous to those for newtonian rheology already studied by Huppert [1982], and by Gratton and Minotti [1988]. In

particular the spreading relations for the currents can also be expressed as power laws in time. The exponents, however, are functions of the rheological index, and thus differ from those corresponding to newtonian liquids. The similarity solutions corresponding to currents, whose volume varies as a power of time, have been investigated. For the spread of a constant volume of liquid, analytic solutions are obtained both for the cartesian, and for the axial symmetry. Solutions of the waiting-time type are found, as well as steady flows from a constant source to a sink at a fixed position. General travelling wave solutions have also been obtained in closed form, and analytic formulae for a simple case are given. A general phase plane formalism, that allows us to systematically investigate the entire family of self similar solutions, is introduced. The application of the Boltzmann transform for solving the governing equations is briefly discussed.

We find a close analogy between the present results and those derived for newtonian liquids. All of our solutions have their counterparts in newtonian rheology. This can be traced to the fact that the constitutive relation (1) introduces a single dimensional parameter [A] into the problem, exactly as in the case of a newtonian liquid [the viscosity coefficient]. In both instances, this dimensional parameter can be scaled out by an appropriate definition of the dependent variable, and thus does not appear in the final governing equations. For this reason one finds similarity solutions whenever the analogous newtonian problem is self similar. The dimensionality of A depends on the rheological index $\imath$, and as a consequence, the spreading relations have rheology-dependent exponents. It is interesting to observe, however, that in most cases this dependence is rather weak, a fact that was already pointed out, in a particular instance, by Gratton [1988].

The present results can be of interest for geophysics since a power law rheology gives an acceptable description of the average mechanical properties of the lithosphere.

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