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Introduction

In order to optimize the extraction scheme used to take antiprotons out of the accumulator, it is necessary to understand the basic processes involved. At present, six antiproton bunches per Tevatron store are removed sequentially by RF unstacking from the accumulator. The phase space dynamics of this process, with its accompanying phase displacement deceleration and phase space dilution of portions of the stack, can be modelled by numerical solution of the longitudinal equations of motion for a large number of particles. We have employed the tracking code ESME[1] for this purpose. In between RF extractions, however, the stochastic cooling system is turned on for a short time, and we must take into account the effect of momentum stochastic cooling on the antiproton energy spectrum. This process is described by the Fokker-Planck equation, which models the evolution of the antiproton stack energy distribution by accounting for the cooling through an applied coherent drag force and the competing heating of the stack due to diffusion, which can arise from intra-beam scattering, amplifier noise and coherent (Schottky) effects. In this note we examine the aspects of the Fokker-Planck in the regime where the nonlinear terms due to Schottky effects are small. This discussion ultimately leads to solution of the equation in terms of an orthonormal set of functions which are closely related to the quantum simple-harmonic oscillator wave-functions.

The Fokker-Planck Equation

The Fokker-Planck equation describing stochastic cooling of the antiproton beam energy distribution $\psi(E)$ is typically written as[2]

$$\frac{\partial\psi(E)}{\partial t} = -\frac{\partial}{\partial E}[F(E)\psi(E) - D(E)\frac{\partial\psi(E)}{\partial E}]. \tag{1}$$

The drag force coefficient $F(E)$ and the diffusion coefficient $D(E)$ are in general not simple functions. In fact, $D(E)$ contains a component due to the stack Schottky noise signal which is proportional to $\psi(E)$, making the equation nonlinear. These complications have led to numerical integration as the favored method of solution of the Fokker-Planck equation in this context. Inside the core of the stack, however, the diffusion is not necessarily Schottky dominated – the heating due to intra-beam scattering, which is nearly a constant component of $D(E)$ over the relevant energy spectrum, can be larger than the component due to Schottky noise. If we assume the approximation that $D = D_0$ is constant, then linear analysis of the core energy spectrum becomes straightforward. Since the RF unstacking process greatly disturbs the core spectrum itself, this analysis is particularly relevant to our extraction optimization problem.

Inside the core, the drag force coefficient can be approximated well by $F(E) = -\alpha(E - E_0)$, where E_0 is the energy of core peak, corresponding to the cenral frequency of the cooling amplifier notch filter. We now write our Fokker-Planck equation for the core energy distribution as

$$\frac{\partial\psi(E)}{\partial t} = \frac{\partial}{\partial E}[\alpha(E - E_0)\psi(E) + D_0\frac{\partial\psi(E)}{\partial E}]. \tag{2}$$

We now examine possible analytical approaches to solution of this equation, starting first with a discussion of the evolution of the moments of the energy distibution. This analysis will serve to illuminate the physical processes implied in the solutions to the Fokker-Planck equation itself, which we present in a later section.

Moments of the Fokker-Planck Equation

As our equation describes a distribution of particles in energy, an obvious starting point is to examine the moments of the distribution. The n-th moment of the distribution is defined as

$$\Psi_n = \frac{1}{N}\int_{-\infty}^{\infty} E^n \psi(E) dE, \; n = 0, 1, 2, \ldots \tag{3}$$

where N is the total number of particles in the stack. The first moment Ψ_1 is merely the average energy of the stack particles $\bar{E}$, while the second is related to the rms energy spread of the distribution by

$$\sigma_E^2 = \frac{1}{N}\int_{-\infty}^{\infty} (E - \bar{E})^2 \psi(E) dE = \Psi_2 - \Psi_1^2. \tag{4}$$

The equations of motion for the moments of the distribution are simply written as

$$\frac{\partial \Psi_n}{\partial t} = \int_{-\infty}^{\infty} E^n \frac{\partial}{\partial E}[\alpha(E - E_0)\psi(E) + D_0 \frac{\partial \psi(E)}{\partial E}] dE. \tag{5}$$

The right hand side of the equation can be integrated by parts to yield

$$\frac{\partial \Psi_n}{\partial t} = n\alpha[E_0 \Psi_{n-1} - \Psi_n] + n(n-1) D_0 \Psi_{n-2}. \tag{6}$$

This expression can easily be generalized to include higher polynomial in E dependence for F and D. As a simple example of the use of these expressions, we take the case $\bar{E} = E_0$, and write the equation for the time evolution of the rms energy spread as

$$\frac{\partial \sigma_E^2}{\partial t} = -2\alpha\sigma_E^2 + 2D_0. \tag{7}$$

The solution to this equation is

$$\sigma_E^2 = \sigma_{eq}^2 + (\sigma_0^2 - \sigma_{eq}^2)\exp(-2\alpha t), \tag{8}$$

where $\sigma_{eq}^2 = D_0/\alpha$, and $\sigma_0^2 = \sigma_E^2(0)$. The cooling time for the energy spread is $\tau_c = (2\alpha)^{-1}$, as could be expected. In the accumulator τ_c is on the order of twenty to thirty minutes.[3]

We now discuss some additional aspects of the evolution of the moments. To simplify the moment equations we can set the constant $E_0 = 0$ for the remainder of our discussion. This allows a hierarchy of equations in which only the moments of the same parity affect each other. Thus the equations can be solved by climbing the even and odd ladders of the moment hierarchy separately. It should be noted in this regard that for symmetric distributions, the odd moments vanish,

$$\Psi_n = 0, \quad n \text{ odd}. \tag{9}$$

Also, for a Gaussian distribution of rms width σ_E, written

$$\psi(E) = \frac{N e^{(-E^2/2\sigma_E^2)}}{\sqrt{2\pi}\sigma_E}, \tag{10}$$

we have the moments

$$\Psi_n = \sigma_E^n (n-1)!!, \quad n \text{ even}. \tag{11}$$

From this equation, we have the relation $\Psi_n/\Psi_{n-2} = \sigma_E^2(n-1)$, and the ratio of the moments increases linearly with n. If we assume the Gaussian is in equilibrium, then $\sigma_E^2 = \sigma_{eq}^2 = D_0/\alpha$.

Now, having established the equilibrium values of the moments, we take as an example perturbation of the distribution the case where a narrow band of $m \ll N$ particles with energy width $\sigma_s \ll \sigma_E$ is removed from the neighborhood of the energy E_s. The change in the moments due to this perturbation is approximately

$$\Delta\Psi_n \simeq -mE_s^n. \tag{12}$$

If we subtract out the equilibrium values of the moments, the equations of motion for the moment perturbations are simply

$$\frac{\partial \Delta\Psi_n}{\partial t} = -n\alpha\Delta\Psi_n + n(n-1)D_0\Delta\Psi_{n-2}. \tag{13}$$

For the first two moments, the solution to this equation is merely a simple exponential,

$$\Delta\Psi_n = \Delta\Psi_n(0)\exp(-n\alpha t), \ n = 1, 2. \tag{14}$$

All higher moments can be solved for straightforwardly by climbing the ladder of moment equation hierarchy. Note that the ratio of drag (cooling) to diffusion (heating) terms in the hierarchy of equations is

$$\frac{n\alpha\Delta\Psi_n}{n(n-1)D_0\Delta\Psi_{n-2}} = \frac{1}{(n-1)}\left(\frac{E_s}{\sigma_{eq}}\right)^2. \tag{15}$$

Thus if $E_s < \sigma_{eq}$, then all moment evolution for $n > 2$ is diffusion dominated. A drastic, local dip in the core distribution is remedied first by the diffusion of particles from the nearby energies, not by coherent drag force effects. If on the other hand $E_s > \sigma_{eq}$, then the perturbation is away from the bulk of the distribution, diffusion is small and a large coherent drag force can be dominant for small n.

The moment approach to the dynamics of the energy spectrum has given us some insight into the mechanisms of core cooling as the evolution of each moment can be calculated explicitly. The moments of the distribution are not a complete, orthogonal set, however, and thus cannot be inverted to give the distribution at a later time. With this in mind, we now turn to the solution of the linear Fokker-Planck equation in terms of such an orthogonal set.

Solution of the Fokker-Planck Equation

The fact that the moments of the Fokker-Planck equation in its assumed form have such simple evolution equations containing only the next lower moment of the same parity, as well the existence of a Gaussian equilibrium solution, strongly suggests that the solutions to this equation are related to the Hermite polynomials. It is straightforward to show that this is the case.

If we assume that the solution to the Fokker-Planck equation is separable in energy and time,

$$\psi(E,t) = \Phi(E)T(t), \tag{16}$$

then the two resultant equations are

$$\frac{dT}{d\tau} = \lambda T, \tag{17}$$

where $\tau = \alpha t$ is the normalized time, λ is the separation constant, and

$$\Phi'' + 2x\Phi' + 2(1-\lambda)\Phi = 0 \tag{18}$$

where $x^2 = (\alpha/2D_0)(E-E_0)^2 = (E-E_0)^2/2\sigma_{eq}^2$ and $'$ denotes a derivative with respect to x, the normalized energy. The formal solution for the time dependent part is of course

$$T(\tau) = \exp(\lambda\tau). \tag{19}$$

From this we can see that for some solution we must have $\lambda = 0$, corresponding to the stationary core profile. All other solutions are expected to give dying exponential behavior, *i.e.* $\lambda < 0$.

In order to solve the energy equation we write it in terms of a new function $\Omega(x) = \Phi(x)\exp(x^2/2)$, and find upon substitution

$$\Omega'' + (1 - 2\lambda - x^2)\Omega = 0. \tag{20}$$

If we tentatively allow the separation constant to take on the only values $\lambda_n = -n$, this equation is identical to that of the Schrodinger equation for the simple harmonic oscillator. It turns out that these are in fact the only allowable solutions, and that this is due to a normalization condition which is similar, but not completely analogous, to the quantum oscillator case. The solutions of our transformed equation are

$$\Omega_n(x) = \exp(-x^2/2)H_n(x), \tag{21}$$

where the H_n are the Hermite polynomials. The Ω_n form a complete, orthogonal set of solutions to the transformed equation, and the equivalent solutions to our original energy equation are

$$\Phi_n(x) = \exp(-x^2/2)\Omega_n(x) = \exp(-x^2)H_n(x). \tag{22}$$

The remaining issue at hand is the normalization of these solutions, examination of which validates the choice of eigenvalues we have made above. We have the particle conservation condition

$$\int_{-\infty}^{\infty} \psi(E,t)dE = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \Phi_n(x)dx \exp(-n\tau) = N \tag{23}$$

In order for this condition to be satisfied at all times $\tau > 0$, we must have

$$\int_{-\infty}^{\infty} \Phi_0(x)dx = N \tag{24}$$

which gives us a normalization constant for Φ_0. Note that this solution is merely the equilibrium core profile,[4] a Gaussian of width $\sigma_{eq} = \sqrt{D_0/\alpha}$,

$$\Phi_0(x)dx = \frac{N}{\sqrt{\pi}} \exp(-x^2)dx = \frac{N}{\sqrt{2\pi}\sigma_{eq}} \exp(-(E - E_0)^2/2\sigma_{eq}^2)dE, \quad (25)$$

as we would expect from our moment analysis. For all higher n the conservation condition yields,

$$\int_{-\infty}^{\infty} \Phi_n(x)dx = 0, \quad n \geq 1. \quad (26)$$

That this condition holds for our solutions is easily shown. The Rodrigues representation of the Hermite polynomials[5] gives us the convenient form

$$H_n(x) = (-)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2). \quad (27)$$

Thus we have simply

$$\int_{-\infty}^{\infty} \Phi_n(x)dx = (-)^n \int_{-\infty}^{\infty} \frac{d^n}{dx^n} \exp(-x^2)dx = (-)^n \frac{d^{n-1}}{dx^{n-1}} \exp(-x^2)|_{-\infty}^{\infty} \equiv 0, \quad (28)$$

for all integer $n \geq 1$. This only holds for our choice of eigenvalues, thus justifying our identification of $\lambda_n = -n$. The normalization for the set of solutions is as follows:

$$\Omega_n(x) \Rightarrow (2^n \pi^{1/2} n!)^{-1/2} \Omega_n(x). \quad (29)$$

The total solution for the evolution of the energy spectrum is thus

$$\psi(E,t) = \sum_{n=0}^{\infty} c_n \exp(-x^2) H_n(x) \exp(-n\tau) \quad (30)$$

where projection of the initial ($\tau = 0$) energy distribution on this complete orthonormal set is given, with $\psi(E,0) = \Phi(x)$, by

$$c_n = \int_{-\infty}^{\infty} \Phi(x) \exp(x^2/2) \Omega_n(x)dx = (2^n \pi^{1/2} n!)^{-1/2} \int_{-\infty}^{\infty} \Phi(x) H_n(x)dx. \quad (31)$$

That these projections should be linear combinations of the moments of the distribution (they might be termed the Hermite moments) is certainly not surprising in light of our previous results. The cooling rate for the second Hermite component is 2α, in agreement with our earlier direct calculation of the cooling of the rms energy spread to its equilibrium value in Eq. 7. Note that for the general solutions the requirement that the distribution $\psi(E,t) \geq 0$ is satisfied for all time if it is satisfied initially.

These solutions have been implemented as a subroutine in ESME, to quickly follow the evolution of the calculated stack energy spectrum in tracking studies of RF unstacking from the accumulator. Comparisons with the numerical solution of the Fokker-Planck equation using the nonlinear terms in $D(E)$ show no significant difference for the cases we have studied. These calculations will be presented in a later communication. We now discuss other possible uses for this formalism.

Perturbation Theory

The solution of the Fokker-Planck equation in terms of an orthonormal basis of functions allows, in analogy with quantum mechanics, the use of perturbation theory to solve for cases close to the one already solved. In particular, one can solve for the asymptotic solution, or static core. We begin by writing the eigenvalues as $\lambda_n = -n + \lambda_n^{(1)}$. As in quantum mechanics, we can write the first order perturbations in the eigenvalues as

$$\lambda_n^{(1)} = \int_{-\infty}^{\infty} \Omega_n(x) K^{(1)} \Omega_n(x) dx, \tag{32}$$

where the perturbed 'Hamiltonian' operator is given in terms of additional energy dependences of the diffusion and drag force coefficients. Writing explicitly $D(E) = D_0(1 + d(E))$ and $F(E) = -\alpha(1 + f(E))(E - E_0)$, with $|d(E)|, |f(E)| \ll 1$, we have the operator

$$K^{(1)} = \frac{\partial}{\partial x}[d(x)\frac{\partial}{\partial x} + f(x)]. \tag{33}$$

For the case $n = 0$, which corresponds the static solution (the "ground state" of the cooling system), we must have $\lambda_0 = 0$ to insure the time independence of the solution. The perturbation treatment does not in general satisfy this requirement for arbitrary $K^{(1)}$, which is a limitation of this method. Fortunately, the changes in eigenvalues are not of interest at this point. However, the changes in the orthonormal basis functions are, and they are given in first order by

$$\Omega_n^{(1)} = \sum_{m=0}^{\infty} \frac{\Omega_m(x)}{(m-n)} \int_{-\infty}^{\infty} \Omega_m(x) K^{(1)} \Omega_n(x) dx, \quad m \neq n. \tag{34}$$

For the static core solution this expression is

$$\Omega_0^{(1)} = \sum_{m=1}^{\infty} \frac{\Omega_m(x)}{m} \int_{-\infty}^{\infty} \Omega_m(x) K^{(1)} \pi^{-1/4} \exp{(-x^2/2)} dx. \tag{35}$$

It would be most beneficial to be able to introduce nonlinear terms into the perturbed 'Hamiltonian', in order to model the effects of the Schottky noise derived diffusion term. This is not a familiar use of the perturbation scheme, and is not in general valid, but it can be used to calculate the new equilibrium state if the nonlinear effects are small. This is accomplished by substitution of the linear static core solution for ψ in the above equation. This process can be iterated - although it is not guaranteed to converge quickly. As an example, the first order change in the static core solution due to a diffusion term proportional to ψ, $d(x) = d_1\psi(x) \ll 1$, can be written as

$$\Omega_0^{(1)} = \sum_{m=1}^{\infty} \Omega_m(x) \frac{N d_1}{\pi^{3/4} m} \int_{-\infty}^{\infty} \Omega_m(x) \frac{\partial}{\partial x} \exp{(-x^2)} \frac{\partial}{\partial x} \exp{(-\frac{x^2}{2})} dx \tag{36}$$

which can be written, after integration by parts, as

$$\Omega_0^{(1)} = \sum_{m=1}^{\infty} \Omega_m(x) \frac{N d_1}{\pi^{3/4} m} \int_{-\infty}^{\infty} x \exp{(-3x^2/2)} \frac{\partial}{\partial x} \Omega_m(x) dx. \tag{37}$$

It can easily be seen from parity arguments that no odd m solutions will contribute to the new core profile. This must be so because an odd m solution is asymmetric, and cannot be generated by diffusion alone. Evaluating the first nonzero component of the perturbed solution, we have for the start of the series solution for the new core:

$$\Phi_0(x) = \tfrac{N}{\sqrt{\pi}} \exp{(-x^2)} [H_0(x) + \tfrac{7}{64\sqrt{2\pi}} N d_1 H_2(x) + \ldots \tag{38}$$

$$= \tfrac{N}{\sqrt{\pi}} \exp{(-x^2)} [1 + \tfrac{7}{64\sqrt{2\pi}} N d_1 (4x^2 - 2) + \ldots \tag{39}$$

This form shows the flattening and expansion of the core due to the Schottky nonlinear diffusion term, as well as the amplitude (N) dependence, explicitly. The added core width due to the first perturbation term is

$$\frac{\Delta\sigma_E^2}{\sigma_E^2} = \frac{7}{16\sqrt{2\pi}} N d_1. \tag{40}$$

The added width is a term proportional to the stack size, added in squares with the calculated linear width.

It is instructive to compare these results to the exact static solution of the Fokker-Planck equation assuming the Schottky noise dominates all other sources of diffusion. If we include only a Schottky term in the diffusion coefficient, $D(E) = D_1\psi(E)$, then the solution of the time-independent Fokker-Planck equation is simply

$$\psi(E) = \rho[1 - \frac{\alpha E^2}{2\rho D_1}], \quad |E| < \sqrt{2\rho D_1/\alpha}, \tag{41}$$

where $\rho = [(3N/4)^2(\alpha/2D_1)]^{1/3}$, and the boundary of the now parabolic distribution is therefore at $|E_m| = (3ND_1/2\alpha)^{1/3}$. Note that this solution implies that adding more anti-protons to the stack in the Schottky-dominated regime increases the maximum spectral density as $N^{2/3}$, instead of the linear N dependence obtained in the constant diffusion coefficient case. The squared width of this parabolic distribution is

$$\sigma_E^2 = \frac{2}{15}(\frac{3}{4})^{5/3}(\frac{2ND_1}{\alpha})^{2/3}. \tag{42}$$

The fact that the squared width does not rise linearly with N, as our perturbation calculation gives, emphasizes the difference between the Schottky-dominated and the nearly constant diffusion coefficient case.

Discussion

This formalism developed in this note has proven to be of use in a specific problem, namely that of quickly determining the evolution of the stack energy distribution under the influence of stochastic cooling after it has been disturbed by the unstacking process. We have discussed the possible uses of perturbation analysis, and pointed out the difficulties arising from inclusion of the nonlinear Schottky diffusion term in the analysis. We have not examined the possible use of the perturbation approach in analyze coherent instabilities in the stack. In this regard, it should be noted that inclusion of imaginary parts in the drag force and diffusion coefficients can give wave-like solutions for the energy distribution which could be unstable. Further consideration of this subject is beyond the scope of the present work.

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